requires a knowledge of high-energy matrix elements. An example of the former is the theorem that the zero energy limit of the *p*-wave effective range extrapolation measures the same coupling constant as the zero-energy limit of the photomeson production amplitude (according to the Kroll-Ruderman theorem). On the other hand the problem of theoretically evaluating the effective range falls in the latter class. Formula (49), for example, shows clearly that the value of r_3 depends on high-energy phenomena.

We have made no serious attempt in this paper to calculate the effective ranges. Presumably the (3,3) effective range could be matched by an appropriate choice of the cut-off energy, whatever method of approximation were used, and the dominant role played by the (3,3) state at low energies guarantees the success of any approach which produces the correct value for r_3. The question naturally arises as to whether one should expect to be able to calculate r_3 and other quantities which involve integrals over high virtual energies with the conventional relativistic form of the Yukawa theory, which has no adjustable cut-off parameter. We think the answer is no, because this theory does not take account of the existence of hyperons and *K*-particles which interact strongly with the pion-nucleon system. Both the cutoff and the local forms of the Yukawa theory are incorrect (or at least incomplete) in the Bev energy region.

Our zero-energy results hold for both theories and we believe they will probably hold in future theories, although this last statement is of course little more than a guess. We also believe that the linear extrapolation of the cotangent of the phase shifts will be maintained because this is essentially a statement of ignorance: the more important are high-energy phenomena, the more nearly constant is the effective-range integral.

We hope to show in the paper on photomeson production, which follows, that many aspects of this latter process fall in the first (low virtual energy) class of phenomena. The same is true for Compton scattering by protons and probably for the nuclear force problem. Phenomena which belong to the second class presumably include *s*-wave scattering, π^0 decay, the charge and current density of nucleons, as well as the fundamental questions concerning the nature and interactions of curious particles.

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Theory of Photomeson Production at Low Energies

G. F. CHEW AND F. E. LOW
Department of Physics, University of Illinois, Urbana, Illinois
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The problem of photomeson production is re-examined using the static model of the pion-nucleon interaction. It is shown that an important part of the low-energy matrix element can be exactly expressed as a function of the scattering phase shifts and the static nucleon magnetic moments. It is argued that the remainder is quite accurately given by the usual Born approximation. Corrections to this result, within the framework of the "one-meson" approximation, are considered.

I. INTRODUCTION

THE purpose of this paper is to extend the theoretical approach of the preceding paper[1] on meson-nucleon scattering to the problem of photomeson production. Extensive use of the notation and results of the scattering paper is necessary, and we shall assume the reader to be familiar with these. The most important conclusion of the present paper is that once the scattering phase shifts are known at a given energy, either experimentally or theoretically, the corresponding photomeson production cross sections can almost unambiguously be predicted.

As in I, the bulk of our discussion will be in terms of the static model, but it may be argued that the important results are probably more general. We begin in Sec. II of this paper by splitting the photomeson production amplitude into three parts, one of which may be written down in an explicit and exact form. Equations satisfied by the other two parts are then derived. Section III deals with these two equations in the one-meson approximation. In Sec. IV, a simple and quite accurate approximation to the total amplitude is proposed, and finally Sec. V compares the simple theoretical amplitude with experiment.

II. PHOTOMESON EQUATIONS

A derivation of the integral equations which we shall apply to photomeson production has already been published.[2] We give here a new derivation which is analogous to that presented in I for scattering. All notations will be the same as in I.

The matrix element for absorption of a photon of type k by a single nucleon, with emission of a meson

[1] G. F. Chew and F. E. Low, preceding paper [Phys. Rev. **101**, 1570 (1956)], hereinafter referred to as I.

[2] F. E. Low, Phys. Rev. **97**, 1392 (1955).

of type q, is

$$\mathcal{H}_k(q)=\left(\Psi_q^{(-)},\int dv(-\mathbf{j}\cdot\mathbf{A}_k)\Psi_0\right). \tag{1}$$

Here the single index k refers to the momentum $\mathbf{k}$ as well as to the polarization $\boldsymbol{\varepsilon}$ of the incident photon, and

$$\mathbf{A}_k(\mathbf{r})=\frac{1}{(2k)^{\frac{1}{2}}}\boldsymbol{\varepsilon}e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2}$$

is the electromagnetic vector potential associated with the absorption of the incident photon. The symbol $\mathbf{j}$ means the total current density operator for the meson-nucleon system.

The conventional way of writing the current density operator $\mathbf{j}$ makes a distinction between component parts which are called "meson current," "nucleon current," and "interaction current" according to whether they depend on meson variables alone (i.e., on φ_1, φ_2, φ_3, the components of the meson field), on nucleon variables alone (i.e., $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$), or on both. For example the conventional meson current operator is

$$j_M=-e(\varphi_1\nabla\varphi_2-\varphi_2\nabla\varphi_1), \tag{3}$$

if e is the electronic charge. However, one of the essential features of the approach to be employed in this paper is the use of a representation in which the basis functions are eigenstates of the complete meson-nucleon Hamiltonian. These basis functions depend on the nucleon and meson variables in a nonfactorable way, so the conventional manner of decomposing the current is not appropriate. We use instead a decomposition which emphasizes the distinction between the ground state Ψ_0 of the meson-nucleon system, which corresponds to the physical ("clothed") nucleon, and excited states which contain one or more real mesons.

We divide the current density operator into three parts:

$$\mathbf{j}=\mathbf{j}_V+\mathbf{j}_S+\mathbf{j}_\pi. \tag{4}$$

The first two terms, $\mathbf{j}_V+\mathbf{j}_S$, are defined to be independent of meson variables (i.e., to commute with the a_q's and $a_q^\dagger$'s) but to have the same matrix elements as the total $\mathbf{j}$ between single *physical* nucleon states. That is,

$$(\Psi_0,[\mathbf{j}_V+\mathbf{j}_S]\Psi_0)=(\Psi_0,\mathbf{j}\Psi_0), \tag{5}$$

and evidently $(\Psi_0,\mathbf{j}_\pi\Psi_0)=0$. The subscripts V and S refer to the isotopic spin character of the respective current densities, V standing for vector and S for scalar. It should be noted that if all currents are associated with the π-meson field (i.e., $\mathbf{j}=\mathbf{j}_M$) then $j_S=0$. It is, however, an experimental fact that the current density of the meson-nucleon system is not an isotopic vector (the neutron and proton charges and magnetic moments are *not* equal and opposite), so we include j_S in our theory. The relativistic theory[2] shows that even in our static limit it is as correct to keep $\mathbf{j}_S$ as to keep $\mathbf{j}_V$.

The two terms $\mathbf{j}_V$ and $\mathbf{j}_S$ are completely determined by the above properties and so, therefore, is $\mathbf{j}_\pi$. Note that $\mathbf{j}_\pi$ is not equal to the conventional meson current operator $\mathbf{j}_M$. Unless this fact is kept in mind, the results obtained below may seem puzzling.

A strong motivation for making this particular decomposition of the current density is that it is possible to express the photoproduction due to $\mathbf{j}_V$ in terms of meson scattering amplitudes and single nucleon magnetic moments. We demonstrate this result immediately: From the definition of $\mathbf{j}_V$, it follows that $\int dv\mathbf{j}_Ve^{i\mathbf{k}\cdot\mathbf{r}}$ must be a polar vector function of $\mathbf{k}$ and $\boldsymbol{\sigma}$ and proportional to τ_3. Finally, since $\operatorname{div}\mathbf{j}_V=0$, we must have

$$\int\mathbf{j}_Ve^{i\mathbf{k}\cdot\mathbf{r}}dv=A\tau_3i\boldsymbol{\sigma}\times\mathbf{k}F(k^2), \tag{6}$$

where $F(0)$ is normalized to unity. The constant A must then be chosen to match the observed difference of proton and neutron static magnetic moments:

$$A=\frac{f_r^{(0)}}{f_r}\cdot\frac{\mu_P-\mu_N}{2}, \tag{7}$$

where we have used the definition of charge renormalization given in Sec. (III-C) of I, together with the defining property of $\mathbf{j}_V$, i.e., that it have the correct one-nucleon expectation. The function $F(k)$ is a linear combination of the proton and neutron form factors such as have been discussed in connection with electron-proton scattering.[3]

The contribution of $\mathbf{j}_V$ to the photomeson matrix element (1) may now be written as follows:

$$\mathcal{H}_k^V(q)=\left(\Psi_q^{(-)},\frac{-A\tau_3}{(2k)^{\frac{1}{2}}}i\boldsymbol{\sigma}\times\mathbf{k}\cdot\boldsymbol{\varepsilon}F(k^2)\Psi_0\right)$$

$$=-\frac{1}{f_r}\frac{\mu_P-\mu_N}{2}\left(\frac{\omega_p}{k}\right)^{\frac{1}{2}}\frac{F(k^2)}{v(k)}(\Psi_q^{(-)},V_p^{(0)}\Psi_0), \tag{8}$$

where $V_p^{(0)}$ is the quantity defined by (I-4) for a meson index p corresponding to momentum $\mathbf{k}\times\boldsymbol{\varepsilon}$ and isotopic variable 3. We now recognize that the matrix element occurring in (8) is precisely that for pion scattering from the state p to the state q introduced in (I-9). Thus the final expression may be written

$$\mathcal{H}_k^V(q)=-\frac{1}{f_r}\frac{\mu_P-\mu_N}{2}\frac{F(k^2)}{v(k)}\left(\frac{\omega_p}{k}\right)^{\frac{1}{2}}T_p(q), \tag{9}$$

which says that $\mathbf{j}_V$ effectively generates neutral mesons of momentum $\mathbf{k}\times\boldsymbol{\varepsilon}$ which are then scattered by the nucleon. The extremely simple result (9) must be regarded, we believe, as fortuitous. The general theory

[3] R. Hofstadter and R. W. McAllister, Phys. Rev. **98**, 217 (1955).

of scattering does require a certain connection between the photomeson amplitude and the scattering amplitude (e.g., the phases of the corresponding partial waves must be the same), but no such detailed result as (9) can be inferred entirely from general principles.

We now turn out attention to the other parts of the current as written in (4). Because $\mathbf{j}_S$ is independent of isotopic spin its contribution to the photomeson matrix element will not be as simply related as that of $\mathbf{j}_V$ to the scattering amplitude. We must treat $\mathbf{j}_S$ (as well as $\mathbf{j}_\pi$) by a different approach. Consider

$$\mathcal{H}_k{}^S(q)=\left(\Psi_q{}^{(-)},\int dv(-\mathbf{j}_S\cdot\mathbf{A}_k)\Psi_0\right). \qquad (10)$$

We substitute for $\Psi_q{}^{(-)}$ the expression (I-13) and then shift the creation operator acting on Ψ_0 from left to right. The shift is possible because by definition $\mathbf{j}_S$ commutes with a_q.

$$\begin{aligned}\mathcal{H}_k{}^S(q)&=\left(\left[a_q{}^\dagger-\frac{1}{H-\omega_q+i\epsilon}V_q{}^{(0)}\right]\Psi_0,\int dv(-\mathbf{j}_S\mathbf{A}_k)\Psi_0\right)\\&=-\left(\Psi_0,V_q{}^{(0)\dagger}\frac{1}{H-\omega_q-i\epsilon}\int dv(-\mathbf{j}_S\cdot\mathbf{A}_k)\Psi_0\right)\\&\quad+\left(\Psi_0,\int dv(-\mathbf{j}_S\cdot\mathbf{A}_k)a_q\Psi_0\right). \qquad (11)\end{aligned}$$

The annihilation operator may be eliminated from the second term of (11) by using the identity (I-19) and the result may then be written in a compact form by introducing the complete set of eigenfunctions $\Psi_n{}^{(-)}$ of the pion-nucleon problem:

$$\begin{aligned}\mathcal{H}_k{}^S(q)&\\&=-\sum_n\left[\frac{(\Psi_0,V_q{}^{(0)\dagger}\Psi_n{}^{(-)})\left(\Psi_n{}^{(-)},\int dv(-\mathbf{j}_S\cdot\mathbf{A}_k)\Psi_0\right)}{E_n-\omega_q-i\epsilon}\right.\\&\qquad\left.+\frac{\left(\Psi_0,\int dv(-\mathbf{j}_S\cdot\mathbf{A}_k)\Psi_n{}^{(-)}\right)(\Psi_n{}^{(-)},V_q{}^{(0)\dagger}\Psi_0)}{E_n+\omega_q}\right]\\&=-\sum_n\left\{\frac{T_q{}^\dagger(n)\mathcal{H}_k{}^S(n)}{E_n-\omega_q-i\epsilon}+\frac{\mathcal{H}_k{}^{S\dagger}(n)T_q(n)}{E_n+\omega_q}\right\}, \qquad (12)\end{aligned}$$

where

$$\mathcal{H}_k{}^S(n)=\left(\Psi_n{}^{(-)},\int dv(-\mathbf{j}_S\cdot\mathbf{A}_k)\Psi_0\right). \qquad (13)$$

As in the scattering problem, the terms in (12) corresponding to $n=0$ may be evaluated explicitly. In fact,

$$\begin{aligned}-&\left\{\frac{T_q{}^\dagger(0)\mathcal{H}_k{}^S(0)}{-\omega_q}+\frac{\mathcal{H}_k{}^{S\dagger}(0)T_q(0)}{\omega_q}\right\}\\&=\frac{1}{\omega_q}\left[V_q,\frac{i\boldsymbol{\sigma}\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})}{(2k)^{\frac{1}{2}}}\right]\frac{\mu_P+\mu_N}{2}G(k^2), \qquad (14)\end{aligned}$$

where $G(k^2)$ is a nuclear form factor normalized to unity at $k=0$.

We consider finally the contribution from $\mathbf{j}_\pi$:

$$\mathcal{H}_k{}^\pi(q)=\left(\Psi_q{}^{(-)},\int dv(-\mathbf{j}_\pi\cdot\mathbf{A}_k)\Psi_0\right). \qquad (15)$$

Repetition of the same procedure as used above for $\mathcal{H}_k{}^S(q)$ leads to an equation for $\mathcal{H}_k{}^\pi(q)$ similar to (12) but with an additional term due to the failure of $\mathbf{j}_\pi$ to commute with a_q:

$$\begin{aligned}\mathcal{H}_k{}^\pi(q)&=\left(\Psi_0,\left[a_q,\int(-\mathbf{j}_\pi\cdot\mathbf{A}_k)dv\right]\Psi_0\right)\\&\quad-\sum_n\left\{\frac{T_q{}^\dagger(n)\mathcal{H}_k{}^\pi(n)}{E_n-\omega_q-i\epsilon}+\frac{\mathcal{H}_k{}^{\pi\dagger}(n)T_q(n)}{E_n+\omega_q}\right\}, \qquad (16)\end{aligned}$$

where now there is no contribution to the sum for $n=0$ because of the defining property that the matrix elements of $\mathbf{j}_\pi$ between single-nucleon states shall vanish. The expectation of the commutator in (16) will now be calculated explicitly. Note that since $\mathbf{j}_V$ and $\mathbf{j}_S$ are defined so as to commute with a_q we may just as well calculate the commutator for the complete current $\mathbf{j}$ and it is convenient now to use the conventional division of $\mathbf{j}$ into meson current, nucleon current and interaction current.

The meson current $\mathbf{j}_M$ is unambiguously defined by Eq. (3). The nucleon current is perhaps not so well defined in the static model, but it commutes with a_q in any case and need not concern us here. All its effects are included in $\mathbf{j}_V$ and $\mathbf{j}_S$. The interaction current depends on both nucleon and meson variables and in the static model is somewhat ambiguous.[4] The most reliable guide to the correct interaction current in the static model is given by starting with the relativistic theory (where there is no interaction current and no ambiguity) and examining the photoproduction matrix element in the zero-energy limit. Comparison term by term with the corresponding limit for the static model then shows what contribution we must get from the interaction current of the latter. The result agrees with that given by the nonrigorous approach of this section, which we make entirely within the framework of the static model.

The interaction energy density of the nucleon-meson system is proportional to the gradient of the meson field. It is well known that gauge invariance requires

[4] R. H. Capps and R. G. Sachs, Phys. Rev. 96, 540 (1954).

that when the electromagnetic field is included in the problem this gradient must be replaced by $\nabla - ie\mathbf{A}$ when acting on the charged meson field φ. If the cut-off factor $v(p)$ is set equal to unity, this substitution suffices to guarantee gauge invariance and evidently gives rise to an interaction energy,

$$\int dv(-\mathbf{j}_{\text{int}}\cdot\mathbf{A}_k)=e_r f_r^{(0)}(\tau_2\varphi_1-\tau_1\varphi_2)\boldsymbol{\sigma}\cdot\mathbf{A}_k, \quad (17)$$

with all fields evaluated at the position of the nucleon. If the cut-off factor is retained, gauge invariance requires additional terms in the interaction energy. These terms, however, are not well defined and seem to have no counterpart in the relativistic theory. Furthermore, reasonable estimates of the size of such terms indicate that they are unimportant for photoproduction at low energies.

In order to evaluate the first term of (16), the commutator of a_q with (17) is needed and is easily found to be

$$\left[a_q, \int dv(-\mathbf{j}_{\text{int}}\cdot\mathbf{A}_k)\right]=\frac{e_r f_r^{(0)}}{(4k\omega_q)^{\frac{1}{2}}}(\tau_2\delta_{q1}-\tau_1\delta_{q2})\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}. \quad (18)$$

Finally, the ordinary meson current j_M gives the following contribution to the commutator in (16):

$$\left[a_q, \int dv(-\mathbf{j}_M\cdot\mathbf{A}_k)\right]=2ie_r\frac{\mathbf{q}\cdot\boldsymbol{\varepsilon}}{(4k\omega_q)^{\frac{1}{2}}}\int dv \times e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}}(\delta_{q1}\varphi_2-\delta_{q2}\varphi_1). \quad (19)$$

We need the matrix elements of (18) and (19) between single-nucleon states. The result for (18) is simply to change the unrenormalized coupling constant $f_r(0)$ to the renormalized f_r. For (19), we first eliminate the meson operators by using the identity (I-19) and then after some simple manipulations obtain

$$\left(\Psi_0, \left[a_q, \int dv(-\mathbf{j}_M\cdot\mathbf{A}_k)\right]\Psi_0\right) = -\frac{2e_r f_r}{(4k\omega_q)^{\frac{1}{2}}}(\tau_2\delta_{q1}-\tau_1\delta_{q2})\frac{\boldsymbol{\sigma}\cdot(\mathbf{q}-\mathbf{k})\mathbf{q}\cdot\boldsymbol{\varepsilon}}{(\mathbf{k}-\mathbf{q})^2+\mu^2}. \quad (20)$$

Inspection of Eq. (16) reveals that the sum over states n produces only p-wave mesons, by virtue of the q dependence of $T_q(n)$. It follows that all partial waves except those for $l=1$ are completely given by the first term of (16). We shall call this term $\mathcal{H}_k^{(0)}(q)$ (it coincides with the usual renormalized Born approximation), which by (18) and (20) has the value,

$$\mathcal{H}_k^{(0)}(q)=\frac{e_r f_r}{(4\omega_q k)^{\frac{1}{2}}}(\tau_2\delta_{q1}-\tau_1\delta_{q2})\left\{\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}-2\frac{\boldsymbol{\sigma}\cdot(\mathbf{q}-\mathbf{k})\mathbf{q}\cdot\boldsymbol{\varepsilon}}{(\mathbf{q}-\mathbf{k})^2+\mu^2}\right\}. \quad (21)$$

The Kroll-Ruderman result[5] is contained in the $\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}$ part of (21), which is pure s-wave. Equation (16) may may then be rewritten

$$\mathcal{H}_k^{\pi}(q)=\mathcal{H}_k^{(0)}(q)-\sum_{n>0}\left\{\frac{T_q^{\dagger}(n)\mathcal{H}_k^{\pi}(n)}{E_n-\omega_q-i\epsilon}+\frac{\mathcal{H}_k^{\pi\dagger}(n)T_q(n)}{E_n+\omega_q}\right\}. \quad (22)$$

To recapitulate, the total photoproduction matrix element has been split into three terms:

$$\mathcal{H}_k(q)=\mathcal{H}_k^{V}(q)+\mathcal{H}_k^{S}(q)+\mathcal{H}_k^{\pi}(q).$$

For the first term, an explicit formula has been given in Eq. (9) in terms of quantities which may be experimentally determined independently of photoproduction measurements. The second term is determined by Eqs. (12) and (14) and corresponds to what Watson[6] calls a nucleon recoil effect. To the extent that neutron and proton magnetic moments are very roughly equal and opposite, this term tends to be small. In addition, since the current responsible is independent of isotopic spin, it cannot produce a final state of isotopic spin 3/2. Both these terms of course produce only p-wave mesons and correspond to magnetic dipole transitions. The third term is determined by Eqs. (21) and (22). It produces a large s-wave (electric dipole) but in addition all higher partial waves.

III. ONE-MESON APPROXIMATION[7]

In this section, the two Eqs. (12) and (22) for the production amplitudes generated by $\mathbf{j}_S$ and $\mathbf{j}_\pi$, respectively, will be discussed in the same approximation as used in Sec. IV of the scattering paper. That is, multimeson terms ($n>1$) will be dropped. As before, this approximation does not destroy any of the fundamental symmetries of the theory. It is valid if multiple meson production is small.

A.

Let us begin with Eq. (12). To eliminate angular and isotopic spin dependence, we note that only states with $I=\frac{1}{2}$ can be produced by $\mathcal{H}^S$. Furthermore, the production is magnetic dipole and so leads only to p-states of total angular momentum $\frac{3}{2}$ or $\frac{1}{2}$. Thus we set

$$\mathcal{H}_k^{S}(q)=\frac{v(q)G(k^2)}{(4k\omega_q)^{\frac{1}{2}}}\tau_q\sum_{J=\frac{1}{2}}^{\frac{3}{2}}P_J M_P^{S}(\omega_q), \quad (23)$$

where

$$P_{\frac{1}{2}}=\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{p}, \quad P_{\frac{3}{2}}=3\mathbf{p}\cdot\mathbf{q}-\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{p}. \quad (24)$$

Equation (12), with neglect of multimeson terms, then

[5] N. M. Kroll and M. A. Ruderman, Phys. Rev. 93, 233 (1954).
[6] K. M. Watson, Phys. Rev. 95, 228 (1954).
[7] The considerations of this section are unavoidably complicated because of the many degrees of freedom in the photoproduction problem. The essential results are summarized in Secs. IV and V, so readers who wish to avoid the proliferation of superscripts and subscripts should skip Sec. III, or at least postpone its reading.

becomes

$$M_J{}^S(\omega_q)=\frac{\gamma_J}{\omega_q}+\frac{1}{\pi}\int_1^\infty d\omega_p p^3v^2(p)\left\{\frac{h_{1J}{}^*(\omega_p)M_J{}^S(\omega_p)}{\omega_p-\omega_q-i\epsilon}+\sum_{J'}C_{JJ'}\frac{M_{J'}{}^{S*}h_{1J}(\omega_p)}{\omega_p+\omega_q}\right\}, \quad (25)$$

where

$$\gamma_J=f_r\frac{\mu_P+\mu_N}{2}\begin{pmatrix}-4/3\\ 2/3\end{pmatrix}\begin{matrix}J=1/2\\ J=3/2\end{matrix}, \quad (26)$$

$$C_{JJ'}=\begin{pmatrix}-1/3 & 4/3\\ 2/3 & 1/3\end{pmatrix}, \quad (27)$$

and $h_{1J}(\omega_p)$ is defined by (I-32) and (I-34).

It is useful now as before to introduce a function of a complex variable. We define the real analytic function $M_J{}^S(z)$ by the following properties:

(A) The limit of $M_J{}^S(z)$ as $z\to\omega_p+i\epsilon$ is $M_J{}^S(\omega_p)=R_J{}^S(\omega_p)\cdot e^{i\delta_{1J}(\omega_p)}$, where δ_{1J} is the scattering phase shift for isotopic spin $\frac{1}{2}$ and angular momentum J at energy ω_p and $R_J{}^S$ is real.

(B) $\sum_{J'}C_{JJ'}M_{J'}{}^S(z)=M_J{}^S(-z)$.

(C) $M_J{}^S(z)$ has a simple pole at the origin of residue γ_J. [Note that $\sum_{J'}C_{JJ'}\gamma_{J'}=-\gamma_J$, so that (C) is consistent with (B).]

(D) At infinity $M_J{}^S(z)$ goes like $1/z$.

(E) $M_J{}^S(z)$ has branch points at $z=\pm1$, with cuts out to $\pm\infty$ along the positive and negative real axes. There are no other singularities.

Using arguments analogous to those of Sec. (III-D) in the preceding paper, we may write

$$M_J{}^S(z)=\frac{\gamma_J}{z}+\frac{1}{\pi}\int_1^\infty dx\left[\frac{F_J{}^S(x)}{x-z}+\frac{G_J{}^S(x)}{x+z}\right], \quad (28)$$

where $F_J{}^S(x)$ and $G_J{}^S(x)$ are real weighting functions related to the jump in $M_J{}^S$ in going across the positive and negative real axes, respectively. In particular,

$$2iF_J{}^S(\omega_p)=M_J{}^S(\omega_p+i\epsilon)-M_J{}^S(\omega_p-i\epsilon), \quad (29)$$

which by condition (A) above leads to

$$F_J{}^S(\omega_p)=R_J{}^S(\omega_p)\sin\delta_{1J}(\omega_p). \quad (30)$$

Use of the crossing condition (B) leads in a similar way to

$$G_J{}^S(\omega_p)=\sum_{J'}C_{JJ'}R_{J'}{}^S(\omega_p)\sin\delta_{1J'}(\omega_p). \quad (31)$$

If the relation (I-30) is again recalled, one sees that we have now produced Eq. (25) exactly. Thus (25) is equivalent to the conditions (A) to (E) above and satisfaction of these conditions solves our problem.

All conditions except the crossing condition (B) are immediately satisfied by the choice,

$$M_J{}^S(z)=\frac{\gamma_J}{z}\frac{1}{g_{1J}(z)}\left[1+\frac{z}{\pi}\int_1^\infty d\omega_p\frac{p^3v^2(p)}{\omega_p{}^2}\frac{K_J{}^S(\omega_p)}{\omega_p+z}\right], \quad (32)$$

where $g_{1J}(z)$ has been defined by (I-41), and $K_J{}^S(\omega_p)$ is a real function as yet undetermined. The weighting function $K_J{}^S(\omega_p)$ must be chosen so as to satisfy the crossing condition. Expanding in a power series and keeping only the first term, we find

$$K_J{}^S(\omega_p)=\frac{8}{3}f^2\begin{pmatrix}1\\ 1\end{pmatrix}+O(f^4).$$

The problem of finding a general expression for $K_J{}^S(\omega_p)$ has not been solved.

B.

We turn now to Eq. (22) for the photoproduction amplitude generated by $\mathbf{j}_\pi$ and begin by separating out the p-wave part of the inhomogeneous term, $\mathcal{H}_k{}^{(0)}(q)$:

$$\mathcal{H}_k{}^{(0)}(q)=\frac{e_rf_r}{(4\omega_qk)^{\frac{1}{2}}}(\tau_2\delta_{q1}-\tau_1\delta_{q2})\{F_M(k,q)(\boldsymbol{\sigma}\times\mathbf{q})\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})+F_Q(k,q)[(\mathbf{q}\cdot\boldsymbol{\varepsilon})(\boldsymbol{\sigma}\cdot\mathbf{k})+(\mathbf{q}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon})]\}+\text{non } p\text{-wave terms}, \quad (33)$$

where

$$F_M(k,q)=\frac{3}{8kq}\left[\frac{2}{u}-\frac{1-u^2}{u^2}\log\left(\frac{1+u}{1-u}\right)\right], \quad (34)$$

$$F_Q(k,q)=\frac{1}{k^2}-\frac{\omega_q{}^2}{k^2}F_M(k,q), \quad (35)$$

$$u=2qk/(\omega_q{}^2+k^2). \quad (36)$$

The two functions F_M and F_Q correspond respectively to magnetic dipole and electric quadrupole transitions. As has been noted before, the non p-wave parts of the photoproduction amplitude are given exactly by $\mathcal{H}_k{}^{(0)}(q)$. That is to say, if (33) is substituted into Eq. (22) it is only the p-wave part which need concern us. Furthermore, the separation into magnetic dipole and electric quadrupole parts corresponds to a distinction between an antisymmetric tensor, $\epsilon_ik_j-\epsilon_jk_i$, and a symmetric tensor, $\epsilon_ik_j+\epsilon_jk_i$, a distinction which is maintained in the sum over states of Eq. (22). We therefore can obtain uncoupled equations for the separate magnetic dipole and electric quadrupole amplitudes.

The electric quadrupole amplitude contains only angular momentum $\frac{3}{2}$ but both $\frac{1}{2}$ and $\frac{3}{2}$ isotopic spins. The magnetic dipole on the other hand contains all four states. In analogy to (23), we write

$$\mathcal{H}_k{}^\pi(q)=\text{non } p\text{-wave part of } \mathcal{H}_k{}^{(0)}(q)+\frac{1}{(4\omega_qk)^{\frac{1}{2}}}\sum_{\alpha=1}^4 P_\alpha M_\alpha{}^\pi(k,q)+\frac{i}{(4\omega_qk)^{\frac{1}{2}}}[(\mathbf{q}\cdot\boldsymbol{\varepsilon})(\boldsymbol{\sigma}\cdot\mathbf{k}+(\mathbf{q}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon})]\times\sum_{I=\frac{1}{2}}^{\frac{3}{2}}P_IQ_I{}^\pi(k,q), \quad (37)$$

where the P_α's are defined by Eq. (I-33) for an initial momentum $\mathbf{k}\times\boldsymbol{\varepsilon}$ and initial charge state 3, and where

$$P_{(I=\frac{1}{2})}=\tfrac{1}{3}\tau_q\tau_3,\quad P_{(I=\frac{3}{2})}=\delta_{q3}-\tfrac{1}{3}\tau_q\tau_3. \tag{38}$$

The first sum in (37) evidently contains the magnetic dipole terms, the second the electric quadrupole. The Q^π's and M^π's are functions of the magnitudes of k and q. As in the scattering case $M_{13}{}^\pi=M_{31}{}^\pi$, so for the magnetic dipole part of the problem we may confine our attention to the three functions $M_1{}^\pi=M_{11}{}^\pi$, $M_2{}^\pi=M_{13}{}^\pi=M_{31}{}^\pi$, $M_3{}^\pi=M_{33}{}^\pi$.

Substituting (37) into (22) yields the following equations for the M^π's and Q^π's:

$$Q_I{}^\pi(k,q)=\eta_I F_Q(k,q)+\frac{1}{\pi}\int_1^\infty d\omega_p p^3 v^2(p)\times\left\{\frac{h_{I3}{}^*(\omega_p)Q_I{}^\pi(k,p)}{\omega_p-\omega_q-i\epsilon}-\sum_{I'}C_{II'}\frac{Q_{I'}{}^*(k,p)h_{I'3}(\omega_p)}{\omega_p+\omega_q}\right\}, \tag{39}$$

where $C_{II'}$ is the same matrix as defined by Eq. (27),

$$\eta_I=e_r f_r\begin{pmatrix}2\\-1\end{pmatrix} \tag{40}$$

and $h_{I3}(\omega_p)$ is defined by (I-32) and (I-34). Similarly, the equation for M^π is

$$M_\alpha{}^\pi(k,q)=\xi_\alpha F_M(k,q)+\frac{1}{\pi}\int_1^\infty d\omega_p p^3 v^2(p)\times\left\{\frac{h_\alpha{}^*(\omega_p)M_\alpha{}^\pi(k,p)}{\omega_p-\omega_q-i\epsilon}+\sum_\beta A_{\alpha\beta}\frac{M_\beta{}^{\pi*}(k,p)h_\beta(\omega_p)}{\omega_p+\omega_q}\right\}, \tag{41}$$

where $A_{\alpha\beta}$ is the matrix defined by (I-36) and

$$\xi_\alpha=e_r f_r\begin{pmatrix}4/3\\-2/3\\1/3\end{pmatrix}. \tag{42}$$

As in the case of $\mathcal{H}_k{}^S(q)$, it is clear that Eqs. (39) and (41) serve to define analytic functions whose limits on the positive real axis are the meson production amplitudes.

We thus introduce the five real analytic functions $\delta Q_I{}^\pi(z)$ and $\delta M_\alpha{}^\pi(z)$, where the dependence on the photon energy k has been suppressed. These have the following properties:

(A) $$\lim_{z\to\omega_p+i\epsilon}\begin{Bmatrix}\delta Q_I{}^\pi(z)\\ \delta M_\alpha{}^\pi(z)\end{Bmatrix}+\begin{Bmatrix}\eta_I F_Q(k,q)\\ \xi_\alpha F_M(k,q)\end{Bmatrix}=\begin{Bmatrix}Q_I{}^\pi(k,q)\\ M_\alpha{}^\pi(k,q)\end{Bmatrix},$$

and

$$\begin{Bmatrix}Q_I{}^\pi(k,q)\\ M_\alpha{}^\pi(k,q)\end{Bmatrix}\text{ has the same phase as }\begin{Bmatrix}\exp[i\delta_{I3}(\omega_q)]\\ \exp[i\delta_\alpha(\omega_q)]\end{Bmatrix},$$

(B) $\sum_{I'}C_{II'}Q_{I'}{}^\pi(z)=-\delta Q_I{}^\pi(-z)$,

$\sum_\beta A_{\alpha\beta}\delta M_\beta{}^\pi(z)=\delta M_\alpha{}^\pi(-z)$.

(C) $\begin{Bmatrix}\delta Q^\pi\\ \delta M^\pi\end{Bmatrix}$ has no poles in the complex plane.

(D) $\begin{Bmatrix}\delta Q^\pi\\ \delta M^\pi\end{Bmatrix}$ goes like $1/z$ at ∞.

(E) $\begin{Bmatrix}\delta Q^\pi\\ \delta M^\pi\end{Bmatrix}$ has branch points at $z=\pm1$, with cuts along the positive and negative real axis to $\pm\infty$, and no other singularities.

The proof that the analytic functions so defined satisfy Eqs. (39) and (41) follows the by now familiar pattern and will not be repeated here. The satisfaction of all the conditions except (B) is trivial, as usual. We find

$$\delta Q_I{}^\pi(z)=\frac{\lambda_{I3}}{\pi g_{I3}(z)}\int_1^\infty d\omega_p\frac{p^3v^2(p)}{\omega_p}\left[\eta_I\frac{F_Q(k,p)}{\omega_p-z}+\frac{G_{QI}(k,p)}{\omega_p+z}\right], \tag{43}$$

$$\delta M_\alpha{}^\pi(z)=\frac{\lambda_\alpha}{\pi g_\alpha(z)}\int_1^\infty d\omega_p\frac{p^3v^2(p)}{\omega_p}\left[\xi_\alpha\frac{F_M(k,p)}{\omega_p-z}+\frac{G_{M\alpha}(k,p)}{\omega_p+z}\right], \tag{44}$$

where the unknown functions G_{QI} and $G_{M\alpha}$ are to be determined to satisfy the crossing theorem, condition (B).

If one expands G_Q and G_M in a power series in f^2, one finds for the first-order terms:

$$G_{QI}(k,p)=-\eta_I F_Q(k,p)+O(f^4)$$

and

$$G_{M\alpha}(k,p)=\zeta_\alpha F_M(k,p)+O(f^4),$$

where

$$\zeta_\alpha=e_r f_r\begin{pmatrix}0\\-2\\-1\end{pmatrix}.$$

General expressions for G_{QI} and $G_{M\alpha}$ have not been obtained.

IV. SIMPLE APPROXIMATION FOR THE COMPLETE PHOTOPRODUCTION AMPLITUDE

We have now presented a recipe for evaluation of the photoproduction amplitude according to the static model in the one-meson approximation. In this section, we wish to point out that certain parts of this amplitude are independent of the details of our model and thus stand on a particularly firm basis. Fortunately these parts also happen to be the dominant ones, so that it is possible to write down a relatively simple and reliable formula for the total amplitude.

The first of the simple parts is of course $\mathcal{H}_k{}^V(q)$, given by Eq. (9), which actually is more general than the one-meson approximation. This result is made even simpler if only the 3–3 part of the scattering amplitude is retained. Our motivation for omitting the "small" phase shifts is twofold: (1) Other terms which we propose to neglect here (e.g., from δM^{π}) contribute about as much to the nonresonant states as does $\mathcal{H}_k{}^V(q)$. (2) Of the p-wave phase shifts, only δ_{33} is known experimentally. In addition, we shall set the form factor $F(k^2)$ in Eq. (7) equal to unity, since both theory[8] and the Stanford experiments[3] indicate a variation of only $\sim 20\%$ over the energy region which we shall consider. Thus we have from (7) and from (I-32) and (I-34):

$$\mathcal{H}_k{}^V(q)\approx\frac{e_rf_r}{(4\omega_qk)^{\frac{1}{2}}}\left(\frac{g_p-g_n}{4Mf^2}\right)(\delta_{q3}-\tfrac{1}{3}\tau_q\tau_3)\times[2\mathbf{q}\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})-i\boldsymbol{\sigma}\cdot\mathbf{q}\times(\mathbf{k}\times\boldsymbol{\varepsilon})]\frac{e^{i\delta_{33}}\sin\delta_{33}}{q^3},\quad(45)$$

where $g_p=2.78$ and $g_n=-1.91$ are the nucleon magnetic moments in units of the nuclear magneton.

The second of the simple terms is $\mathcal{H}_k{}^{(0)}$, given by Eq. (20), which we rewrite here for convenience:

$$\mathcal{H}_k{}^{(0)}(q)=\frac{-ie_rf_r}{(4\omega_qk)^{\frac{1}{2}}}\left(\frac{\tau_3\tau_q-\tau_q\tau_3}{2}\right)\times\left[\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}-2\frac{[\boldsymbol{\sigma}\cdot(\mathbf{q}-\mathbf{k})](\mathbf{q}\cdot\boldsymbol{\varepsilon})}{(\mathbf{k}-\mathbf{q})^2+\mu^2}\right].\quad(46)$$

We now wish to argue that the sum of (45) and (46) in the low-energy region ($k\lesssim 2\mu$) is a good first approximation to the total amplitude. Let us recall the terms omitted.

First there is $\mathcal{H}^S$ as given by Eq. (12). The principal justification for its omission is that the coefficient $\frac{1}{2}(\mu_P+\mu_N)$, which occurs in this term, is five times smaller than $\frac{1}{2}(\mu_P-\mu_N)$, which occurs in $\mathcal{H}^V$; also the resonant 3–3 state does not occur in $\mathcal{H}^S$ at all. Quantitative estimates based on a power-series evaluation of Eq. (32) bear out these qualitative arguments. Neglect of $\mathcal{H}^S$ causes an error of $\sim 20\%$ in the p-wave amplitude near threshold ($k\approx\mu$) and less than 10% near resonance ($k\approx 2\mu$). An experimental measurement which might somehow select the p-wave photoproduction amplitude produced by the isotopic scalar part of the current would of course not tolerate neglect of $\mathcal{H}^S$.

We found in Sec. III that $\mathcal{H}_k{}^{\pi}(q)$ could be written in the form

$$\mathcal{H}_k{}^{\pi}(q)=\mathcal{H}_k{}^{(0)}(q)+\frac{1}{(4\omega_qk)^{\frac{1}{2}}}\Big\{\sum_{\alpha=1}^{4}P_\alpha\delta M_\alpha{}^{\pi}(k,q)+i[(\mathbf{q}\cdot\boldsymbol{\varepsilon})(\boldsymbol{\sigma}\cdot\mathbf{k})+(\mathbf{q}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon})]\sum_{I=\frac{1}{2}}^{\frac{3}{2}}P_I\delta Q_I{}^{\pi}(k,q)\Big\},$$

where the projection operators P_α and P_I are defined by (38) and the preceding sentence. Here δM and δQ stand for magnetic dipole and electric quadrupole respectively.

[8] G. Salzman, Phys. Rev. 99, 619 (1955).

Our proposal, then, is to neglect δM and δQ. This neglect is made on an entirely different basis from that of $\mathcal{H}^S$. Consider $\delta M_\alpha{}^{\pi}$ as given by Eq. (44) and compare to the corresponding term in $\mathcal{H}_k{}^{(0)}$, which by Eq. (41) is $\xi_\alpha F_M$. It is to be expected that the unknown part of (44) which is proportional to G_M is of the same order as the known part which is proportional to F_M. Thus we may compare δM^{π} to $\mathcal{H}_k{}^{(0)}$ by comparing

$$\frac{\lambda_\alpha}{g_\alpha(\omega_q)}\frac{1}{\pi}\int_1^\infty d\omega_p\frac{p^3v^2(p)}{\omega_p}\frac{F_M(k,p)}{\omega_p-\omega_q-i\epsilon}\quad(47)$$

to $F_M(k,q)$. The integral in (47) is only logarithmically dependent on the cut-off energy, $\omega_{\max}$, so one may say that the ratio of δM^{π} to $\mathcal{H}_k{}^{(0)}$ is $\sim\lambda_\alpha[g_\alpha(\omega_q)]^{-1}$, which is always small except for the (3,3) state near resonance. An estimate of $\delta M_{33}{}^{\pi}$ near resonance based on a power series evaluation of Eq. (44) shows that this term, while not necessarily negligible compared to the (3,3) part of $\mathcal{H}_k{}^{(0)}$, is never more than 20% of $\mathcal{H}_k{}^V$ as given by Eq. (45). Entirely similar considerations also justify the neglect of δQ^{π}. The physical reason behind the smallness of these correction terms to $\mathcal{H}_k{}^{(0)}$ might be stated as follows: $\mathcal{H}_k{}^{(0)}$ itself represents the absorption of the incident photon by a virtual pion (already present) in the field of the physical nucleon. For reasons of kinematics and the uncertainty principle, this absorption takes place on the average at a large distance, of the order μ^{-1}, from the origin and the meson has a good chance to escape without further interaction. Thus the terms δM^{π} and δQ^{π} which represent the secondary scattering of these mesons are small. This situation is to be contrasted to the absorption of the photon by the nucleon magnetic moment. This mechanism produces mesons much closer to the origin, and these mesons then undergo strong secondary scattering. Mathematically the difference shows up in the fact that the integral in (47) has only a logarithmic cut-off dependence while the corresponding integrals in $\mathcal{H}^S$ and $\mathcal{H}^V$ are linearly dependent on $\omega_{\max}$.

It is unfortunately the case that although $\mathcal{H}_k{}^{(0)}$ is large compared to δM^{π} and δQ^{π} (except in the 3–3 state), and quite comparable to $\mathcal{H}^V$, there is an enormous coherence in the various parts of $\mathcal{H}_k{}^{(0)}$, which conspires to suppress the effects of this term in most experimental measurements. For example, $\mathcal{H}_k{}^{(0)}$ gives no contribution whatsoever to neutral photoproduction. Even for charged photopions, the observable effects of $\mathcal{H}_k{}^{(0)}$ (except for the s-wave) are disappointingly small. The s–p interference, for example, vanishes completely and the p^2 terms are largely canceled by s–d and s–s interference. Thus the non s-wave parts of $\mathcal{H}_k{}^{(0)}$, although fully as large as $\mathcal{H}^V$, are hard to find experimentally.

In spite of these experimental difficulties the advantages of keeping only $\mathcal{H}_k{}^{(0)}$ and $\mathcal{H}^V$ are great. These terms depend neither on the cutoff nor the one-meson approximation. $\mathcal{H}_k{}^{(0)}$ certainly is correct even in a relativistic theory and $\mathcal{H}^V$ has such a general appearance that we would be surprised to find it much altered by a relativistic treatment. Even if the accuracy of the experimental information requires the inclusion of other terms, one feels confident that terms corresponding to $\mathcal{H}_k^{(0)}$ and $\mathcal{H}^V$ must be present in nature and must be important. In the following section, therefore, we shall compare the sum of (45) and (46) to experiment.

Our approach ideally would be to obtain, from the experimental information, amplitudes in the sense of Watson[9] and then to compare these experimental amplitudes with theory. Such a procedure would circumvent the difficulties due to cancellations mentioned above. Unfortunately the accuracy of the data does not yet permit a complete analysis, so we shall be forced to compromise. The spirit of our approach however will be a comparison of amplitudes, not of cross sections.

A word may be in order at this point to relate the results of this paper to earlier and less complete results reported by one of the authors.[10] Formulas (6) and (6′) of the earlier paper are almost equivalent to the sum of Eqs. (45) and (46) above. The differences are: (1) The terms proportional to M_1 in the earlier work represent the sum of the 3–3 part of $\mathcal{H}^V$ *plus* δM_{33}^π. An attempt was made by Salzman[11] to calculate δM_{33}^π but it is now believed that only the order of magnitude of the result is reliable. Salzman also calculated $\mathcal{H}^V$, but by less reliable methods than used here, and got somewhat too small a value. (2) The terms in the earlier paper proportional to E_2 correspond to δQ_{33}^π here. Again Salzman's estimate is only good for an order of magnitude and possibly for sign. At the time of the earlier work it was expected that δM_{33}^π and δQ_{33}^π would be very important and $\mathcal{H}^V$ a minor term, and although calculations gave just the opposite result there was great reluctance to omit terms which had been expected to dominate. Nevertheless, the numerical calculations reported in the earlier paper correspond to small values for δM_{33}^π and δQ_{33}^π and to a value for $\mathcal{H}^V$, which is not seriously wrong. The success of the previous comparison to experiment is therefore comparable to what we achieve here.

V. COMPARISON WITH EXPERIMENT

A. Charged s-Wave Amplitude at Threshold and the Coupling Constant

It was first pointed out by Kroll and Ruderman[5] that the s-wave amplitude for charged photomeson production at threshold ($\mathbf{q}=0$) is very simply related to the renormalized coupling constant. Their result is represented by the first term in the bracket of formula (46) which for charged photomeson production becomes

$$\pm\sqrt{2}\frac{ief_re_r}{(4\omega_q k)^{\frac{1}{2}}}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}, \tag{48}$$

the plus sign going with positive mesons and the minus with negative. Kroll and Ruderman also showed that the first-order relativistic corrections to (48), i.e., terms of order $1/M$ where M is the nucleon mass, are equal for positive and negative production. One thus may obtain the experimental amplitude corresponding to (48) correct to order $1/M^2$ by using as a basis the *average* of the positive and negative production cross sections at threshold. Using data from both hydrogen and deuterium targets and making appropriate corrections for Coulomb effects, Bernardini[12] finds that the value of the coupling constant needed to fit the threshold experiments is

$$f^2=0.073\pm0.007, \tag{49}$$

a result in satisfactory agreement with that obtained in the preceding paper from measurements of pion-nucleon scattering.

B. Neutral p-Wave Amplitude

Since $\mathcal{H}_k^{(0)}$ vanishes for neutral production, one has a direct test of $\mathcal{H}^V$ in the neutral p-wave amplitude. Furthermore, it is known experimentally that the neutral s-wave amplitude is small, so that the total cross section is a good measure of the p-wave amplitude. Finally, we note that the neglect of the small p-phase shifts should be least important at the energy of the 3–3 resonance, and so make our first test of formula (45) by asking what value of the coefficient

$$C_0=\left(\frac{g_p-g_n}{4M}\right)\frac{e}{f}, \tag{50}$$

which occurs in (45) is needed to reproduce the measured total neutral production cross section at a photon lab energy of 325 Mev, where $\delta_{33}=90°$. Using California Institute of Technology data,[13] we find

$$C_0=0.050\pm0.003,$$

while the theoretical value, using f as given by (49), is

$$C_0=0.06\pm0.01.$$

It is known from phenomenological analyses[9] that an amplitude of the form (45) leads to the correct energy dependence for the total neutral cross section. A simple and direct check of this point can be made by noting that Eq. (9) implies the following relation between the total neutral photomeson cross section $\sigma_{\gamma\to0}$ and the total neutral-to-neutral scattering cross section $\sigma_{0\to0}$ at

[9] K. M. Watson, Phys. Rev. **98**, 234 (1955).
[10] G. F. Chew, Phys. Rev. **94**, 1748 (1954).
[11] F. Salzman (private communication).
[12] G. Bernardini (private communication).
[13] D. C. Oakley and R. L. Walker, Phys. Rev. (to be published).

the same energy:

$$\sigma_{\gamma\to 0}=\left(\frac{k}{q}\right)\left(\frac{e^2}{f^2}\right)\left(\frac{g_p-g_n}{4M}\right)^2\sigma_{0\to 0}. \quad (51)$$

Koester and Mills[14] have recently verified this relation quantitatively for energies up to the resonance.

The angular distribution of neutral photopions on the other hand is not so well described by (45) which leads to the familiar energy-independent form, $2+3\sin^2\theta$. Deviations from this distribution due to δQ^π and to the $J=\frac{1}{2}$ magnetic dipole transitions will be much more important, because of interference, than the corresponding deviations in the total cross section, where interference terms cancel out.

C. Charged p-Wave Amplitude

At the 3–3 resonance energy, formulas (45) and (46) predict that the total charged cross section is very closely the sum of the contribution from (45), which is p-wave, and the first part of (46), which is s-wave and known from the threshold measurement as discussed above. Interference terms drop out. We may thus test the charged p-wave predicted by (45) if we subtract the s-wave part from the measured total cross section at 325-Mev photon lab energy. Combining in this way the California Institute of Technology measurement of the resonance cross section with the Illinois measurement of the threshold cross section, we find that the value of the coefficient C_0 defined by (50) should be 0.046. As noted above the theoretical value is 0.06. The sign of the charged p-wave amplitude is checked by the asymmetry in the angular distribution at energies below the 3–3 resonance. The interference between (45) and the s-wave part of (46) should produce preferentially backward positive photopions, and such an asymmetry actually is observed. The magnitude of the observed asymmetry is somewhat smaller than that predicted, but the discrepancy could easily be due to the neglect of the small phase shifts and δQ^π.

[14] L. J. Koester and F. Mills, Phys. Rev. (to be published).

D. Nonthreshold Part of $\mathcal{H}_k^{(0)}$

It will be noticed that we have as yet made no test of the second part of $\mathcal{H}_k^{(0)}$, which contains s,p and higher partial waves; in fact we have ignored any contributions which this term might make to the charged cross section. Explicit calculation shows that these contributions are almost everywhere small, even though the amplitude itself is large. The reasons for this unusual situation are as follows: (1) The low-energy p part of this term does not interfere with the main s-wave term for unpolarized target nucleons. (2) Interference of the s and d parts with the main s-wave term almost cancels out the square of the p-wave except very near threshold. Here a negative term is of some importance. Relative to the main s-wave contribution one finds

$$\frac{|\mathcal{H}_k^{(0)}(q)|^2}{|\mathcal{H}_k^{(0)}(0)|^2}=1-\frac{q^2}{2k^4}\frac{\sin^2\theta}{[1-(q/k)\cos\theta]^2}. \quad (52)$$

The negative term actually determines the q^2 dependence near threshold but it is unimportant for $q\gtrsim 1$. If the excitation function can be accurately measured a minimum in the square of the matrix element should be found 10–15 Mev above threshold. Bernardini[12] has assumed the form (52) in determining the coupling constant.

Another manifestation of the second part of $\mathcal{H}_k^{(0)}$ is its interference with $\mathcal{H}^V$ which leads to a term with angular dependence $\sim\sin^2\theta$ and phase-shift dependence $\sim\sin\delta_{33}\cos\delta_{33}$. This term has its maximum near $\delta_{33}=45°$ and thus should shift the maximum in the charged total cross section down in energy relative to the neutral. Such a shift has been experimentally observed, although its magnitude is too small to allow a quantitative test of the theory.