PHYSICAL REVIEW VOLUME 101, NUMBER 5 MARCH 1, 1956

Effective-Range Approach to the Low-Energy p-Wave Pion-Nucleon Interaction

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(Received November 28, 1955)

The theory of p-wave pion-nucleon scattering is reexamined using the formalism recently proposed by one of the authors (F.E.L.). On the basis of the cut-off Yukawa theory without nuclear recoil it is found, for not too high values of the coupling constant, that: (a) For each p-wave phase shift a certain function of the cotangent should be approximately linear at low energies and should extrapolate to the Born approximation at zero total energy. The value of the renormalized unrationalized coupling constant determined in this way from experiment is $f^2=0.08$. A special feature of the predicted energy dependence of the phase shifts is that δ_{33} is positive and the other p phase shifts are negative. (b) The so-called "crossing theorem" requires a relation between the four p phase shifts, so that in addition to the coupling constant only two further constants are needed to completely specify the low-energy behavior. (c) The direction of the energy variation in the (3,3) state is such that a resonance will occur for a sufficiently large cut-off $\omega_{\max}$. Rough estimates indicate that $\omega_{\max}\approx 6$ will produce a resonance at the energy required by experiment. It is argued that the results (a) and (b) are very probably also consequences of a relativistic theory but that (c) may not be.

I. INTRODUCTION

THE ability of the Yukawa theory to describe quantitatively the pion-nucleon interaction is still uncertain. The recent discovery of new particles, the hyperons and K-mesons, makes it unlikely that this theory can be valid in the multi-Bev energy region, but there still remains an interesting and important question: Can a theory of the Yukawa type quantitatively correlate experiments in the sub-Bev region, below the threshold for production of "curious" particles? Recently, it has been shown[1] that a crude static model of the pion-nucleon interaction, based on the Yukawa idea, is quite powerful in correlating certain experiments; however, the relation of the model to a true theory has been obscure. One of the main purposes of this paper and the one following (which will be concerned with photomeson production) is to show that the most important predictions of the model are actually independent of its details and thus may also be predictions of a "true" theory.

The results to be presented are based on a new set of equations[2] which can be applied both to the static model and to the relativistic Yukawa theory, and which exhibit the low-energy properties of both in a clear and useful way. Some new and quite general predictions about p-wave pion-nucleon scattering have been achieved without explicit solution of the equations, and the second purpose of this paper is to report these new predictions.

Throughout the first six sections of this paper, arguments and derivations will be given in terms of the static model. The simplification achieved by eliminating antinucleons and recoil is enormous. In Sec. II, a new and simplified derivation of the equations for meson scattering is presented. In Sec. III, those properties of the scattering are discussed which can be deduced without explicit solution of the equations. Section IV deals with the equations in what might be called the "one-meson approximation." In Sec. V, an effective-range treatment of the scattering problem is presented. Section VI deals with certain total-cross-section sum rules, including a sum rule for the renormalization constants of the theory. Finally, in Sec. VII, we discuss the possible extension of our results to more complicated cases such as the relativistic pseudoscalar theory.

II. DERIVATION OF THE SCATTERING EQUATIONS

We present here a much simpler derivation of the scattering equations than is given in reference 2. We follow a method suggested by Wick[3] which unfortunately applies to the fixed-source theory only. For a derivation of the appropriate expression when nucleon recoil is to be included, see reference 2.

We take as our Hamiltonian[4]

$$H=H_0+H_I, \tag{1}$$

where

$$H_I=\sum_k V_k^{(0)}a_k+V_k^{(0)\dagger}a_k^\dagger, \tag{2}$$

$$H_0=\sum_k a_k^\dagger a_k\omega_k, \tag{3}$$

and

$$V_k^{(0)}=if_{(r)}^{(0)}(\boldsymbol{\sigma}\cdot\mathbf{k}/\sqrt{2}\omega_k)\tau_k v(k). \tag{4}$$

Here $a_k^\dagger$ and a_k are, respectively, creation and annihilation operators for single mesons, $\omega_k=(1+k^2)^{\frac{1}{2}}$, $\boldsymbol{\sigma}$ is the nucleon spin vector, and τ_k is the kth component of the nucleon isotopic spin operator. In our notation, the meson quantum numbers are all described by a single symbol (k) which includes the three components of momentum and the isotopic spin. Also, $f_{(r)}^{(0)}$ is the rationalized but unrenormalized coupling constant.

The Hamiltonian (1) has a complete set of eigenstates Ψ_n. These states include the four single-nucleon states

[1] G. F. Chew, Phys. Rev. **95**, 1669 (1954).
[2] F. E. Low, Phys. Rev. **97**, 1392 (1955). Closely related equations have also been derived by Lehmann, Symanzik, and Zimmerman, Nuovo cimento **1**, 1 (1955).
[3] G. C. Wick, Revs. Modern Phys. **27**, 339 (1955).
[4] We take $\hbar=c=\mu=1$; μ is the meson rest mass.

Ψ_0 (we suppress the spin and isotopic spin indices), the one-meson states Ψ_q, two-meson states, etc. We are of course particularly interested in the one-meson states with outgoing waves, $\Psi_q^{(+)}$. These are solutions of the Schrödinger equation

$$(H-E_q)\Psi_q^{(+)}=0,$$

with

$$E_q=E_0+\omega_q, \tag{5}$$

where E_0 is the single-nucleon energy. Following Wick, we set

$$\Psi_q^{(+)}=a_q^\dagger\Psi_0+\chi^{(+)}, \tag{6}$$

where

$$(H-E_q)\chi^{(+)}=-(H-E_q)a_q^\dagger\Psi_0.$$

Now

$$[H_0,a_q^\dagger]=\omega_q a_q^\dagger$$

and

$$[H_I,a_q^\dagger]=V_q^{(0)},$$

so that

$$(H-E_q)a_q^\dagger\Psi_0=V_q^{(0)}\Psi_0$$

and

$$(H-E_q)\chi^{(+)}=-V_q^{(0)}\Psi_0.$$

Dividing through by $(H-E_q)$, we have

$$\chi^{(+)}=-\frac{1}{H-E_q-i\epsilon}V_q^{(0)}\Psi_0, \tag{7}$$

where the $-i\epsilon$ is inserted to produce outgoing waves in $\chi^{(+)}$.[5] The reader will recall that $1/(a-i\epsilon)=\mathrm{P}(1/a)+i\pi\delta(a)$,[6] where $\mathrm{P}(1/a)$ stands for the Cauchy principal value. We rewrite Eq. (7) as:

$$\chi^{(+)}=-\sum_n \frac{1}{E_n-E_q-i\epsilon}\Psi_n^{(-)}\langle\Psi_n^{(-)},V_q^{(0)}\Psi_0\rangle, \tag{8}$$

where the $\Psi_n^{(-)}$ are the complete orthonormal set of *incoming* wave eigenstates. We assume in writing Eq. (8) that there are no bound states.

The next problem is to relate the result (8) to the scattering matrix. Wick[7] has been able to show that the S-matrix is directly related to the coefficient of $\Psi_n^{(-)}$ in the sum in (8) by the equation

$$\langle n|S|q\rangle=\delta_{nq}-2\pi i\delta(E_q-E_n)T_q(n), \tag{9}$$

where $T_q(n)=(\Psi_n^{(-)},V_q^{(0)}\Psi_0)$.

The following simple proof of (9) was kindly communicated to us by B. S. de Witt. The starting point is a well-known formula for the S-matrix:

$$\langle n|S|q\rangle=(\Psi_n^{(-)},\Psi_q^{(+)}). \tag{10}$$

Now rewrite $\Psi_q^{(+)}$ so that the corresponding incoming wave solution appears explicitly:

$$\begin{aligned}\Psi_q^{(+)}&=a_q^\dagger\Psi_0-\frac{1}{H-E_q-i\epsilon}V_q^{(0)}\Psi_0\\&=\Psi_q^{(-)}+\left\{\frac{1}{H-E_q+i\epsilon}-\frac{1}{H-E_q-i\epsilon}\right\}V_q^{(0)}\Psi_0\\&=\Psi_q^{(-)}-2\pi i\delta(H-E_q)V_q^{(0)}\Psi_0.\end{aligned} \tag{11}$$

Substitution of (11) into (10) clearly leads to the desired result (9).

The total cross section for mesons of momentum and isotopic spin q is then

$$\sigma_q=\frac{2\pi}{v_q}\sum_n \delta(E_n-E_q)|T_q(n)|^2, \tag{12}$$

where $v_q=q/\omega_q$ is the incident meson velocity.

It will be seen by (9) that for $E_q=E_n$, the quantity $T_q(n)$ is the conventional T matrix of scattering theory, but for $E_q\neq E_n$ this is no longer the case. $T_q(n)$ remains everywhere closely related to the energy-conserving T matrix at energy E_n, since it depends on the variable q only in a trivial way. In contrast, the conventional T matrix usually depends on its two indices in a non-factorable way and its values off and on the energy shell are not simply related. The trivial dependence of $T_q(n)$ on the variable q is an important simplification achieved by the present method of calculation.

In order to investigate the properties of $T_q(p)$, we note that since $\Psi_p^{(-)}$ can be written

$$\Psi_p^{(-)}=a_p^\dagger\Psi_0-\frac{1}{H-E_p+i\epsilon}V_p^{(0)}\Psi_0, \tag{13}$$

it follows that

$$T_q(p)=(a_p^\dagger\Psi_0,V_q^{(0)}\Psi_0)-\left(\frac{1}{H-E_p+i\epsilon}V_p^{(0)}\Psi_0,\ V_q^{(0)}\Psi_0\right) \tag{14}$$

$$=(\Psi_0,V_q^{(0)}a_p\Psi_0)-\left(\Psi_0,\ V_p^{(0)\dagger}\frac{1}{H-E_p-i\epsilon}V_q^{(0)}\Psi_0\right), \tag{15}$$

where we have made use of the fact that $V_q^{(0)}$ and a_p commute.

Let us normalize H (by subtracting the nucleon self-energy) so that

$$H\Psi_0=0, \tag{16}$$

$$H\Psi_p=\omega_p\Psi_p, \tag{17}$$

etc. The annihilation operator a_p may be eliminated from the first term of (15) by making use of its commutator with the Hamiltonian:

$$[a_p,H]=\omega_p a_p+V_p^{(0)\dagger}.$$

[5] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[6] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, New York, 1947), third edition, p. 198.
[7] G. C. Wick, Revs. Modern Phys. **27**, 339 (1955).

Thus,

$$[a_pH-Ha_p-\omega_pa_p-V_p^{(0)\dagger}]\Psi_0=0$$

or

$$(H+\omega_p)a_p\Psi_0=-V_p^{(0)\dagger}\Psi_0, \quad (18)$$

so that

$$a_p\Psi_0=-\frac{1}{H+\omega_p}V_p^{(0)\dagger}\Psi_0, \quad (19)$$

and Eq. (15) becomes

$$T_q(p)=-\left(\Psi_0,\, V_p^{(0)\dagger}\frac{1}{H-\omega_p-i\epsilon}V_q^{(0)}\Psi_0\right) \quad (20)$$
$$-\left(\Psi_0,\, V_q^{(0)}\frac{1}{H+\omega_p}V_p^{(0)\dagger}\Psi_0\right).$$

Finally, since $V_p^{(0)\dagger}=-V_p^{(0)}$, we have

$$T_q(p)=\left(\Psi_0,\left[V_p^{(0)}\frac{1}{H-\omega_p-i\epsilon}V_q^{(0)} + V_q^{(0)}\frac{1}{H+\omega_p}V_p^{(0)}\right]\Psi_0\right). \quad (21)$$

We may write Eq. (21) in a somewhat more familiar form by reintroducing the complete orthonormal set of states $\Psi_n^{(-)}$:

$$T_q(p)=\sum_n(\Psi_0,V_p^{(0)}\Psi_n^{(-)})(\Psi_n^{(-)},V_q^{(0)}\Psi_0)/(E_n-\omega_p-i\epsilon)$$
$$+\sum_n(\Psi_0,V_q^{(0)}\Psi_n^{(-)})(\Psi_n^{(-)},V_p^{(0)}\Psi_0)/(E_n+\omega_p). \quad (22)$$

Although Eq. (22) is strongly reminiscent of second-order perturbation theory, it is an exact result. The difference from perturbation theory lies, of course, in the use of initial, intermediate, and final states which are exact eigenstates of the total Hamiltonian H rather than of the free Hamiltonian H_0.

A more compact writing of (22) evidently is achieved by the use of the matrix element,

$$T_q(n)=(\Psi_n^{(-)},V_q^{(0)}\Psi_0), \quad (23)$$

already introduced in Eq. (9), and the complex conjugate matrix element,

$$T_q^*(n)=(V_q^{(0)}\Psi_0,\Psi_n^{(-)})$$
$$=(\Psi_0,V_q^{(0)\dagger}\Psi_n^{(-)})$$
$$=-(\Psi_0,V_q^{(0)}\Psi_n^{(-)}). \quad (24)$$

Even more notational simplification results if we consider (22) to be an operator equation with respect to the nucleon spin and isotopic spin variables. To illustrate how this works out, we write these variables explicitly in a typical matrix element. Take a final state β, initial state α, and 4 intermediate states Ψ_n^γ, $\gamma=1,\cdots 4$. Then

$$\sum_\gamma\langle\beta|V_p^{(0)}|n,\gamma\rangle\langle n,\gamma|V_q^{(0)}|\alpha\rangle$$
$$=-\sum_\gamma\langle\gamma|T_p(n)|\beta\rangle^*\langle\gamma|T_q(n)|\alpha\rangle$$
$$=-\sum_\gamma\langle\beta|T_p^\dagger(n)|\gamma\rangle\langle\gamma|T_q(n)|\alpha\rangle$$
$$=-\langle\beta|T_p^\dagger(n)T_q(n)\alpha\rangle.$$

Thus, if indices corresponding to initial and final nucleon states are as usual suppressed, Eq. (22) may be re-expressed as

$$T_q(p)=-\sum_n\left[\frac{T_p^\dagger(n)T_q(n)}{E_n-\omega_p-i\epsilon}+\frac{T_q^\dagger(n)T_p(n)}{E_n+\omega_p}\right], \quad (25)$$

a form of the equation which exhibits clearly its most important general properties.

III. GENERAL PROPERTIES OF THE EQUATION

A. Unitarity of the S-Matrix

We begin by examining the well-known requirement that the scattering matrix shall be unitary. From the relation (9), it follows that the unitarity condition $S^\dagger S=1$ is equivalent to the following statement:

$$T_p^\dagger(q)-T_q(p)=2\pi i\sum_n\delta(E_n-\omega_q)T_p^\dagger(n)T_q(n), \quad (26)$$

when $\omega_p=\omega_q=\omega$. If one takes the conjugate transpose of (25) to obtain $T_p^\dagger(q)$, the only change to occur on the right-hand side is the replacement of $-i\epsilon$ in the first denominator by $+i\epsilon$. Then, since

$$\frac{1}{E_n-\omega-i\epsilon}-\frac{1}{E_n-\omega+i\epsilon}=2\pi i\delta(E_n-\omega),$$

the unitarity condition (26) is evidently satisfied by any matrix function satisfying (25). It would seem, therefore, that one novel feature of (25), the quadratic rather than linear dependence on T of the right-hand side, is largely a reflection of the unitarity requirement. The second term of the right-hand side, however, has nothing to do with unitarity. Its presence has rather to do with a second and a quite different general property of the theory, which we discuss next.

B. Crossing Theorem

Gell-Mann and Goldberger[8] have pointed out an important symmetry possessed by theories of the Yukawa type. In terms of Feynman diagrams for meson-nucleon scattering, one may express this symmetry by saying that for any given diagram, another must exist which is obtained from the first by exchanging the incoming and outgoing meson lines. This exchange is not a simple time reversal because the nucleon line is not inverted.

In the present formulation of the theory this "crossing symmetry," as it is sometimes called, can be simply expressed in terms of a certain matrix function of a complex variable, z, which we define by

$$t_{qp}(z)=-\sum_n\left[\frac{T_p^\dagger(n)T_q(n)}{E_n-z}+\frac{T_q^\dagger(n)T_p(n)}{E_n+z}\right]. \quad (27)$$

[8] M. Gell-Mann and M. L. Goldberger, in *Proceedings of the Fourth Annual Rochester Conference on High Energy Nuclear Physics* (University of Rochester Press, Rochester, 1954).

Note that $t_{qp}(z)$ is a Hermitian matrix function of z in the sense that

$$t_{qp}(z)=t_{pq}^{\dagger}(z^*).$$

Note further that the dependence on both p and q is now trivial. Only the dependence on z is unknown.

Clearly, the limit of $t_{qp}(z)$ as z approaches the positive real axis from above ($z\to\omega_p+i\epsilon$) is $T_q(p)$; but for an expression of the crossing symmetry we must keep the functional dependence on p and z separate. The symmetry is expressed by the relation

$$t_{qp}(z)=t_{pq}(-z), \tag{28}$$

and it becomes apparent that the reason for the second term in (27) is precisely to satisfy (28).

C. Pole of the Function $t_{q,p}(z)$ at the Origin

It is helpful to get clearly in mind the nature and location of the singularities of the function $t_{qp}(z)$. This is possible from inspection of (27) because the energy eigenvalues of the Hamiltonian, E_n, are known even though the eigenfunctions are not.

The lowest eigenvalue (after the self-energy subtraction) is zero, belonging to the four zero-meson states. These states, then give rise to a simple pole at $z=0$, with residue $R_{qp}=[T_p^{\dagger}(0)T_q(0)-T_q^{\dagger}(0)T_p(0)]$. This statement is an analog of the Kroll-Ruderman theorem[9] and provides a method for measuring the coupling constant by scattering experiments. The point is that $T_q(0)$ contains the zero-meson wave functions only in a matrix element,

$$(\Psi_0^{(\alpha)},\sigma_q\tau_q\Psi_0^{(\beta)}),$$

which for reasons of invariance must be a q-, α-, and β-independent multiple, say Z, of the matrix element

$$(u_\alpha,\sigma_q\tau_q u_\beta),$$

where u_α and u_β are normalized Pauli spinors ("bare-nucleon" wave functions). If we now *define* $f_{(r)}=Zf_{(r)}^{(0)}$ and call $f_{(r)}$ the renormalized (rationalized) coupling constant, it follows that

$$T_q(0)=V_q,$$

where V_q is obtained from $V_q^{(0)}$ by replacing $f_{(r)}^{(0)}$ by $f_{(r)}$. The function $t_{qp}(z)$ in the neighborhood of $z=0$ is therefore completely determined by the renormalized coupling constant and vice versa. Since our theory is a finite one, we are of course free to define the coupling constant in the most convenient way. The present definition coincides with those previously given by Chew[10] and Lee[11]; most important, it is also appropriate (in the sense of the Kroll-Ruderman theorem) to the calculation of threshold photomeson production.

[9] N. M. Kroll and M. A. Ruderman, Phys. Rev. **93**, 233 (1954).
[10] G. F. Chew, Phys. Rev. **94**, 1748 (1954).
[11] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

D. Behavior at Infinity

It is clear from inspection of (27) that as $z\to\infty$, $t_{qp}(z)$ behaves like $1/z$. It is an interesting but not very useful fact that the coefficient of $1/z$ at infinity is a multiple of the residue at the origin which is independent of q and p and of the nucleon variables. The proof of this relation is given by reference back to Eq. (22). As z (or ω_p) approaches infinity, the dependence on the energy eigenvalue E_n in the denominator is removed and closure may be applied to evaluate the sum over states, giving

$$\lim_{z\to\infty} z t_{qp}(z)=-(\Psi_0,[V_p^{(0)},V_q^{(0)}]\Psi_0),$$

which is to be compared to the residue at the origin,

$$R_{qp}=-[V_p,V_q].$$

By using the standard commutation properties of the σ's and τ's, these two coefficients are easily shown to differ only by a factor which is independent of p and q as well as of the nucleon spin and isotopic spin, but the factor is not unity. The point is that here one is dealing essentially with matrix elements of σ_q or τ_q rather than the product $\sigma_q\tau_q$ as in the preceding section.

E. Location of Branch Points and Cuts

From the form of (27) it is clear that, in addition to the pole at the origin, all the other singularities of (27) also lie along the real axis. The next lowest eigenvalue of the energy corresponds to a single meson at rest; then there is a continuous distribution in energy of one-meson states up to $+\infty$. At an energy equal to two meson rest masses, a continuous distribution of two-meson states begins, and so on, for all higher numbers of mesons. It follows that, because of the one-meson states, the function $t_{qp}(z)$ has a branch point at $z=1$ and a cut along the real axis for $z>1$. The two-meson states produce a branch point at $z=2$ and a cut for $z>2$, etc. The "crossed" terms in (27) obviously produce similar singularities in the left half-plane.

It will now be shown that the conditions listed under A, B, C, D, E, above, together with the factorability of the T matrix which is implied by Eq. (9), are completely equivalent to Eq. (27). Consider the condition that except for a simple pole of residue R_{pq} at the origin all the singularities of $t_{qp}(z)$ are confined to branch points with cuts running along the real axis for $z>+1$ and $z<-1$. If $t_{qp}(z)$ goes like $1/z$ at infinity, it may be expanded in the form

$$t_{qp}(z)=\frac{R_{qp}}{z}+\int_1^\infty dx'\left[\frac{F_{qp}(x')}{x'-z}+\frac{G_{qp}(x')}{x'+z}\right], \tag{29}$$

where $F_{qp}(x)$ and $G_{qp}(x)$ are weighting functions defined for $x\geqslant 1$. Evidently the functions F_{qp} and G_{qp} are given by the jump in the function t_{qp} going across the real axis in the right and left half-planes, respectively. We have, in fact,

$$2\pi i F_{qp}(x) = \lim_{z\to x+i\epsilon} t_{qp}(z) - \lim_{z\to x-i\epsilon} t_{qp}(z) \quad (30)$$

for $x \geqslant 1$, and

$$2\pi i G_{qp}(x) = \lim_{z\to -x-i\epsilon} t_{qp}(z) - \lim_{z\to -x+i\epsilon} t_{qp}(z), \quad (31)$$

where x is still $\geqslant 1$. Now, if we define $T_q(p)$ as the limit of $t_{qp}(z)$ as $z\to\omega_p+i\epsilon$ and further impose the reality condition, $t_{qp}(z)=t_{pq}^\dagger(z^*)$, then (30) leads to

$$F_{qp}(\omega_p) = \frac{1}{2\pi i}[T_q(p) - T_p^\dagger(q)]_{\omega_p=\omega_q}. \quad (30')$$

Imposing the crossing relation (28) allows (31) to be written

$$G_{qp}(\omega_p) = \frac{1}{2\pi i}[T_p(q) - T_q^\dagger(p)]_{\omega_p=\omega_q}. \quad (31')$$

Finally the unitarity condition (26) transforms Eq. (29) via (30′) and (31′) into our original Eq. (27) for the special case $\omega_p=\omega_q$. The latter restriction is of no consequence, however, since we have noted before that one may move off the energy shell at will.

To summarize, if it is possible to find a Hermitian matrix function, $t_{qp}(z)$, which has a simple pole at the origin of residue R_{qp}, goes to zero like $1/z$ at ∞, has otherwise only branch points and cuts along the real axis for $z>1$ and $z<1$, and which satisfies unitarity as well as the crossing relation, then one has a solution of Eq. (27). Unfortunately, the formulation of the unitarity condition involves multimeson (two-meson and higher) states and cannot be written down on the basis of *a priori* arguments in terms of $t_{qp}(z)$. However, if multimeson states are neglected, as in the next section, then the above conditions form a practical basis for solving the scattering problem.

IV. ONE-MESON APPROXIMATION

If we assume that the inelastic cross sections are small compared to the elastic ones (for all values of the energy) then, as a first approximation, the contributions of multimeson states to the unitarity condition (26) may be neglected. In this case it is convenient to re-express the conditions on the matrix t_{qp} in terms of phase shifts.

We set

$$t_{qp}(z) = -v(q)v(p)\frac{4\pi}{(4\omega_q\omega_p)^{\frac{1}{2}}}\sum_{\alpha=1}^{4} P_\alpha(p,q)h_\alpha(z), \quad (32)$$

where

$$\begin{aligned} P_{11} &= \tfrac{1}{3}\tau_p\tau_q(\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{q}),\\ P_{13} &= \tfrac{1}{3}\tau_p\tau_q[3\mathbf{p}\cdot\mathbf{q} - (\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{q})],\\ P_{31} &= (\delta_{pq} - \tfrac{1}{3}\tau_p\tau_q)(\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{q}),\\ P_{33} &= (\delta_{pq} - \tfrac{1}{3}\tau_p\tau_q)[3\mathbf{p}\cdot\mathbf{q} - (\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{q})]. \end{aligned} \quad (33)$$

The P_α's are projection operators for the four eigenstates of total angular momentum and isotopic spin. In the subscript $\alpha=(2I,2J)$, I is the total isotopic spin and J the total angular momentum.

The new functions $h_\alpha(z)$ are simply related to phase shifts. Application of the unitarity condition shows that

$$\lim_{z\to\omega_p+i\epsilon} h_\alpha(z) = e^{i\delta_\alpha(p)}\sin[\delta_\alpha(p)]/p^3v^2(p), \quad (34)$$

where the $\delta_\alpha(p)$ are real and identical with the conventionally defined phase shifts for $\omega_p \leqslant 2$. It is well known that the scattering phase shifts of the (1,3) and (3,1) states are equal in this theory, so we henceforth confine our attention to the three functions $h_1=h_{11}$, $h_2=h_{13}=h_{31}$, and $h_3=h_{33}$.

The conditions on the function $t_{qp}(z)$ given in the preceding section may be translated into conditions on the functions $h_\alpha(z)$. In addition to unitarity, which is expressed by (34), we have the crossing relation

$$h_\alpha(z) = \sum_{\beta=1}^{3} A_{\alpha\beta}h_\beta(-z), \quad (35)$$

where

$$A = \begin{bmatrix} 1/9 & -8/9 & 16/9 \\ -2/9 & 7/9 & 4/9 \\ 4/9 & 4/9 & 1/9 \end{bmatrix}. \quad (36)$$

The condition that $t_{qp}(z)$ be a Hermitian matrix function of z implies that $h_\alpha(z)$ is a real function of z in the sense that $h_\alpha(z^*)=h_\alpha^*(z)$. The boundary condition at infinity is that h_α behave like $1/z$, while at the origin h_α should have a simple pole of residue λ_α, where

$$\lambda_\alpha = \tfrac{2}{3}f^2\begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix}. \quad (37)$$

Here f^2 is the nonrationalized coupling constant. We may at this point mention two useful properties of the crossing matrix $A_{\alpha\beta}$:

(1) $$\textstyle\sum_\beta A_{\alpha\beta}A_{\beta\gamma} = \delta_{\alpha\gamma}, \quad (38)$$

(2) $$\textstyle\sum_\beta A_{\alpha\beta}\lambda_\beta = -\lambda_\alpha. \quad (39)$$

Finally, we require that all other singularities of h_α be confined to two branch points at $z=\pm 1$, with cuts along the real axis to $\pm\infty$. This set of conditions is equivalent to the following equations for $h_\alpha(\omega)$:

$$h_\alpha(\omega) = \frac{\lambda_\alpha}{\omega} + \frac{1}{\pi}\int d\omega_p p^3 v^2(p)\left\{\frac{|h_\alpha(\omega_p)|^2}{\omega_p-\omega-i\epsilon} + \sum_\beta A_{\alpha\beta}\frac{|h_\beta(\omega_p)|^2}{\omega_p+\omega}\right\}. \quad (40)$$

Let us eliminate the pole at the origin by introducing a new (real) function

$$g_\alpha(z) = \frac{\lambda_\alpha}{z}[h_\alpha(z)]^{-1}. \quad (41)$$

The boundary condition at $z=0$ is now that $g_\alpha(0)=1$. Furthermore, g_α behaves like a constant at infinity. The unitarity condition (34) implies that

$$\lim_{z\to\omega_p+i\epsilon} g_\alpha(z)-\lim_{z\to\omega_p-i\epsilon} g_\alpha(z)=-2i\frac{\lambda_\alpha p^3}{\omega_p}v^2(p), \quad (42)$$

for $\omega_p\geqslant 1$. The crossing relation becomes

$$\sum_{\beta=1}^{3} B_{\alpha\beta}\frac{1}{g_\beta(z)}=\frac{1}{g_\alpha(-z)}, \quad (43)$$

where

$$B=\begin{bmatrix} -1/9 & 2/9 & 8/9 \\ 8/9 & -7/9 & 8/9 \\ 8/9 & 2/9 & -1/9 \end{bmatrix}. \quad (44)$$

Finally, consider the location of the singularities of $g_\alpha(z)$. Clearly, just as for $h_\alpha(z)$, there are branch points at $z=\pm 1$ with cuts along the real axis to $\pm\infty$. If $h_\alpha(z)$ has no zeros, these will be the only singularities of $g_\alpha(z)$.

With f^2 sufficiently small, there are certainly no zeros in $h_\alpha(z)$. For this case, the boundary conditions on $g_\alpha(z)$, together with the nature of its singularities, imply that this function may be written

$$g_\alpha(z)=1-\frac{z}{\pi}\int_1^\infty dx'\left[\frac{F_\alpha(x')}{x'-z}+\frac{G_\alpha(x')}{x'+z}\right], \quad (45)$$

where $F_\alpha(x)$ and $G_\alpha(x)$ are real weighting functions defined for $x\geqslant 1$. The functions $F_\alpha(x)$ and $G_\alpha(x)$ are power series in f^2, whose coefficients are continuous differentiable functions of the variable x. Since $F_\alpha(x)$ gives the jump in $g_\alpha(z)$ in going across the real axis for $z\geqslant 1$, the first of the two weighting functions is completely determined by the unitarity condition (42):

$$F_\alpha(\omega_p)=\lambda_\alpha p^3/\omega_p{}^2 v^2(p). \quad (46)$$

The final condition to be satisfied is the crossing relation (43), which is just sufficient to determine the second weighting function $G_\alpha(x)$. Thus, for values of f^2 which are such that the power series for $\int G_\alpha(x)dx/(x+z)$ converges, we have

$$h_\alpha(z)=\frac{\lambda_\alpha/z}{1-\dfrac{z}{\pi}\lambda_\alpha\displaystyle\int\frac{d\omega_p}{\omega_p{}^2}\frac{p^3v^2(p)}{\omega_p-z}-\dfrac{z}{\pi}\displaystyle\int\frac{d\omega_p}{\omega_p{}^2}\frac{p^3v^2(p)}{\omega_p+z}H_\alpha(\omega_p)}, \quad (47)$$

where

$$G_\alpha(\omega_p)=p^3\frac{v^2(p)}{\omega_p{}^2}H_\alpha(\omega_p).$$

For larger values of f^2, the function that replaces $\int H_\alpha(p)d\omega_p p^3v^2/(\omega_p+z)\omega_p{}^2$ in (47) must be determined by analytic continuation. We have been unable to do this for the symmetric pseudoscalar theory.[12]

It will be seen that in the region of convergence of $\int G_\alpha(x)dx/(x+z)$ there can be no zeros of $h_\alpha(z)$, so that the sign of the phase shifts must be the same as that of their Born approximations.

It has been pointed out by Castillejo, Dalitz, and Dyson[13] that (47) is probably not the only solution of (40). The extra solutions found by them, however, are not analytic continuations of the perturbation theory power series, as is our expression (47). If one assumes that the solution of the original field theoretic problem is unique and is the analytic continuation of the power series, then it is clear that our solution is the only one of physical interest.

G. Salzman has solved Eq. (40) by a numerical method which he will discuss in detail in a forthcoming paper. Here we concentrate on features of the solution which may be deduced from general considerations.

It should be noted that the real part of g_α is essentially the cotangent of the phase shift. More precisely,

$$\mathrm{Re}g_\alpha(\omega_p)=\frac{\lambda_\alpha p^3v^2(p)}{\omega_p}\cot\delta_\alpha(\omega_p) \quad (48)$$

[the imaginary part of $g_\alpha(\omega_p)$ is fixed by (42)]. It seems natural in relating theory to experiment to discuss as the primary experimental quantity the right-hand side of (48). One is then led, by analogy to the corresponding situation in nucleon-nucleon scattering theory, to what might be called an "effective-range" treatment of pion-nucleon scattering.

V. EFFECTIVE-RANGE APPROXIMATION

We consider

$$\mathrm{Re}g_\alpha(\omega)=1-\omega\left\{\frac{\lambda_\alpha}{\pi}\mathrm{P}\int\frac{d\omega_p p^3v^2(p)}{\omega_p{}^2(\omega_p-\omega)}+\frac{1}{\pi}\int\frac{d\omega_p}{\omega_p{}^2}p^3v^2(p)\frac{H_\alpha(\omega_p)}{\omega_p+\omega}\right\}. \quad (49)$$

The effective range approximation is based on the weak dependence of $\mathrm{Re}g_\alpha(\omega)$ on the ω occurring in the denominators of the integrands in (49). Neglect of this dependence seems reasonable *a priori* for values of ω which are small compared to $\omega_{\max}$, the maximum energy effectively allowed by the cutoff factor $v(p)$, provided only that $H_\alpha(\omega_p)$ maintains the type of smoothness indicated by the first few terms of the power series. One would suppose the error incurred to be of the order $\omega/\omega_{\max}$, which may not be excessive for pion kinetic

[12] It has been kindly pointed out to us by Dr. T. D. Lee that an exact solution of the one-meson approximation can be obtained for the charged scalar theory, in which case one finds essentailly $H_\alpha=-\lambda_\alpha$, and the problem of analytic continuation becomes trivial.

[13] Castillejo, Dalitz, and Dyson, Phys. Rev. **101**, 453 (1956).

energies less than 200 Mev, since $\omega_{\max}$ is in the neighborhood of 1 Bev. Explicit calculation of the first integral in (49) verifies this conjecture.

Referring back to (48), we see that the combination

$$\lambda_\alpha p^3 \cot\delta_\alpha/\omega_p \tag{50}$$

can be written in the form

$$1-\omega r_\alpha(\omega), \tag{51}$$

where $r_\alpha(\omega)$ is almost a constant for small ω. The effective-range approximation corresponds to a complete neglect of the energy dependence of $r_\alpha(\omega)$. It can be tested experimentally by plotting $(p^3/\omega)\cot\delta_\alpha(\omega)$ against ω. According to (50) and (51), one should find a straight line with intercept at zero energy equal to λ_α^{-1}. Lindenbaum and Yuan[14] have made such a plot for δ_{33} (where, for reasons to be discussed in Sec. VII, ω_p has been replaced by $\omega_p^*=\omega_p+p^2/2M$, M being the nucleon mass). The expected linear dependence has been found and the intercept leads to a value for the renormalized (unrationalized) coupling constant of $f^2=0.08$.

The effective-range approximation that $r_\alpha(\omega)\approx r_\alpha(0)$ for $\omega\ll\omega_{\max}$ may seem superficially equivalent to the statement that an expansion of $r_\alpha(\omega)$ in powers of ω has a radius of convergence $\gtrsim\omega_{\max}$. Such is not the case, of course, because the branch points in $g_\alpha(z)$ at $z=\pm1$ give a radius of convergence equal to 1. Serber and Lee[15] have pointed out that the part of $g_\alpha(z)$ which is not analytic at $z=+1$ can be isolated and evaluated, and one may extend their approach to separate also the part which is most singular at $z=-1$. The reason that the effective-range approximation works is that the remaining part of $g_\alpha(z)$, which has no singularity in the low-energy region, is larger than the nonanalytic parts by a factor of order $\omega_{\max}$.

If the experimental data were sufficiently accurate to warrant the effort, one could improve the coupling constant determination by correcting for the small terms which cannot be extrapolated from the physical region ($\omega>1$) to the point $\omega=0$. The recipe turns out to be the following: Introduce a quantity

$$\Gamma_\alpha(\omega_p)=\frac{p^3}{\omega_p}\cot\delta_\alpha(\omega_p)-\frac{1}{\omega_p}+\frac{3}{2}\omega_p+\begin{pmatrix}-\frac{1}{2}\\-2\\+1\end{pmatrix}\omega_p \times\left[\frac{p^2}{\omega_p^2}\left(\frac{p}{\pi}\log\frac{1}{\omega_p-p}+\frac{1}{2}-\frac{\omega_p}{\pi}\right)+\frac{1}{4}-\frac{\omega_p}{3\pi}\right]. \tag{52}$$

One can show[16] that to an accuracy of order $1/\omega_{\max}^2$ the new quantity $\Gamma_\alpha(\omega_p)$ will be of the following form:

$$\Gamma_\alpha(\omega_p)=\frac{1}{\lambda_\alpha}[1-r_\alpha\omega_p+P_\alpha\omega_p^2], \tag{53}$$

where the coefficient P_α should be smaller than r_α by a factor of order $1/\omega_{\max}$. That is, $\Gamma_\alpha(\omega_p)$ is an almost linear function in the low-energy region and extrapolates to the value λ_α^{-1} at $\omega_p=0$.

Note that there is no point in making the above refinement unless the extrapolation is at least quadratic, because the term in (53) proportional to P_α is presumably of the same order as the modifications made by formula (52) to the original effective-range approximation. Existing experimental data probably do not warrant the refined extrapolation procedure.

Plots for δ_{11}, δ_{13}, and δ_{31} should of course lead to the same value of f^2, but unfortunately the experimental information on these phase shifts is only that they are small compared to δ_{33}. Our theory definitely predicts that they should all be negative for not too large values of f^2, and the first few terms of the power series for H_α suggests strongly that the phase shifts should be small.

The crossing relation (43) makes a prediction about the coefficients of the term proportional to ω in an expansion of g_α about $\omega=0$. When translated into a statement about the effective range $r_\alpha(0)$, the prediction is that

$$r_\alpha=r_3\begin{pmatrix}-1+\frac{1}{4}x\\-x\\1\end{pmatrix}, \tag{54}$$

where x is an unknown parameter.

In the following section, it will be proved that for the (3,3) state the coefficient of the linear term is negative definite, that is to say, that r_3 is positive definite. One might say, then, that the theory "predicts" a resonance in the 33 state provided the coupling is sufficiently strong. Estimates based on the power series for H_α suggest that resonance will indeed occur at the right energy with the known value of f^2 if the cutoff energy $\omega_{\max}$ is in the neighborhood of 6.

It actually can be shown from the form of the complete scattering equation, that is, the equation before the multi-meson terms are dropped, that the neglected terms will not interfere with the effective-range approach. If anything the energy dependence of integrals over the multi-meson states will be weaker than that of the one-meson terms considered in this section. The value of the effective ranges for a given cutoff is of course altered, but the relation (54), which depends only on the crossing symmetry, is preserved.

VI. TOTAL CROSS SECTION SUM RULES

Returning to the general problem, we now wish to point out some important relations involving total cross sections which may be derived in the static model

[14] S. J. Lindenbaum and L. C. L. Yuan, Phys. Rev. **100**, 306 (1955).

[15] R. Serber and T. D. Lee (private communication). See also Friedman, Lee, and Christian, Phys. Rev. **100**, 1494 (1955).

[16] A proof of this statement will be given in a forthcoming review article by G. F. Chew, *Encyclopedia of Physics* [Springer-Verlag, Berlin (to be published)], Vol. 43.

without any approximation. The basis of these relations is formula (12) for the total cross section, or rather a generalization of (12) which we now write down.

The notation employed so far can accommodate arbitrary initial and final nucleon spin—isotopic-spin states. The meson states, however, must be such that the linear momentum and isotopic variable are well defined. If we wish to consider a more general set of one-meson states, say $\phi_a(q)$, then the quantity which describes the scattering from state a to state n is

$$T_a(n)=\sum_q T_q(n)\phi_a(q), \tag{55}$$

while that to another state b, of the same set, is

$$\langle b|T|a\rangle=\sum_{q,p}\phi_b^*(p)T_q(p)\phi_a(q), \tag{56}$$

if the standard normalization

$$\sum_q \phi_b^*(q)\phi_a(q)=\delta_{ba} \tag{57}$$

is employed. The total cross section for the state a is

$$\sigma_a=\frac{2\pi}{v_a}\sum_n \delta(E_n-\omega_a)T_a^\dagger(n)T_a(n), \tag{58}$$

where an expectation with respect to the initial nucleon state is understood. One may also write down the equation for $\langle b|T|a\rangle$ corresponding to (25):

$$\langle b|T|a\rangle=-\sum_n\left[\frac{T_b^\dagger(n)T_a(n)}{E_n-\omega_a-i\epsilon}+\frac{T_{a^*}^\dagger(n)T_{b^*}(n)}{E_n+\omega_a}\right], \tag{59}$$

where a^* and b^* refer to states which are the complex conjugates of ϕ_a and ϕ_b, respectively.

For the special case $b=a$ and the same initial and final nucleon states, it is clear that (58) allows (59) to be rewritten in terms of total cross sections. One finds easily that

$$\langle a|T|a\rangle=\langle a|T^0|a\rangle-\frac{1}{2\pi}\frac{q_a^2v^2(q_a)}{\omega_a}\int_1^\infty\frac{dE}{q_Ev^2(q_E)}\times\left[\frac{\sigma_a(E)}{E-\omega_a-i\epsilon}+\frac{\sigma_{a^*}(E)}{E+\omega_a}\right], \tag{60}$$

where q_E is the momentum of a meson of energy E. The operator T^0 represents the zero-meson part of the sum over states in (59). It is also the zero-energy limit of T. The notation $\sigma_a(E)$ means the total cross section for an incident meson whose energy is E but whose other variables (isotopic spin and angular momentum) are those of the state a.

We concentrate here on two special cases of (60) although other applications may also be interesting. The first case is when ϕ_a and ϕ_a^* represent the same state of the incident meson. A neutral meson with well-defined linear momentum is the simplest example of this situation. [Note that for a charged meson, $\phi_a\neq\phi_a^*$.] With no distinction between a and a^*, (60) reduces to

$$\langle a|T|a\rangle=\langle a|T^0|a\rangle-\frac{q_a^2v^2(q_a)}{\pi\omega_a}\int_0^\infty\frac{dq_E}{v^2(q_E)}\frac{\sigma_a(E)}{q_E^2-q_a^2-i\epsilon}, \tag{61}$$

a result which is almost identical with the well-known dispersion relation.[17] Apart from the cut-off factor, it differs from the usual dispersion relation in that ϕ_a may be an arbitrary real (in this representation) state. If all important contributions to the integral come below $\omega_{\max}$, the cutoff factors are unimportant. It is interesting that a result so close to the dispersion relation is obtained, because the latter is a consequence of causality and with an extended source the static model is of course not causal.

A second special initial state of particular interest is one which is an eigenstate of total angular momentum and total isotopic spin. This can occur only in the (3,3) case for the kind of states we have considered, where for example ϕ_a may be chosen as a positive meson with its orbital angular momentum up. The nucleon must then be a proton with its spin up. ϕ_a^* then corresponds to a negative meson with orbital angular momentum down. For this special case, (60) shows that the quantity $h_3(\omega)$ defined by Eqs. (32) and (33) is given by

$$h_3(\omega)=\frac{\lambda_3}{\omega}+\frac{1}{12\pi^2}\int_1^\infty\frac{dE}{q_Ev^2(q_E)}\left[\frac{\sigma_+(E)}{E-\omega-i\epsilon}+\frac{\sigma_-(E)}{E+\omega}\right], \tag{62}$$

where σ_+ is the total cross section for the state ϕ_a and σ_- that for ϕ_a^*. This result shows among other things that the quantity r_3, defined in connection with the effective-range discussion, is certainly positive. Comparison with (51) and (41) leads to

$$r_3f^2=\frac{1}{(4\pi)^2}\int_1^\infty\frac{dE}{Eq_Ev^2(q_E)}[\sigma_+(E)+\sigma_-(E)], \tag{63}$$

a relation which may be used to determine r_3 experimentally if the appropriate parts of the total cross section can be isolated. It should be noted that according to (63) the contribution to r_3 from *each* type of state ($n=1$, 2, etc.) is individually positive. Thus the one-meson approximation probably requires an unnecessarily high cut-off energy to produce the required value of r_3.

We now derive a different type of sum rule to obtain a relation between the renormalized and unrenormalized coupling constants. Consider the expectation of the operator $\sigma_q\tau_q\sigma_q\tau_q=1$ taken with respect to a single-nucleon state. If the latter is normalized, the expected value is of course unity. Starting with this apparently trivial fact and using the completeness of the set of

[17] R. Karplus and M. A. Ruderman, Phys. Rev. **98**, 771 (1955); Goldberger, Miyazawa, and Oehme, Phys. Rev. **99**, 979 (1955).

states $\Psi_n^{(-)}$, we obtain the following:

$$1=(\Psi_0,\sigma_q\tau_q\cdot\sigma_q\tau_q\Psi_0) \quad (64)$$

$$=\sum_n(\Psi_0,\sigma_q\tau_q\Psi_n^{(-)})(\Psi_n^{(-)},\sigma_q\tau_q\Psi_0) \quad (65)$$

$$=\sum_n\left[f_r^{(0)}\frac{iqv(q)}{(2\omega_q)^{\frac{1}{2}}}\right]^{-2}(\Psi_0,V_q^{(0)}\Psi_n^{(-)})(\Psi_n^{(-)},V_q^{(0)}\Psi_0) \quad (66)$$

$$=[f_r^{(0)}]^{-2}\sum_n\frac{2\omega_q}{q^2v^2(q)}T_q^\dagger(n)T_q(n), \quad (67)$$

where an expectation with respect to the nucleon state is understood. Next multiply (67) by $(f_r^0)^2$ and separate the $n=0$ terms from the sum. The result is

$$(f_r^{(0)})^2=f_r^2+\sum_{n>0}\frac{2\omega_q}{q^2v^2(q)}T_q^\dagger(n)T_q(n); \quad (68)$$

finally, we employ (58) to introduce the total cross section and find

$$(f_r^{(0)})^2=f_r^2+\frac{1}{\pi}\int\frac{dE\sigma_q(E)}{q_Ev^2(q_E)}. \quad (69)$$

The result (69) demonstrates that $(f_r^0)^2>f_r^2$, a fact already shown by Lee.[11] It also gives in principle a method for measuring experimentally the unrenormalized coupling constant. Note that the cross section involved could be that for a neutral pion or it could be the average for positive and negative pions. The target may be either a neutron or a proton.

VII. CONCLUSIONS AND POSSIBLE EXTENSION OF RESULTS

We summarize here the results of the preceding sections which we feel are the most significant:

(a) The function $p^3/\omega_p \cot\delta_\alpha$ should be approximately linear at low energies and should extrapolate to λ_α^{-1} at $\omega=0$. Here $\lambda_{11}=-(8/3)f^2$, $\lambda_{13}=\lambda_{31}=-(2/3)f^2$, $\lambda_{33}=(4/3)f^2$. This implies among other things that the p-wave phase shifts maintain the sign of the Born approximation, that is, δ_{33} is positive and all the others are negative.

(b) The effective ranges for the various p-states, defined by (50) and (51), are not completely independent but must obey the relation (54).

(c) The effective range in the (3,3) state is certainly positive, so that a resonance will occur in this state if the coupling is sufficiently strong.

These results have been derived on the basis of a theory which completely neglects relativistic effects. The question naturally arises as to whether they will be maintained when nucleon recoil and nucleon pair formation, as well as the effects of other particles, are taken into account. It is not possible to give an unqualified answer to this question, but recent and independent investigations of the relativistic theory, based on such general requirements as Lorenz invariance, give some indications as to the validity of the statements (a)–(c).

In the first place, it has been rigorously demonstrated for the relativistic Yukawa theory that the p-wave scattering amplitude approaches the renormalized Born approximation in the limit $\omega_q^*\to0$, where

$$\omega_q^*=\omega_q+(q^2/2M)+O(1/M^2), \quad (70)$$

if M is the nucleon mass. That the functional form of the individual phase shifts is as given in statement (a) has not really been proved but seems extremely likely to us in view of recent work by Thirring[18] and Oehme.[19] These authors have shown or have promised to show that the first derivatives of the scattering amplitude with respect to $\sin\theta$ and $\cos\theta$, evaluated at $\theta=0$, have an analytic form which corresponds to condition (E) of Sec. III above. Conditions (A) and (B), which correspond to unitarity and crossing, respectively, are certainly general. Condition (C) is the zero-energy limit theorem which, as stated above, has been proved to be general. At low energies it seems legitimate to neglect orbital angular momenta higher than one, in which case the Thirring-Oehme amplitude derivatives can be identified to order $1/M$ with the p-wave amplitude alone, and one seems almost to have reproduced the equations of the cut-off theory from a general point of view. The difficulty of course is that at high energies orbital angular momenta larger than one certainly contribute, and high energies are important under the integrals which occur in the scattering equations.

One cannot, then, make clear-cut statements about the behavior of the p-wave amplitude in the entire complex plane. It is hard to imagine, however, a form for the amplitude which differs at low energies from that in statement (a) above and still manages to be unitary, to satisfy the Thirring-Oehme equations and to approach the correct zero-energy limit.

Statement (b), which connects the various p-wave effective ranges, is not rigorously true in the relativistic case because s-wave and p-wave amplitudes occur together in the term linear in the energy when the appropriate covariant expansion about zero energy is made.[20] However, the s-wave amplitudes experimentally are sufficiently small so that Eq. (54) remains a good approximation.

We do not have confidence that statement (c) above, although known experimentally to be correct, is necessarily a consequence of the relativistic Yukawa theory because it depends sensitively on the behavior of the scattering at high energies.

In conclusion we should like to emphasize a distinction between two classes of meson phenomena: those that depend only on low-energy matrix elements (in the sense of this paper) and those whose calculation

[18] W. Thirring, private communication.
[19] R. Oehme, Phys. Rev. 100, 1503 (1955).
[20] F. Low (to be published).

requires a knowledge of high-energy matrix elements. An example of the former is the theorem that the zero energy limit of the p-wave effective range extrapolation measures the same coupling constant as the zero-energy limit of the photomeson production amplitude (according to the Kroll-Ruderman theorem). On the other hand the problem of theoretically evaluating the effective range falls in the latter class. Formula (49), for example, shows clearly that the value of r_3 depends on high-energy phenomena.

We have made no serious attempt in this paper to calculate the effective ranges. Presumably the (3,3) effective range could be matched by an appropriate choice of the cut-off energy, whatever method of approximation were used, and the dominant role played by the (3,3) state at low energies guarantees the success of any approach which produces the correct value for r_3. The question naturally arises as to whether one should expect to be able to calculate r_3 and other quantities which involve integrals over high virtual energies with the conventional relativistic form of the Yukawa theory, which has no adjustable cut-off parameter. We think the answer is no, because this theory does not take account of the existence of hyperons and K-particles which interact strongly with the pion-nucleon system. Both the cutoff and the local forms of the Yukawa theory are incorrect (or at least incomplete) in the Bev energy region.

Our zero-energy results hold for both theories and we believe they will probably hold in future theories, although this last statement is of course little more than a guess. We also believe that the linear extrapolation of the cotangent of the phase shifts will be maintained because this is essentially a statement of ignorance: the more important are high-energy phenomena, the more nearly constant is the effective-range integral.

We hope to show in the paper on photomeson production, which follows, that many aspects of this latter process fall in the first (low virtual energy) class of phenomena. The same is true for Compton scattering by protons and probably for the nuclear force problem. Phenomena which belong to the second class presumably include s-wave scattering, π^0 decay, the charge and current density of nucleons, as well as the fundamental questions concerning the nature and interactions of curious particles.

PHYSICAL REVIEW VOLUME 101, NUMBER 5 MARCH 1, 1956

Theory of Photomeson Production at Low Energies

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The problem of photomeson production is re-examined using the static model of the pion-nucleon interaction. It is shown that an important part of the low-energy matrix element can be exactly expressed as a function of the scattering phase shifts and the static nucleon magnetic moments. It is argued that the remainder is quite accurately given by the usual Born approximation. Corrections to this result, within the framework of the "one-meson" approximation, are considered.

I. INTRODUCTION

THE purpose of this paper is to extend the theoretical approach of the preceding paper[1] on meson-nucleon scattering to the problem of photomeson production. Extensive use of the notation and results of the scattering paper is necessary, and we shall assume the reader to be familiar with these. The most important conclusion of the present paper is that once the scattering phase shifts are known at a given energy, either experimentally or theoretically, the corresponding photomeson production cross sections can almost unambiguously be predicted.

As in I, the bulk of our discussion will be in terms of the static model, but it may be argued that the important results are probably more general. We begin in Sec. II of this paper by splitting the photomeson production amplitude into three parts, one of which may be written down in an explicit and exact form. Equations satisfied by the other two parts are then derived. Section III deals with these two equations in the one-meson approximation. In Sec. IV, a simple and quite accurate approximation to the total amplitude is proposed, and finally Sec. V compares the simple theoretical amplitude with experiment.

II. PHOTOMESON EQUATIONS

A derivation of the integral equations which we shall apply to photomeson production has already been published.[2] We give here a new derivation which is analogous to that presented in I for scattering. All notations will be the same as in I.

The matrix element for absorption of a photon of type k by a single nucleon, with emission of a meson

[1] G. F. Chew and F. E. Low, preceding paper [Phys. Rev. **101**, 1570 (1956)], hereinafter referred to as I.

[2] F. E. Low, Phys. Rev. **97**, 1392 (1955).