Remarks on Scattering Theory*

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Some consequences of the construction of potentials from phase shifts are derived. The question of selecting a unique potential by asymptotic requirements, and the connection with analyticity of the S-matrix is discussed. An expression for the variational derivative of the potential with respect to the phase shift is derived. A simple exact relation between the value of the potential at the origin and the first moment of the phase shift emerges as a result. The variational derivative is made the basis of a "phase-shift perturbation method." One "unperturbed" S-matrix suggests itself naturally, and is here called the "intrinsic" S-matrix of the bound state. The "intrinsic" potential is discussed. Variational derivatives of one phase shift with respect to another, as well as an extension of the phase-shift perturbation method near the intrinsic S-matrix to the presence of a tensor force, are given.

INTRODUCTION

IN a number of recent papers[1,2] the complete procedure has been established for obtaining all potentials that cause a given phase shift (of one angular momentum, as a function of the energy) and given bound states. It is the purpose of the present note to discuss some consequences of that construction, which are obtained by varying the quantities that determine the potential.

In the first part we outline the Gel'fand-Levitan method and discuss the consequences of a variation of the parameters C_n, upon which the potential depends in the presence of bound states.[1] The result shows that in certain cases a requirement on the asymptotic decay of the potential can serve to select the latter uniquely, even in the presence of bound states; in other cases it cannot. It is shown that no finite piece of the phase shift allows a decision on whether the potential decays exponentially or not.

In the second section, we perform infinitesimal variations of the phase shift and derive an expression for the variational derivative of potential and wave function with respect to it. As a consequence we obtain the main new result of the present paper: a simple, exact relation between the value of the potential at the origin and the first moment of the phase shift.

In Sec. III the variational derivative is made the basis of a "phase-shift perturbation method." A special case is the inversion of the Born approximation, while others include the possibility of bound states. A special "unperturbed" phase shift suggests itself naturally, and is here called the "intrinsic" phase shift of the bound state. The set of "intrinsic" potentials is discussed.

Since the phase shift of one angular momentum (plus the bound states and normalizations there) already uniquely determines a central potential, all other phase shifts of the latter are functionals of the former. That fact is used in Sec. IV to derive an expression for the variational derivative of one phase shift with respect to another. Two phase shifts are found to be most sensitive to one another's small local variations at equal energies, except when they differ by an odd integral multiple of $\pi/4$.

Section V, finally, generalizes the phase-shift perturbation method near the intrinsic S-matrix to the case of the presence of a tensor force, allowing for variations of two phase shifts and the "mixture angle." There is an appendix containing the proof of a special sufficient condition for exponential decay of the asymptotic potential.

I. CHANGES IN THE PARAMETER C

We recall the essentials of the Gel'fand Levitan[1,3] method. Let $\phi(E,r)$ be a regular radial wave function of the angular momentum l whose r-dependence near the origin is independent of E such that

$$(2l+1)!!\lim_{r\to 0} r^{-(l+1)}\phi(E,r)=1,$$

and whose spectral function (weight function in the completeness relation) is $\rho(E)$. Corresponding to an arbitrary change in the spectral function, $\Delta\rho(E)$, there is a function

$$g(s,r)=\int d\Delta\rho(E)\phi(E,s)\phi(E,r), \tag{1}$$

which is used both as kernel and inhomogeneity in the Fredholm equation

$$K(s,r)+g(s,r)+\int_0^r dt\, g(s,t)K(t,r)=0. \tag{2}$$

This integral equation always has a unique solution

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[1] R. Jost and W. Kohn, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 9 (1953); also previous papers by R. Jost and W. Kohn, Phys. Rev. **87**, 977 (1952); **88**, 382 (1952).

[2] Generalization to the tensor force: R. G. Newton and R. Jost, Nuovo cimento **1**, 590 (1955); R. G. Newton, Phys. Rev. **100**, 412 (1955).

[3] I. M. Gel'fand and B. M. Levitan, Doklady Akad. Nauk. S.S.S.R. Ser. **77**, 557 (1951), and Izvest. Akad. Nauk. S.S.S.R. **15**, 309 (1951).

$K(s,r)$, which yields the change in the wave function,

$$\Delta\phi(E,r)=\int_0^r dt\phi(E,t)K(t,r), \tag{3}$$

and the change in the potential,

$$\Delta V(r)=2\frac{d}{dr}K(r,r). \tag{4}$$

The lth eigenvalue of the S-matrix can be written (uniquely) as

$$S(k)=f(k)/f(-k),$$

where $f(k)$ is analytic in the lower half-plane, has (simple) zeros there only at a number of discrete points on the negative imaginary axis, each corresponding to a bound state,[4] and tends in that half-plane to unity at infinity. Then the spectral function is given (with $k^2=E$) by

$$\rho(-\infty)=0,$$

$$\frac{d\rho(E)}{dE}=\begin{cases}\pi^{-1}k^{2l+1}|f(k)|^{-2}, & E>0,\\ \sum_n C_n\delta(E+E_n), & E\leq 0.\end{cases}$$

The non-negative numbers E_n are the bound-state energies, and the positive numbers C_n are the reciprocal normalizations of the bound-state wave functions,[1]

$$C_n^{-1}=\int_0^\infty dr\phi^2(-E_n,r).$$

Both E_n and C_n are independent of the S-matrix. (The *normalized* bound state wave function, $\Phi_n(r)$, satisfies the relation: $(2l+1)!!\lim r^{-(l+1)}\Phi_n(r)=C_n^{\frac{1}{2}}$, as $r\to 0$.)

The result of a (finite) change in one of the normalization parameters C_n, say in C_m, was derived by Jost and Kohn[5]:

$$\Delta V(r)=-2\frac{d^2}{dr^2}\log\left[1+\Delta C_m\int_0^r dt\phi_m^2(t)\right].$$

The asymptotic behavior of this potential change is

$$\Delta V(r)\sim\text{const}\times\exp(-2K_m r),\quad K_m^2=E_m.$$

It follows that if there is one potential with given phase shift *and* given binding energies E_n, and with the property that

$$\lim_{r\to\infty}V(r)e^{2Kr}=0,\quad K^2=\max E_n, \tag{5}$$

then this potential is the only one with that property. Furthermore, it follows that if there is one potential with the property that

$$\lim_{r\to\infty}V(r)e^{2Kr},\quad K^2=\min E_n,$$

does not exist, then every potential with the same phase shift and bound states has it. In the first instance there is a natural possibility of selecting a *unique* potential out of the family of those equivalent with respect to both phase shift and binding energies.

As is well known, it follows from (5) that the S-matrix is analytic in a strip of width K above the real axis, except for a number of poles there on the imaginary axis corresponding to all the bound states.[6] Therefore, it is a *necessary* condition for the existence of a *unique* potential [in the sense of (5)] that the S-matrix is in fact analytic in a strip above the real axis which contains all the bound states except for the points corresponding to the latter, so that all of them are recognizable as poles and no poles are "redundant." What additional criteria are *sufficient* for the existence of such a unique potential is not known.

Among the family of Bargmann[7] potentials, i.e., those for which the (lth eigenvalue of the) S-matrix is a rational function of k, it is sufficient for exponential decay of the potential that near $k=0$, $f(k)-f(0)=O(k^{2l})$.[8] In that case $\lim V(r)\times\exp(2\alpha r)$, as $r\to\infty$, exists for all α smaller than or equal to the imaginary part of the point in the upper half-plane at which the S-matrix has its closest pole to the real axis, and for *one* potential α can, in fact, go as high as the imaginary part of the first pole that does not correspond to a bound state. One may then conjecture the following general theorem: A sufficient criterion for the existence of a potential $V(r)$ for which $V(r)\exp[2(\alpha-\epsilon)r]$ is integrable at infinity for all $\epsilon>0$ is the analyticity of the S-matrix in a strip of width α above the real axis, except for simple poles corresponding to the bound states, together with uniform approach to unity at infinity in that strip and $f(k)-f(0)=O(k^{2l})$ near $k=0$. The author has not, however, succeeded as yet in proving this conjecture.

It is clear from the fact that analyticity of the S-matrix is necessary for exponential decay of the potential, that no finite piece of the phase shift can render information about whether the potential decreases asymptotically like an exponential. In fact, the smallest "kink" in the phase shift at high energies can destroy that analyticity and thereby radically change the asymptotic behavior of the potential.

II. CHANGES IN THE PHASE SHIFT

Instead of varying the parameter C, we shall now perform infinitesimal variations of the spectral function in the region of positive energies. Then, clearly, the

[4] "Points corresponding to the bound states" is always meant in the sense: $\pm iK$, if the binding energy is K^2.

[5] See reference 1, (2.9).

[6] See reference 1, Sec. 3.

[7] V. Bargmann (unpublished). The author is greatly indebted to Professor Bargmann for showing him his manuscript of the explicit construction of all potentials for which the lth eigenvalue of the S-matrix is a rational function of k.

[8] This theorem was first proved by Dr. Thomas Fulton. A proof, different from his, can be found in the appendix.

first order in the variation,

$$K(s,r)=-g(s,r)=-\int\phi(E,s)\phi(E,r)d\delta\rho(E),$$

and the corresponding first variation of the potential,

$$\delta V(r)=-2\frac{d}{dr}\int\phi^2(E,r)d\delta\rho(E).$$

The variation in the spectral function which we are contemplating is due to a small change in the phase shift. Now, in the presence of bound states with energies K_n^2 we can write

$$f(k)=e^{i\xi(k)}f_1(k)=|f(k)|\exp i[\xi(k)+\eta_1(k)],$$

where

$$\xi(k)=2\sum_n\tan^{-1}(K_n/k) \tag{6}$$

is the "intrinsic phase shift" associated with the bound states, and in the lower half-plane $f_1(k)$ is again analytic, tends to unity as $|k|\to\infty$, and does not vanish anywhere. Therefore its logarithm is analytic in the lower half-plane and

$$\log|f(k)|=\log|f_1(k)|=-\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{dk'\eta_1(k')}{k'-k}$$
$$=-\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{dk'k'\eta_1(k')}{k'^2-k^2},$$

if $\eta_1(k)$ is defined as an odd function of k, and $\mathcal{P}$ indicates the Cauchy principal value. Variations in the phase shift will, of course, be such as not to change the "intrinsic" one, and therefore

$$\frac{\delta|f(k)|}{|f(k)|}=-\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{dk'k'\delta\eta(k')}{E-E'}.$$

The variation of the spectral function concomitant with that in the phase shift is then, for $E>0$,

$$\delta\frac{d\rho(E)}{dE}=-(2/\pi)k^{2l+1}\delta|f(k)|/|f(k)|^3$$
$$=-\frac{d\rho(E)}{dE}\frac{2}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{dk'k'\delta\eta(k')}{E-E'},$$

and zero for $E\leq 0$. Therefore, after an interchange in the order of integration,

$$K(s,r)=-\frac{2}{\pi}\int_{-\infty}^{\infty}dk'k'\delta\eta(k')\mathcal{P}\int_0^{\infty}\frac{d\rho(E)}{E'-E}\phi(E,s)\phi(E,r).$$

We now recognize the second integral as part of a Green's function:

$$\mathcal{G}(E';s,r)$$
$$=\mathcal{P}\int_{-\infty}^{\infty}\frac{d\rho(E)}{E'-E}\phi(E,s)\phi(E,r)$$
$$=\mathcal{P}\int_0^{\infty}\frac{d\rho(E)}{E'-E}\phi(E,s)\phi(E,r)+\sum_n C_n\frac{\phi_n(s)\phi_n(r)}{E'+E_n}. \tag{7}$$

The variational derivative of the potential at r with respect to the phase shift at k can therefore be written

$$\frac{\delta V(r)}{\delta\eta(k)}=-\frac{4}{\pi}\frac{d}{dr}\left[\mathcal{G}(E;r,r)-\sum_n C_n\frac{\phi_n^2(r)}{E+E_n}\right], \tag{8}$$

and that of the wave function,

$$\frac{\delta\phi(E',r)}{\delta\eta(k)}=-\frac{2}{\pi}\int_0^r dt\phi(E',t)$$
$$\times\left[\mathcal{G}(E;t,r)-\sum_n C_n\frac{\phi_n(t)\phi_n(r)}{E+E_n}\right]. \tag{9}$$

The Green's function $\mathcal{G}(E;s,r)$ vanishes at $s=0$ for arbitrary r and is symmetric in the two variables. It can therefore also be written

$$\mathcal{G}(E;s,r)=\begin{cases}\phi(E,s)\psi(E,r), & s\leq r,\\ \phi(E,r)\psi(E,s), & s\geq r,\end{cases}$$

where $\psi(E,r)$ is an "irregular" solution of the Schrödinger equation whose Wronskian with $\phi(E,r)$ is unity[9]:

$$[\phi;\psi]=1.$$

We therefore have

$$\frac{\delta V(r)}{\delta\eta(k)}=-\frac{4}{\pi}\frac{d}{dr}\left[\phi(k^2,r)\psi(k^2,r)-\sum_n C_n\frac{\phi_n^2(r)}{k^2+E_n}\right], \tag{10}$$

and

$$\frac{\delta\phi(E',r)}{\delta\eta(k)}=-\frac{2}{\pi}\int_0^r dt\phi(E',t)$$
$$\times\left[\phi(k,^2t)\psi(k^2,r)-\sum_n C_n\frac{\phi_n(t)\phi_n(r)}{k^2+E_n}\right]. \tag{11}$$

The "irregular" solution $\psi(E,r)$ is, so far, not uniquely determined. We obtain it by expressing $\phi(E,s)$ in (7), for $s>r$, in terms of $f(k)$ and $f(k,s)$:

$$\phi(E,s)=(2i)^{-1}k^{-(l+1)}[(-1)^l f(k)f(-k,s)-f(-k)f(k,s)].$$

Then

$$\mathcal{G}(E;s,r)=\frac{i}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{dk'k'^{l+1}}{k^2-k'^2}\frac{f(k',s)}{f(k')}\phi(k',r)$$
$$+\sum_n C_n\frac{\phi_n(s)\phi_n(r)}{E+E_n}.$$

[9] The bracket symbol is used for the Wronskian: $[\phi;\psi]=\phi\psi'-\phi'\psi$.

The path of integration can be supplemented by a large semicircle in the lower half-plane, whose contribution vanishes with increasing radius because $|\phi|\sim|k|^{-(l+1)}\exp(\mathrm{Im}kr)$ as $|k|\to\infty$, and $s>r$. The value of the integral is therefore $2\pi i$ times the sum of the residues in the lower half-plane, plus πi times the sum of the residues on the real axis. The former just cancel the bound state terms, and the remainder, divided by $\phi(E,r)$, is

$$\psi(E,s)=-\tfrac{1}{2}k^l\left[\frac{f(k,s)}{f(k)}+(-1)^l\frac{f(-k,s)}{f(-k)}\right]=-k^l\,\mathrm{Re}\frac{f(k,s)}{f(k)}.$$

The variation of the potential with respect to the phase shift can therefore also be expressed as

$$\frac{\delta V(r)}{\delta\eta(k)}=\frac{2}{\pi}\frac{d}{dr}\left\{\mathrm{Im}[S(k)f^2(-k,r)]+2k\sum_n C_n\frac{\phi_n^2(r)}{E_n+k^2}\right\}. \quad (12)$$

The variation of the wave function on the other hand, can be somewhat simplified by the use of the identity

$$[\phi(E,r);\phi(E',r)]=(E-E')\int_0^r dt\phi(E,t)\phi(E',t),$$

which is readily proved from the Schrödinger equation and the boundary condition of $\phi(E,r)$. Thus,

$$\frac{\delta\phi(E',r)}{\delta\eta(k)}=-\frac{2}{\pi}k\left\{\psi(k^2,r)\frac{[\phi(E',r);\phi(k^2,r)]}{E'-k^2}-\sum_n C_n\frac{\phi_n(r)[\psi(k^2,r);\phi_n(r)]}{(k^2+E_n)^2}\right\}. \quad (13)$$

The first observation is that (12) again shows that a small change in the phase shift at high energies may in general change the asymptotic behavior of the potential completely. Equations (10) or (12) can, however, not be used to determine that asymptotic behavior in any simple way (such as by replacing ψ or f by their asymptotic values). Parts of the k-integrand which are asymptotically (in r) negligible, may yield dominant contributions to the integral.

Equation (10) has one very simple consequence. Since, near $r=0$,

$$\phi(E,r)\sim r^{l+1}/(2l+1)!!,\quad \psi(E,r)\sim -r^{-l}(2l-1)!!,$$

we obtain directly

$$\delta V(0)/\delta\eta_l(k)=4k/\pi(2l+1). \quad (14)$$

If $V(0)$ and $\int_0^\infty dk\,k\eta_l(k)$ both exist, and there are no bound states, then it follows immediately that

$$V(0)=\frac{8}{\pi(2l+1)}\int_0^\infty dk\,k\eta_l(k). \quad (15)$$

It is worth emphasizing that this is not an approximation in any sense, but an exact result.

In general, however, the first moment of the phase shift will not exist. Since

$$\lim_{k\to\infty}k\eta(k)=-\frac{1}{2}\int_0^\infty dr\,V(r), \quad (16)$$

if both sides exist, the finiteness of the right-hand side of (15) implies that the integral of the potential is zero. If it is not zero, then we can conclude from (14) merely that if V and V_0 have the same bound states, then

$$V(0)-V_0(0)=\frac{8}{\pi(2l+1)}\int_0^\infty dk\,k[\eta_l(k)-\eta_l^{(0)}(k)].$$

We can readily construct, by the Bargmann[7] procedure, a simple potential $V_0(r)$ whose phase shift $\eta_l^{(0)}(k)$ asymptotically approaches $\eta_l(k)$. The phase shift

$$\eta_l^{(0)}(k)=\tan^{-1}(\beta/k)-\tan^{-1}(\alpha/k)+2\textstyle\sum_n\tan^{-1}(K_{\sigma l}/k),$$

where α and β are positive constants, and the bound states with energies $E_{nl}=K_{nl}^2$, belong to a potential $V_0(r)$ whose value at the origin is

$$V_0(0)=(\alpha^2-\beta^2)/(l+\tfrac{1}{2}).$$

It is, however easily checked that

$$4\pi^{-1}\int^\infty dk[k\tan^{-1}(\beta/k)-\beta]=-\beta^2.$$

From this it follows that

$$V(0)=\frac{8}{\pi(2l+1)}\int_0^\infty dk\left[k\eta_l(k)+\frac{1}{2}\int_0^\infty drV(r)\right]+\frac{4}{2l+1}\sum_n E_{nl}, \quad (15')$$

if both sides exist. Equation (15′) is again an exact result, and not an approximation.

It is well known that for potentials with finite $2(l+1)$th moment, the phase shifts of increasing l are asymptotically equal and decrease near $E=0$ (as $E^{l+\frac{1}{2}}$). Equation (15′) shows that nevertheless, in the absence of bound states, their first moments (after subtraction of their asymptotic values) *increase* as $(2l+1)$. This indicates that the maxima of the phase shifts move to higher energies with increasing l.

The case of a square well potential affords a simple illustration. The two parameters, depth a and range b, are directly obtainable as

$$a=\frac{8}{\pi(2l+1)}\int_0^\infty dk[k\eta_l(k)-\lim_{k'\to\infty}k'\eta_l(k')]+\frac{4}{2l+1}\sum_n E_{nl},$$

$$ab=2\lim_{k\to\infty}k\eta_l(k).$$

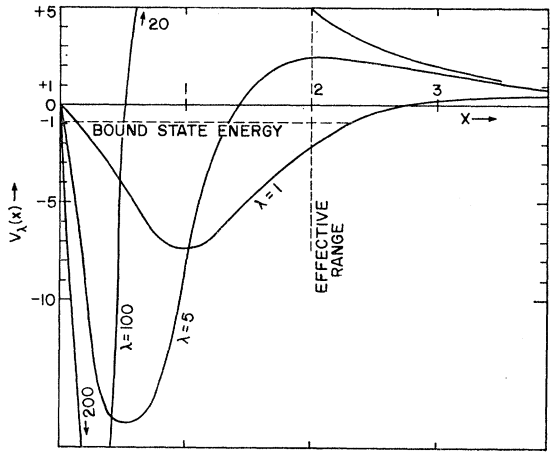

FIG. 1. Potentials "intrinsic" to a single bound state of angular momentum zero, in units of the binding energy; distance units are equal to twice the "size of the bound state."

In general, the numbers a and b may be interpreted as equivalent depth and range.

III. A PHASE-SHIFT PERTURBATION METHOD

Equations (10) or (12) can be used as the basis for a "phase-shift perturbation theory." If there are no bound states, one obtains the small phase-shift approximation

$$V(r)=-\frac{4}{\pi}\int_0^\infty dk\eta(k)\frac{d}{dr}\,\mathrm{Im}\; w_l^2(kr), \qquad (17)$$

provided that the right-hand side exists. (Otherwise one must again subtract the asymptotic value and a potential belonging to it.) This is an inversion of the Born approximation and is in principle obtainable from the latter. (For $l=0$, this is very easily checked.) Just as (16) indicates the fact that the Born approximation becomes exact in the limit of high energies (in the non-relativistic case), so (15) or (15′) shows that the small phase shift approximation becomes exact at the origin.

The present method can, however, be extended to areas where the Born approximation is not applicable. First, one can use the small phase-shift approximation also in the presence of bound states:

$$V(r)=V_0(r)+\frac{4}{\pi}\int_0^\infty dk\eta(k)\frac{d}{dr}\times\left\{\mathrm{Im}f_0^2(-k,r)+2k\sum_n C_n\frac{\phi_{n0}^2(r)}{k^2+E_n}\right\},$$

where V_0, f_0, and ϕ_0 belong to the bound states E_n and normalizations C_n, and with *zero* phase shift. They can be constructed explicitly by the Bargmann method.[7] The potential $V_0(r)$ will, of course, not be of short range (in the sense of the existence of the second moment) since the value of the phase shift at zero energy is not π times the number of bound states. This kind of approximation will therefore not be of much physical interest.

It is more natural, in the presence of bound states, to use as the "unperturbed" phase shift the one "intrinsic" to the binding energies, given by (6). In the case of a single bound state of binding energy K^2, this "intrinsic" phase shift satisfies

$$k\cot\xi(k)=-\tfrac{1}{2}K+\tfrac{1}{2}k^2K^{-1}.$$

In other words, the "shape-independent approximation" is *exactly* correct for this phase shift. The only quantity which enters is the "size" of the bound state, $R=K^{-1}$. Scattering length and effective range are related to it by $a=2R$ and $r_0=R$.[10]

Potentials and wave function corresponding to (6) are readily constructed by Bargmann's method. It may be of interest to see what the family of potentials "intrinsic" to a single bound state of angular momentum zero looks like. If the distance is measured in multiples of twice the "bound state size," and the potentials in units of the binding energy, then the "intrinsic" potentials are (see Fig. 1)

$$V_\lambda(x)=-8\frac{d^2}{dx^2}\log[1+\lambda(\sinh x-x)],$$

($x=2Kr$, $V_\lambda=K^{-2}V_I$, $\lambda=C/2K^3$). V_λ vanishes at $x=0$, starts with the slope -8λ, has a minimum which shifts toward the origin and becomes deeper with increasing λ, vanishes at a point which shifts toward the origin, has a maximum of increasing height, which also shifts toward the origin and finally goes asymptotically as $16xe^{-x}$, i.e., positive and independent of λ. (The last aspect affords an illustration of the statements following (5); since the asymptotic decay is slower than e^{-x}, it must be independent of λ, and vice versa.) As a consequence of (16), the area under the potential is -8, independently of λ.

An alternative description of these potentials is this: they have a repulsive center, an attractive shell, and a relatively weak repulsive tail. In spite of the fact that they all have an "effective range" of $x_0=2$, repulsive core and attractive shell can be made small (and strong) or large (and weak) by varying λ. It is, in principle, within the domain of the approximation method to make the repulsive core stronger and possibly to eliminate the repulsive tail.

[10] If, as an illustration, one wanted to apply this method to the case of the deuteron, the "intrinsic" phase shift would not, by itself, be a good enough quantity. The effective range of the nuclear potential in the triplet state is only about one third of the "deuteron size." That fact does not, however, necessarily mean that the experimental phase shift and the "intrinsic" phase shift differ so much that the "phase-shift perturbation method" may not yield a good approximation.

IV. DEPENDENCE OF ONE PHASE SHIFT UPON ANOTHER

The inverse of (10) is the variation of the phase shift with respect to the potential. A simple expression for that quantity is a well-known result of perturbation theory:

$$\frac{\delta\eta_l(k)}{\delta V(r)}=-k^{2l+1}\left|\frac{\phi_l(k,r)}{f_l(k)}\right|^2=-k^{-1}|\Phi_l(k,r)|^2, \quad (18)$$

where $\Phi_l(k,r)$ is the solution which is asymptotic to $\sin(kr+\eta_l-\frac{1}{2}\pi l)$ at infinity.

The two variations, (10) and (18), can now be combined into the variational derivative of one phase shift with respect to another, via the existence of a central potential that causes them both. Thus,

$$\frac{\delta\eta_l(k)}{\delta\eta_L(k')}=\int_0^\infty dr\frac{\delta\eta_l(k)}{\delta V(r)}\frac{\delta V(r)}{\delta\eta_L(k')}=\frac{4}{\pi}\frac{k'k^{2l+1}}{|f_l(k)|^2}\int_0^\infty dr\phi_l^2(k,r)$$
$$\times\frac{d}{dr}\left\{\phi_L(k',r)\psi_L(k',r)-\sum_n C_n\frac{\phi_{Ln}^2(r)}{k'^2+E_n}\right\}. \quad (19)$$

Naturally it must be true that for $l=L$,

$$\frac{\delta\eta_l(k)}{\delta\eta_l(k')}=\tfrac{1}{2}[\delta(k-k')-\delta(k+k')]=k\,\mathrm{sgn}k'\delta(k^2-k'^2).$$

In order to isolate the δ-function part of (19), we subtract from the integrand its asymptotic value and integrate it separately. The result is

$$\frac{\delta\eta_l(k)}{\delta\eta_L(k')}=(-1)^{l+L}k\,\mathrm{sgn}k'\delta(k^2-k'^2)\cos2[\eta_l(k)-\eta_L(k)]$$
$$-(-1)^{l+L}\pi^{-1}(k'/k)\frac{\sin2[\eta_l(k)+\eta_L(k')]}{k+k'}$$
$$+(-1)^L(\pi k)^{-1}\sin2\eta_L(k')$$
$$+2(\pi k)^{-1}\int_0^\infty dr\left\{[\mathrm{Im}\chi_l(k,r)]^2\frac{d}{dr}\right.$$
$$\times\left[\mathrm{Im}(\chi_L(k',r))^2-2k'\sum_n C_n\frac{\phi_{Ln}^2(r)}{k'^2+E_n}\right]$$
$$+2k'(-1)^L\sin^2[kr+\eta_l(k)-\tfrac{1}{2}\pi l]$$
$$\left.\times\cos2[k'r+\eta_L(k')]\right\}. \quad (20)$$

Except when η_l and η_L differ by an odd integral multiple of $\pi/4$, $\eta_l(k)$ is therefore most sensitive to small local changes in η_L at the same energy, k^2. Equations (19) or (20) can again be used as a basis of a phase shift perturbation method which directly determines one phase shift from another, if they are both caused by the same central potential.

V. GENERALIZATION OF THE PHASE SHIFT PERTURBATION METHOD TO THE TENSOR FORCE CASE

In the presence of the tensor force we shall limit ourselves from the start to variations near the "intrinsic" quantities,[11] for which[12]

$$F^T(k)F(-k)=\mathbf{1}.$$

Then

$$\delta[F^T(k)F(-k)]=\delta F^T(k)F^{T-1}(k)+F^{-1}(-k)\delta F(-k).$$

The first term is analytic in the lower half-plane and the second, in the upper. Therefore

$$\delta[F^T(k)F(-k)]$$
$$=-\frac{1}{i\pi}\mathcal{P}\int_{-\infty}^\infty\frac{dk'}{k'-k}[\delta F^T(k')F^{T-1}(k')-F^{-1}(-k')\delta F(-k')$$
$$=-\frac{2}{i\pi}\mathcal{P}\int_0^\infty\frac{dk'k'}{k'^2-k^2}[F^{-1}(k')\delta F(k')-F^{-1}(-k')\delta F(-k')]_s,$$

where the subscript s denotes the symmetric part.

On the other hand

$$F^{-1}(k)\delta S(k)F(-k)=F^{-1}(k)\delta F(k)-F^{-1}(-k)\delta F(-k),$$

and for the "intrinsic" quantities the left-hand side is

$$F^T(-k)\delta S(k)F(-k),$$

and therefore clearly symmetric since $\delta S(k)$ must be so. We can thus drop the s in the integral above and write,[13] for the variation of the derivative of the spectral function,

$$\delta\frac{dP(E)}{dE}=\frac{4}{\pi}\mathcal{P}\int_0^\infty\frac{dk'k'}{k'^2-k^2}\frac{k}{\pi}\Upsilon_l(k)\delta M(k')\Upsilon_l(k), \quad E>0,$$

where

$$\delta M(k')\equiv(2i)^{-1}F^T(-k')\delta S(k')F(-k').$$

Let us write the S-matrix in the following form:

$$S=UsU^{-1}, \quad (21)$$

where

$$U=\begin{pmatrix}\cos\epsilon & \sin\epsilon\\ -\sin\epsilon & \cos\epsilon\end{pmatrix}, \quad (22)$$

and

$$s=\begin{pmatrix}\exp(2i\eta_l) & 0\\ 0 & \exp(2i\eta_{l+2})\end{pmatrix}. \quad (23)$$

[11] The reason is not so much that variations near the intrinsic quantities are believed to be the only interesting ones, but general variations present, in the matrix case, difficulties due to the lack of commutativity of the functions involved. The "intrinsic" case happens to be simpler in that respect, and it may have applications by itself.

[12] The notation is the same here as in references 2. The superscript T stands for the transpose: $F(k)$ is the matrix generalization of $f(k)$.

[13] $\Upsilon_l(k)=k^l\binom{10}{0k^2}$; in the second paper of reference 2, $\Upsilon_l(k)$ was called $\mathcal{K}_l(k)$.

The quantities η_l, η_{l+2}, and ϵ are the "eigenphase shifts" and the "mixture angle," respectively, U being the transformation to the "eigenstates of the scattering."[14] Then

$$\delta S=\delta U U^{-1}S-S\delta U U^{-1}+U\delta s U^{-1},$$

$$\delta S=U\{(\Xi s-s\Xi)\delta\epsilon+\delta s\}U^{-1}$$
$$=U\begin{pmatrix}2i\delta\eta_l\exp(2i\eta_l) & \\ \delta\epsilon[\exp(2i\eta_{l+2})-\exp(2i\eta_l)] & 2i\delta\eta_{l+2}\exp(2i\eta_{l+2})\end{pmatrix}U^{-1}. \quad (24)$$

For the "intrinsic" quantities we have, in the presence of only a single bound state,

$$F_I(k)=\mathbf{1}-P_0+P_0\left(\frac{k+iK}{k-iK}\right),$$

$$S_I(k)=\mathbf{1}-P_0+P_0\left(\frac{k+iK}{k-iK}\right)^2.$$

The projection P_0 is defined by

$$P_0=\begin{pmatrix}\cos^2\epsilon & -\sin\epsilon\cos\epsilon\\ -\sin\epsilon\cos\epsilon & \sin^2\epsilon\end{pmatrix}$$
$$=U_I(\mathbf{1}-P)U_I^{-1},\quad P\equiv\begin{pmatrix}0&0\\0&1\end{pmatrix},$$

where $\tan\epsilon$ is the asymptotic ratio of the two components of the bound state wave function. The function

$$\xi_l(k)=2\tan^{-1}(K/k)$$

is the "intrinsic" eigenphase shift. For variations near the "intrinsic" S-matrix for one bound state we then have

$$\delta M(k)=U_I\begin{pmatrix}\delta\eta_l & -\delta\epsilon\sin\xi_l\\ -\delta\epsilon\sin\xi_l & \delta\eta_{l+2}\end{pmatrix}U_I^{-1}$$
$$=U_I\begin{pmatrix}\delta\eta_l(k) & -\delta\epsilon(k)\dfrac{2kK}{k^2+K^2}\\ -\delta\epsilon(k)\dfrac{2kK}{k^2+K^2} & \delta\eta_{l+2}(k)\end{pmatrix}U_I^{-1}. \quad (25)$$

In terms of the solution

$$G_e(k^2,r)\equiv\Upsilon_0(k)G(k^2,r),$$

we obtain for the operator $K(s,r)$

$$K(s,r)=-\int G^T(E,s)\delta d\mathrm{P}(E)G(E,r)$$
$$=-\frac{4}{\pi}\int_0^\infty dk k^{-1}\mathcal{P}\int_{-\infty}^\infty\frac{dk'k'^{2(l+1)}}{k'^2-k^2}$$
$$\times G_e^T(k'^2,s)\delta M(k)G_e(k'^2,r).$$

and

$$\delta U U^{-1}=\delta\epsilon\,\Xi,\quad \Xi=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Since the matrix Ξ commutes with U, we obtain

This can now not be written in the form of a Green's function, but it can be evaluated just as before, for $s<r$, by closing the k'-contour in the lower half-plane. In contrast to the scalar case, a contribution also arises from $k'=0$, after $G_e(k'^2,r)$ has been expressed in terms of $F(k',r)$ and $F(-k',r)$. The result is, for one bound state with energy E_0, for $s<r$,

$$K(s,r)=-4\pi^{-1}\int_0^\infty dk k\{G_e^T(k^2,s)\delta M(k)H(k^2,r)$$
$$-k^{-2}G^T(0,s)QF_e(0,r)\delta\gamma(k)$$
$$-(k^2+E_0)^{-1}G_e^T(-E_0,s)\delta M(k)C'G_e(-E_0,r)\},$$

where[15] the irregular solution is

$$H(k^2,r)=-\tfrac{1}{2}k^l[F^{-1}(k)F(k,r)+(-1)^lF^{-1}(-k)F(-k,r)]$$
$$=-k^l\,\mathrm{Re}[F^{-1}(k)F(k,r)],$$

and $\delta\gamma(k)$ is obtained from

$$(\mathbf{1}-P)\delta M(k)(\mathbf{1}-2P_0)P\equiv-Q\delta\gamma(k),\quad Q=\begin{pmatrix}0&1\\0&0\end{pmatrix},$$

as

$$\delta\gamma(k)=\tfrac{1}{2}[\delta\eta_l(k)+\delta\eta_{l+2}(k)]\sin2\epsilon+\frac{2kK}{k^2+K^2}\delta\epsilon(k). \quad (26)$$

The potential associated with an S-matrix that is not very different from the "intrinsic" S-matrix of the one bound state of binding energy $K^2=E_0$, is therefore

$$V(r)=V_I(r)-\frac{8}{\pi}\frac{d}{dr}\int_0^\infty dk k\{G_{eI}^T(k^2,r)\delta M(k)H_I(k^2,r)$$
$$-k^{-2}G_I(0,r)QF_{eI}(0,r)\delta\gamma(k)$$
$$-c(k^2+E_0)^{-1}G_{eI}^T(-E_0,r)\delta M_0(k)G_{eI}(-E_0,r)\}. \quad (27)$$

We have here made use of the fact that in this special "intrinsic" case of one bound state, C' must be a multiple of P_0,

$$C'=cP_0,$$

and therefore

$$\delta M_0(k)=U_I\begin{pmatrix}\delta\eta_l(k) & 0\\ -\delta\epsilon(k)\dfrac{2kK}{k^2+K^2} & 0\end{pmatrix}U_I^{-1}. \quad (28)$$

[14] See, for example, J. M. Blatt and V. F. Weissk opf, *Theoretica Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952) p. 111.

[15] $C'=\Upsilon_0^{-1}(-iK_0)C\Upsilon_0^{-1}(-iK_0)$, if C is one of the matrixes C_n of reference 2.

The "intrinsic" quantities are readily explicitly calculated as

$$V_I(r)=-2\frac{d}{dr}\left\{\mathfrak{U}_K(r)C'\left[1+\int_0^r dt\mathfrak{U}_K^2(t)C'\right]^{-1}\mathfrak{U}_K(r)\right\},$$

$$G_{eI}(-E_0,r)=\left[1+\int_0^r dt\mathfrak{U}_K^2(t)C'\right]^{-1}\mathfrak{U}_K(r),$$

$$G_{eI}(k^2,r)=k^{-(l+1)}\mathfrak{u}(kr)-\frac{k^{-(l+1)}}{k^2+E_0}[\mathfrak{u}(kr);\mathfrak{U}_K(r)]\times C'\left[1+\int_0^r dt\mathfrak{U}_K^2(t)C'\right]^{-1}\mathfrak{U}_K(r),$$

where[16]

$$\mathfrak{U}_K(r)=\begin{pmatrix}V_K{}^{(l)}(r) & 0\\ 0 & -K^2V_K{}^{(l+2)}(r)\end{pmatrix}$$

and

$$\mathfrak{u}(kr)=\begin{pmatrix}u_l(kr) & 0\\ 0 & u_{l+2}(kr)\end{pmatrix}.$$

The functions $H_I(k^2,r)$ and $F_I(-k)F_I(k,r)$ are obtained from G_{eI} by replacing[16] $u_l(kr)$ and $u_{l+2}(kr)$ by $k^{l+1}v_l(kr)$ and $k^{l+1}v_{l+2}(kr)$, $k^{2l+1}w_l(kr)$ and $k^{2l+1}w_{l+2}(kr)$, respectively.

APPENDIX

The purpose of this appendix is to prove the following *theorem*: A sufficient condition for a central potential to be asymptotically exponentially decreasing is that for the lth angular momentum, $f(k)$ be a rational function of k which, near $k=0$ behaves as follows:

$$f(k)-f(0)=O(k^{2l}). \tag{A.1}$$

[In addition, of course, $f(k)$ must have all the properties previously given.]

Proof.—Let us write

$$f(k)=\prod_1^\nu\frac{(k-i\alpha_j)}{(k-i\beta_j)}=1+i\sum_{j=1}^\nu R_j(k-i\beta_j)^{-1}, \tag{A.2}$$

where[17]

$$R_j=\frac{\prod(\beta_j-\alpha_l)}{\prod'(\beta_j-\beta_l)}. \tag{A.3}$$

Clearly, then the first $2l-1$ derivatives of $f(k)$ at $k=0$ vanish, and therefore (A.1) is satisfied, if and only if

$$C_m\equiv\sum_{j=1}^\nu R_j\beta_j{}^{-m}=0,\quad m=2,3,\cdots,2l. \tag{A.4}$$

We must therefore show that (A.4) is sufficient for exponential decay of the potential.

Now, as Bargmann has shown, the potential belonging to $f(k)$ of (A.2) is the following:

$$V(r)=-2(d/dr)^2\log X(r),\quad X=\|x_{\alpha\beta}\|,\quad x_{\alpha\beta}=[V_\beta{}^l;U_\alpha{}^l]/(\alpha^2-\beta^2), \tag{A.5}$$

where the determinant is indicated by the double vertical bars. The asymptotically leading term in X, in the sense that the remainder vanishes exponentially with respect to it, is

$$Y=\|y_{\alpha\beta}\|,\quad y_{\alpha\beta}=[U_{-\beta}{}^l;U_\alpha{}^l]/(\alpha^2-\beta^2), \tag{A.6}$$

which can be written

$$Y=Z\exp[\textstyle\sum(\beta-\alpha)r],\quad Z=\|z_{\alpha\beta}\|,$$
$$z_{\alpha\beta}=(\beta-\alpha)^{-1}\{h_{-\beta}h_\alpha-(\beta+\alpha)^{-1}[h_{-\beta};h_\alpha]\},\quad h_\alpha(r)=e^{\alpha r}U_\alpha{}^l(r). \tag{A.7}$$

Since $h_\alpha(r)$ is a polynomial in r^{-1}, so is Z. In order that the potential be exponentially decreasing at infinity it must therefore be shown that Z is in fact constant.

The function $h_\alpha(r)$ is a polynomial in $(\alpha r)^{-1}$ of degree l, and $[h_{-\beta};h_\alpha]/(\alpha+\beta)$ is a polynomial in r^{-1} whose coefficients are homogeneous polynomials in α^{-1} and β^{-1}. Since

$$\lim_{\beta\to\alpha}[U_{-\beta}{}^l;U_\alpha{}^l]/(\alpha+\beta)=-1, \tag{A.8}$$

then

$$z_{\alpha\beta}=(\beta-\alpha)^{-1}+\sum_{n=1}^{2l-1}r^{-n}\sum_{i+j=n+1}A_{ij}\alpha^{-i}\beta^{-j},\ i\geq1,j\geq1. \tag{A.9}$$

If we mean by $\|A;B,C,\cdots\|$ the sum of all the determinants obtained by replacing one row (column) of A by one of B, another row (column) of A by one of C, etc., then we readily see that a sufficient condition for Z to be constant is that

$$\|(\beta-\alpha)^{-1};\alpha^{-n_1}\beta^{-m_1},\alpha^{-n_2}\beta^{-m_2},\cdots,\alpha^{-n_L}\beta^{-m_L}\|=0,\ L=1,\cdots,\nu-1;\ n_j+m_j\leq2l. \tag{A.10}$$

In order to prove that it follows from (A.4) that (A.10) is satisfied, we require the following *lemma*:

$$\sum_{i=1}^{n+1}\prod_{k=1}^{n}(\delta_k-\gamma_i)/\prod'(\gamma_k-\gamma_i)=1. \tag{A.11}$$

The proof of this lemma consists simply in considering the left-hand side as a rational function of, say, γ_1. The residues at the apparent poles all vanish, and the value at $\gamma_1=\infty$ is 1. The function must therefore be identically equal to unity.

[16] The notation is as follows: $u_l(x)=xj_l(x)$, $v_l(x)=xn_l(x)$, $w_l(x)=(-1)^{l+1}xh_l{}^{(1)}(x)$, $V_\alpha{}^{(l)}(r)=(-i)^{l+1}(i\alpha r)j_l(i\alpha r)=(-i)^{l+1}u_l(i\alpha r)$, where j, n, and $h^{(1)}$ are the customary spherical Bessel functions, spherical Neumann functions, and spherical Hankel functions, respectively.

[17] The prime on the product sign indicates that the term with $l=j$ is missing.

Furthermore, we need the fact that

$$\|(\beta-\alpha)^{-1}\| = (-1)^{\frac{1}{2}\nu(\nu-1)} \prod_{i<l}(\alpha_i-\alpha_l) \prod_{i<l}(\beta_i-\beta_l) \times \prod_{i,l}(\beta_l-\alpha_i)^{-1}, \quad \text{(A.12)}$$

which is readily proved by induction on ν and development of the determinant after the jth column. If $\|A\|_{ij}$ is the (ij) subdeterminant of $\|A\|$ (without the sign factor), then

$$\|(\beta-\alpha)^{-1}\| = \sum_i (-1)^{i+j}(\beta_j-\alpha_i)^{-1}\|(\beta-\alpha)^{-1}\|_{ij}$$

$$= \sum_i (-1)^{i+j+\frac{1}{2}(\nu-1)(\nu-2)}(\beta_j-\alpha_i)^{-1} \times \prod_{\substack{k<l \\ k,\,l\neq j}}(\beta_k-\beta_l) \prod_{\substack{k<l \\ k,\,l\neq i}}(\alpha_k-\alpha_l) \prod_{\substack{k\neq j \\ l\neq i}}(\beta_k-\alpha_l)$$

$$= \frac{\prod_{k<l}(\beta_k-\beta_l)\prod_{k<l}(a_k-\alpha_l)}{\prod(\beta_k-\alpha_l)} \cdot (-1)^{\frac{1}{2}(\nu-1)(\nu-2)} \frac{\prod_l(\beta_j-\alpha_l)}{\prod_l'(\beta_j-\beta_l)} \sum_i \frac{\prod_{k\neq j}(\beta_k-\alpha_i)}{\prod_k'(\alpha_i-\alpha_k)(\beta_j-\alpha_i)}.$$

In the same manner in which the lemma was proved, one sees that the last sum equals

$$(-1)^{\nu-1}\prod_l'(\beta_j-\beta_l)/\prod_l(\beta_j-\alpha_l),$$

which proves (A.12).

We now proceed as in the above proof of (A.12):

$$\|(\beta-\alpha)^{-1};\alpha^{-n}\beta^{-m}\| = \sum_{i,j}(-1)^{i+j}a_i^{-n}\beta_j^{-m}\|(\beta-\alpha)^{-1}\|_{ij}$$

$$= \|(\beta-\alpha)^{-1}\| \sum_j \frac{\prod_l(\beta_j-\alpha_l)}{\prod_l'(\beta_j-\beta_l)} \cdot \frac{1}{\beta_j^m} \sum_i \frac{\prod_k(\beta_k-\alpha_i)}{\prod_k'(\alpha_k-\alpha_i)} \cdot \frac{1}{\alpha_i^n}\frac{1}{\beta_j-\alpha_i}$$

$$= \|(\beta-\alpha)^{-1}\|\left\{C_{m+n}+\sum_{j=1}^{n} C_{m+j} \times\left[\sum_i \frac{\prod_k(\beta_k-\alpha_i)}{\prod_k'(\alpha_k-\alpha_i)} \cdot \frac{1}{\alpha_i^{n-j+1}}\right]\right\},$$

by the use of partial fractions on $\alpha_i^{-n}(\beta_j-\alpha_i)^{-1}$ and the lemma. Equations (A.4), and therefore (A.1), imply, consequently, that

$$\|(\beta-\alpha)^{-1};\alpha^{-n}\beta^{-m}\| = 0, \quad 2\leq m+n\leq 2l.$$

Similarly,

$$\left\|\frac{1}{\beta-\alpha};\frac{1}{\alpha^n\beta^m},\frac{1}{\alpha^r\beta^s}\right\| = \sum_{ijkl}\frac{(-1)^{i+j+k+l}}{\alpha_i^n\alpha_k^r\beta_j^m\beta_l^s}\left\|\frac{1}{\beta-\alpha}\right\|_{ik,\,jl}$$

$$= \left\|\frac{1}{\beta-\alpha}\right\| \sum_i \frac{\prod_q(\beta_j-\alpha_q)}{\prod_q'(\beta_j-\beta_q)} \cdot \frac{1}{\beta_j^m} \sum_l \frac{\prod_q(\beta_l-\alpha_q)}{\prod_q'(\beta_l-\beta_q)} \cdot \frac{1}{\beta_l^s} \cdot \sum_i \frac{\prod_q(\beta_q-\alpha_i)}{\prod_q'(\alpha_i-\alpha_q)} \cdot \frac{1}{\alpha_i^n(\beta_j-\alpha_i)(\beta_l-\alpha_i)}$$

$$\times \sum_k \frac{\prod_q(\beta_q-\alpha_k)}{\prod_q'(\alpha_k-\alpha_q)} \frac{(\beta_j-\beta_l)(\alpha_i-\alpha_k)}{\alpha_k^r(\beta_j-\alpha_k)(\beta_l-\alpha_k)}$$

$$= 2\left\|\frac{1}{\beta-\alpha}\right\| \left\{\sum_{p=1}^{n-1}\sum_{z=1}^{r} C_{m+n-1+z-p}C_{s+r+1-z+p} - C_{m+n+z-p}C_{s+r-z+p} + \sum_{t=1}^{p}\left(\sum_i \frac{\prod_q(\beta_q-\alpha_i)}{\prod_q'(\alpha_i-\alpha_q)}\frac{1}{\alpha_i^t}\right)(C_{m+n-1+z-p}C_{s+r-t+2-z+p} - C_{m+n+z-p}C_{s+r-t+1-z+p}) + \sum_{x=1}^{z}\left(\sum_i \frac{\prod_q(\beta_q-\alpha_i)}{\prod_q'(\alpha_i-\alpha_q)}\frac{1}{\alpha_i^x}\right)(C_{m+n-x+z-p}C_{s+r+1-z+p} - C_{m+n-x+1+z-p}C_{s+r-z+p}) + \sum_{t=1}^{p}\sum_{x=1}^{z}\left(\sum_i \frac{\prod_q(\beta_q-\alpha_i)}{\prod_q'(\alpha_i-\alpha_q)}\frac{1}{\alpha_i^t}\right)\left(\sum_i \frac{\prod_q(\beta_q-\alpha_i)}{\prod_q'(\alpha_i-\alpha_q)}\frac{1}{\alpha_i^x}\right) \times (C_{m+n-x+z-p}C_{s+r-t+2-z+p} - C_{m+n-x+1+z-p}C_{s+r-t+1-z+p})\right\},$$

again by the use of partial fractions and the lemma. Therefore, if, $C_i=0$ for $i\leq[\frac{1}{2}(m+n+s+r)]$, then

$$\left\|\frac{1}{\beta-\alpha};\frac{1}{\alpha^n\beta^m},\frac{1}{\alpha^r\beta^s}\right\| = 0.$$

Because of (A.4), then, certainly

$$\left\|\frac{1}{\beta-\alpha},\frac{1}{\alpha^n\beta^m}\frac{1}{\alpha^r\beta^s}\right\| = 0$$

for

$$n+m\leq 2l,$$
$$r+s\leq 2l.$$

The other cases are proved similarly and the details are left to the reader.