Anomalous Magnetic Moment of the Nucleon and the Pion-Nucleon Scattering*

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The anomalous magnetic moment of the nucleon is calculated using the method developed by Chew and Low. The static model is used, i.e., recoil and nucleon pair creation are omitted. The anomalous moments are expressed exactly in terms of the renormalized coupling constant and pion-nucleon scattering cross sections. The result shows a satisfactory agreement for the magnetic moment arising from the pion current. The contribution from the nucleon current, however, turns out to be too large. It is concluded that the Sachs' mirror condition cannot be satisfied with the simple static model.

1. INTRODUCTION

THE theory of Chew and Low[1] for the P-wave pion-nucleon interaction has proved successful for the scattering and photoproduction of pions.[2] The coupling constant was determined with some accuracy and the approximate value of the cut-off momentum was also obtained. It is the purpose of this paper to determine whether these constants are also capable of explaining the anomalous magnetic moments of the nucleons. It is found that the Chew-Low theory applied to this problem allows one to express the nucleon moments in terms of the renormalized coupling constant and the total cross section for pion-nucleon scattering.

In this paper, arguments are given in terms of the static model with a fixed nucleon. This presupposes that the magnetic moment anomaly is a low-energy phenomenon, meaning that the major contribution comes from virtual pions of small momenta. This may be the case for the part coming from the pion current since pions of low energy carry large orbital magnetic moments. The bare proton is assumed to have unit nuclear magneton, and the interaction current is omitted completely. This is the model discussed by Sachs.[3]

The anomalous magnetic moments consist of two parts, which are, for the sake of convenience, called vector and scalar parts. The former has the same magnitude but opposite sign for proton and neutron. It comes from the pion current and the τ_3 part of the nucleon charge, which form vectors in isotopic space. The scalar part is the same for both proton and neutron and comes from the isotopic scalar part of the nucleon charge. Experimentally the scalar part μ_s is very small (0.06) compared with the vector part μ_v (1.85). The smallness of μ_s leads us to suppose that the nucleon current contributes little to the anomaly and the largest part of μ_v comes from the pion current. Theoretically, however, all calculations have given too large values for μ_s. This is characteristic of the pseudoscalar pion theory, for in this theory virtual pions of high energy are important and they have small magnetic moments. The large value of μ_v therefore requires a large amount of dissociation probability, which also makes μ_s large.

2. FORMULATION OF THE PROBLEM

The problem to be worked out is the evaluation of the expectation values of the magnetic moment operators in the physical one-proton state. In order to express the magnetic moment operators, it is convenient to describe the pion field in terms of spherical waves instead of the more conventional plane waves. Only P-waves need be considered since they alone interact with the nucleon in the static model. With the introduction of creation and annihilation operators $a_{k,m}^*$ and $a_{k,m}$ for positive, $b_{k,m}^*$ and $b_{k,m}$ for negative, and $c_{k,m}^*$ and $c_{k,m}$ for neutral pions with momentum k and magnetic quantum number m, the magnetic moment operator can be expressed as[3]

$$\mathfrak{M}=\mathfrak{M}_1+\mathfrak{M}_2+\mathfrak{M}_3,$$

$$\mathfrak{M}_1=\frac{e}{2}\sum_k\frac{1}{\omega_k}(a_{k,1}^*a_{k,1}-a_{k,-1}^*a_{k,-1}-b_{k,1}^*b_{k,1}+b_{k,-1}^*b_{k,-1}+a_{k,1}^*b_{k,-1}^*-a_{k,-1}^*b_{k,1}^*-b_{k,1}a_{k,-1}+b_{k,-1}a_{k,1}), \tag{1}$$

$$\mathfrak{M}_2=(e/4M)\tau_3\sigma_3, \tag{2}$$

$$\mathfrak{M}_3=(e/4M)\sigma_3. \tag{3}$$

Here ω_k is the energy of the pion with momentum k, and M is the nucleon mass. The expectation values are to be taken in the nucleon state with spin pointing in the z-direction. The expectation values of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are equal but have opposite sign for proton and neutron, and those of $\mathfrak{M}_3$ are equal for both. Therefore, only the proton moment need be calculated.

The Hamiltonian of the pion-nucleon system is

$$H=H_0+V,$$

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[1] G. F. Chew and F. Low, this issue [Phys. Rev. **101**, 1570 (1956)].
[2] G. F. Chew and F. Low, this issue [Phys. Rev. **101**, 1579 (1956)].
[3] R. G. Sachs, Phys. Rev. **87**, 1100 (1952).

$$H_0=\sum_{k,m}\omega_k(a_{k,m}{}^*a_{k,m}+b_{k,m}{}^*b_{k,m}+c_{k,m}{}^*c_{k,m}),$$

$$V=\frac{f_0}{\mu}\frac{1}{(6\pi R)^{\frac{1}{2}}}\sum_k\frac{k^2}{(2\omega_k)^{\frac{1}{2}}}\{\tfrac{1}{2}[\tau_+\sigma_+(a_{k,1}+b_{k,-1}{}^*)+\tau_+\sigma_-(a_{k,-1}+b_{k,1}{}^*)+\tau_-\sigma_+(a_{k,-1}{}^*+b_{k,1})+\tau_-\sigma_-(a_{k,1}{}^*+b_{k,-1})]+(1/\sqrt{2})[\tau_+\sigma_3(a_{k,0}+b_{k,0}{}^*)+\tau_-\sigma_3(a_{k,0}{}^*+b_{k,0})+\tau_3\sigma_+(c_{k,-1}{}^*+c_{k,1})+\tau_3\sigma_-(c_{k,1}{}^*+c_{k,-1})]+\tau_3\sigma_3(c_{k,0}{}^*+c_{k,0})\},$$

where R is the radius of a very large sphere introduced to make the wave functions discrete. We write these expressions simply as

$$H_0=\sum_p\omega_pd_p{}^*d_p,\quad V=\sum_p[G(d_p)d_p+G^*(d_p)d_p{}^*],$$

where d^* and d are creation and annihilation operators, and p stands for the magnetic quantum number m and the charge quantum number as well as the momentum p.

The anomalous moments are defined by

$$\mu_v=\mu_1+\mu_2,\quad \mu_s=\mu_3,$$

$$\mu_1=\frac{2M}{e}\langle|\mathfrak{M}_1|\rangle,\quad \mu_2=\left\langle\left|\frac{\tau_3\sigma_3-1}{2}\right|\right\rangle,\quad \mu_3=\left\langle\left|\frac{\sigma_3-1}{2}\right|\right\rangle,$$

where $|\rangle$ is the proton state with spin up. These are expressed in terms of nuclear magnetons. $-\frac{1}{2}$ for μ_2 and μ_3 are the normal moments of the proton.

3. EXPECTATION VALUES OF THE BILINEAR QUANTITIES IN THE PION OPERATOR

For the evaluation of $\langle|\mathfrak{M}_1|\rangle$, we need quantities like

$$\langle|d^*d|\rangle,\quad \langle|d^*d^*|\rangle,\quad \langle|dd|\rangle. \tag{4}$$

These quantities can be transformed according to the method of Chew and Low.[1]

Let us normalize H so that

$$H|\rangle=0. \tag{5}$$

From the commutation of H and d_p,

$$[H,d_p]=-\omega_pd_p-G^*(d_p),$$

and using (5),

$$d_p|\rangle=-\frac{1}{H+\omega_p}G^*(d_p)|\rangle. \tag{6}$$

Similarly,

$$\langle|d_p{}^*=-\langle|G(d_p)\frac{1}{H+\omega_p}. \tag{7}$$

In the same way we have

$$d_pd_q|\rangle=\frac{1}{H+\omega_p+\omega_q}\left[G^*(d_p)\frac{1}{H+\omega_q}G^*(d_q)+G^*(d_q)\frac{1}{H+\omega_p}G^*(d_p)\right]|\rangle, \tag{8}$$

and

$$\langle|d_p{}^*d_q{}^*=\langle|\left[G(d_p)\frac{1}{H+\omega_p}G(d_q)+G(d_q)\frac{1}{H+\omega_q}G(d_p)\right]\frac{1}{H+\omega_p+\omega_q}. \tag{9}$$

Thus we can write (4) as

$$\langle|d_p{}^*d_p|\rangle=\sum_n\frac{\langle|G(d_p)|n\rangle\langle n|G^*(d_p)|\rangle}{(E_n+\omega_p)^2}, \tag{10}$$

$$\langle|d_pd_q|\rangle=\sum_n\frac{1}{\omega_p+\omega_q}\left[\frac{\langle|G^*(d_p)|n\rangle\langle n|G^*(d_q)|\rangle}{E_n+\omega_q}+\frac{\langle|G^*(d_q)|n\rangle\langle n|G^*(d_p)|\rangle}{E_n+\omega_p}\right], \tag{11}$$

$$\langle|d_p{}^*d_q{}^*|\rangle=\sum_n\frac{1}{\omega_p+\omega_q}\left[\frac{\langle|G(d_p)|n\rangle\langle n|G(d_q)|\rangle}{E_n+\omega_p}+\frac{\langle|G(d_q)|n\rangle\langle n|G(d_p)|\rangle}{E_n+\omega_q}\right], \tag{12}$$

where $|n\rangle$ are the complete orthonormal set of incoming wave eigenstates. These equations are exact, although they resemble the results of the second-order perturbation theory.

The matrix element

$$T_q(n)=\langle n|G(d_q)|\rangle \tag{13}$$

is related to the scattering matrix.[1] For $E_n=\omega_q$, $T_p(n)$ is equal to the transition matrix element of the process in which a pion of type p is scattered by a proton into the state n. The use of the spherical wave for the incoming particle is somewhat unfamiliar to us. The relation to the more conventional expression using a plane wave can be obtained by means of transformation functions. If $T(\mathbf{k},n)$ is the conventional matrix element for the incident pion with momentum $\mathbf{k}$ and $T_{k,m}(n)$ is the one used in this paper, then,

$$T_{\mathbf{k}}(n)=\frac{i}{k}(6\pi R)^{\frac{1}{2}}\left\{\cos\Theta T_{k,0}(n)+\frac{\sin\Theta}{\sqrt{2}}[e^{i\Phi}T_{k,1}(n)+e^{-i\Phi}T_{k,-1}(n)]\right\}, \tag{14}$$

where Θ, Φ are the polar angles of the vector $\mathbf{k}$.

It is more convenient to express the T matrix in terms of four independent elements, $T_{ij}(k;n)$, corresponding to four eigenstates of total isotopic spin $i/2$ and total angular momentum $j/2$. k is the momentum of the incoming pion. $T_k(n)$ and T_{ij} are simply related by the familiar Clebsch-Gordan coefficients. With this notation we have

$$\langle|G(a_1)|n\rangle\langle n|G^*(a_1)|\rangle$$
$$=\langle|G(a_1)|n\rangle\langle n|G(b_{-1})|\rangle$$
$$=\tfrac{1}{9}[|T_{33}(n)|^2+2|T_{31}|^2+2|T_{13}|^2+4|T_{11}|^2],$$
$$\langle|G(a_{-1})|n\rangle\langle n|G^*(a_{-1})|\rangle$$
$$=\langle|G(a_{-1})|n\rangle\langle n|G(b_1)|\rangle=\tfrac{1}{3}(|T_{33}|^2+2|T_{13}|^2),$$
$$\langle|G(b_1)|n\rangle\langle n|G^*(b_1)|\rangle$$
$$=\langle|G^*(a_{-1})|n\rangle\langle n|G^*(b_1)|\rangle=\tfrac{1}{3}(|T_{33}|^2+2|T_{31}|^2), \quad (15)$$
$$\langle|G(b_{-1})|n\rangle\langle n|G^*(b_{-1})|\rangle$$
$$=\langle|G^*(a_1)|n\rangle\langle n|G^*(b_{-1})|\rangle=|T_{33}|^2,$$
$$\langle|G(c_1)|n\rangle\langle n|G^*(c_1)|\rangle$$
$$=\tfrac{1}{9}(2|T_{33}|^2+4|T_{31}|^2+|T_{13}|^2+2|T_{11}|^2),$$
$$\langle|G(c_{-1})|n\rangle\langle n|G^*(c_{-1})|\rangle=\tfrac{1}{3}(2|T_{33}|^2+|T_{31}|^2).$$

The arguments of T_{ij} are omitted in these equations.

Consider the term $a_{k,1}{}^*a_{k,1}$ as an example.

$$\langle|a_{k,1}{}^*a_{k,1}|\rangle=\sum_n\frac{\langle|G(a_{k,1})|n\rangle\langle n|G^*(a_{k,1})|\rangle}{(E_n+\omega_k)^2}.$$

In this summation we separate the term $n=0$ corresponding to the nucleon state. The matrix element

$$f_0\langle 0|\tau_-\sigma_-|\rangle$$

is equal to

$$f(u_0\tau_-\sigma_-u),$$

where f is the renormalized coupling constant and u_0 and u are normalized Pauli spinors. Thus

$$\sum_{n=0}\frac{|\langle n|G^*(a_{k,1})|\rangle|^2}{(E_n+\omega_k)^2}=\frac{f^2}{\mu^2}\frac{1}{3\pi R}\frac{k^4}{\omega_k{}^3}.$$

We assume that there is no bound state except the nucleon state. The remaining term is

$$\frac{1}{9}\sum_{n>0}\frac{1}{(E_n+\omega_k)^2}[|T_{33}(k,n)|^2+2|T_{31}|^2+2|T_{13}|^2+4|T_{11}|^2].$$

The summation

$$\sum_{n>0}\frac{|T_{ij}|^2}{(E_n+\omega_k)^2} \quad (16)$$

can be expressed in terms of the cross sections. The total cross section for positive pions of momentum k incident on a proton is

$$\sigma^+(k)=\frac{2\pi}{v_k}\sum_n\delta(E_n-\omega_k)|T_{\mathbf{k}}(n)|^2,$$

where $v_k=k/\omega_k$. Using (14) and (13), we get

$$\sigma^+(k)=\frac{12\pi^2R}{v_kk^2}\sum_n\delta(E_n-\omega_k)\left|\cos\Theta\langle n|G(a_{k,0})|\rangle+\frac{\sin\Theta}{\sqrt{2}}[e^{i\Phi}\langle n|G(a_{k,1})|\rangle+e^{-i\Phi}\langle n|G(a_{k,-1})|\rangle]\right|^2.$$

We average this quantity over the direction of k in order to get the cross section for an unpolarized proton. Observing that $G(a_{k,1})=G^*(b_{k,-1})$, etc., and using (15), we obtain

$$\sigma^+(k)=4\pi^2\frac{\omega_kR}{k^3}\sum_n\delta(E_n-\omega_k)[2|T_{33}(k,n)|^2+|T_{31}|^2].$$

We write this as

$$\sigma^+(k)=\tfrac{1}{3}[2\sigma_{33}(k)+\sigma_{31}(k)].$$

Similarly, the total cross section for negative pions is

$$\sigma^-(k)=\tfrac{1}{9}[2\sigma_{33}(k)+4\sigma_{13}(k)+\sigma_{31}(k)+2\sigma_{11}(k)],$$

where

$$\sigma_{ij}(k)=12\pi^2R\frac{\omega_k}{k^3}\sum_n\delta(E_n-\omega_k)|T_{ij}(k,n)|^2 \quad (17)$$

is the cross sections due to the pure state of isotopic spin $i/2$ and angular momentum $j/2$. Below the threshold of pion production (two pions in the final state) σ_{ij} takes the simple form

$$\sigma_{ij}(k)=\frac{12\pi}{k^2}\sin^2\delta_{ij},$$

where the δ_{ij} are the usual phase shifts.

In order to compare (16) with (17), we first observe that $|T_{ij}(k,n)|^2$ depends on k only in a trivial way, namely

$$|T_{ij}(k,n)|^2=\frac{k^4}{\omega_k}\frac{\omega_l}{l^4}|T_{ij}(l,n)|^2.$$

Thus (17) can be written as

$$\sigma_{ij}(k)=12\pi^2R\frac{\omega_l\omega_k}{l^4}\sum_n\delta(k_n-k)|T_{ij}(l,n)|^2, \quad (18)$$
$$k_n{}^2+\mu^2=E_n{}^2.$$

where the relation

$$k\delta(E_n-\omega_k)=\omega_k\delta(k_n-k)$$

has been used. Multiplying (18) by

$$\frac{1}{12\pi^2R}\frac{l^4}{(\omega_l+\omega_k)^\nu\omega_k\omega_l}, \quad (\nu=1,2)$$

and integrating over k, we obtain

$$\frac{1}{12\pi^2 R}\frac{l^4}{\omega_l}\int\frac{\sigma_{ij}(k)dk}{(\omega_l+\omega_k)^\nu\omega_k}=\sum_{n>0}\frac{|T_{ij}(l,n)|^2}{(\omega_l+E_n)^\nu}. \quad (19)$$

Inserting (15) and (19) into (10), (11), (12), we have the required expectation values:

$$\begin{aligned}
\langle|a_{k,1}{}^*a_{k,1}|\rangle&=\frac{f^2}{\mu^2}\frac{1}{3\pi R}\frac{k^4}{\omega_k{}^3}+\frac{1}{9}[M_{33}(k)+2M_{31}+2M_{13}+4M_{11}],\\
\langle|a_{k,-1}{}^*a_{k,-1}|\rangle&=\tfrac{1}{3}(M_{33}+2M_{13}),\\
\langle|b_{k,1}{}^*b_{k,1}|\rangle&=\tfrac{1}{3}(M_{33}+2M_{31}),\\
\langle|b_{k,-1}{}^*b_{k,-1}|\rangle&=M_{33},\\
\langle|c_{k,1}{}^*c_{k,1}|\rangle&=\frac{f^2}{\mu^2}\frac{1}{6\pi R}\frac{k^4}{\omega_k{}^3}+\frac{1}{9}(2M_{33}+4M_{31}+M_{13}+2M_{11}),\\
\langle|c_{k,-1}{}^*c_{k,-1}|\rangle&=\tfrac{1}{3}(2M_{33}+M_{31}),\\
\langle|a_{k,1}{}^*b_{k,-1}{}^*|\rangle=\langle|a_{k,1}b_{k,-1}|\rangle&=\frac{f^2}{\mu^2}\frac{1}{6\pi R}\frac{k^4}{\omega_k{}^3}+\tfrac{1}{9}[5N_{33}(k)+N_{31}+N_{13}+2N_{11}],\\
\langle|a_{k,-1}{}^*b_{k,1}{}^*|\rangle=\langle|a_{k,-1}b_{k,1}|\rangle&=\tfrac{1}{3}(N_{33}+N_{13}),
\end{aligned}\quad (20)$$

where

$$M_{ij}(k)=\frac{1}{12\pi^2 R}\frac{k^4}{\omega_k}\int\frac{\sigma_{ij}(l)dl}{(\omega_k+\omega_l)^2\omega_l},$$

$$N_{ij}(k)=\frac{1}{12\pi^2 R}\frac{k^4}{\omega_k{}^2}\int\frac{\sigma_{ij}(l)dl}{(\omega_k+\omega_l)\omega_l}.$$

The summation over k can be replaced by an integral, using the formula

$$\sum_k\to\frac{R}{\pi}\int dk.$$

Thus we have for the anomalous magnetic moment due to the pion current (expressed in the units of nuclear magneton):

$$\mu_1=\frac{2M}{e}\langle|\mathfrak{M}_1|\rangle=\frac{f^2}{4\pi}\frac{8}{3\pi}\frac{M}{\mu^2}\int^K\frac{k^4}{\omega_k{}^4}dk+\tfrac{1}{9}[4P_{33}(K)-2P_{31}-2P_{13}+4P_{11}], \quad (21)$$

with

$$P_{ij}(K)=2M[I_{ij}(2,2;K)+I_{ij}(3,1;K)],$$

$$I_{ij}(\mu,\nu;K)=\frac{1}{12\pi^3}\int\frac{\sigma_{ij}(k)l^4dkdl}{\omega_k\omega_l{}^\mu(\omega_k+\omega_l)^\nu}.$$

TABLE I. Values of I_{33} for various cutoffs K.

K	$I_{33}(2.1;K)$	$2MI_{33}(2.2;K)$	$2MI_{33}(3.1;K)$
5μ	0.13	0.32	0.50
6μ	0.21	0.45	0.68

The first term in (21) is equal to the result of the lowest-order perturbation calculation but with the renormalized coupling constant. This result is exact. No approximation is involved except the static approximation which was introduced in the first stage.

4. NUMERICAL VALUES

In order to calculate the integral I_{ij}, we must know the σ_{ij} which are not directly observed by experiment. However, a very simple situation prevails, at least for low-energy scattering. It has been predicted by theory and confirmed by experiments that σ_{33} is the dominant cross section:

$$\sigma_{33}\gg\sigma_{11},\ \sigma_{13},\ \sigma_{31}.$$

This relation holds for pion energies up to about 250 Mev. We simply assume that all σ_{ij}'s are zero except σ_{33}, since the high-energy part contributes little to the integral. In this case

$$\sigma_{33}=\tfrac{3}{2}\sigma^+,\quad \sigma_{11},\ \sigma_{13},\ \sigma_{31}=0. \quad (22)$$

Errors involved in this approximation will be discussed in Sec. 6.

Calculations are made in the barycentric system. That is, $\sigma(k)$ is defined as the total cross section for pions with momentum k in the barycentric system. Some values of I_{ij} for various cutoffs K are shown in Table I.

The magnetic moment due to the pion current is calculated with $f^2/4\pi=0.08$.

$$\mu_1=1.36+0.36=1.72 \quad (K=5\mu),$$
$$\mu_1=1.80+0.50=2.30 \quad (K=6\mu).$$

The first numbers in these equations represent the first term in (21) and the second numbers are the remaining terms.

5. MAGNETIC MOMENTS DUE TO PROTON CHARGE

The scalar part of the magnetic moment, μ_s, can be evaluated in the same way as in Sec. 3. By the conservation of the total angular momentum,

$$\tfrac{1}{2}\sigma_3=J_3-(J_{\text{pion}})_3=\tfrac{1}{2}-(J_{\text{pion}})_3,$$

where J_{pion} is the angular momentum of the pion field only and is given by

$$(J_{\text{pion}})_3=\sum\nolimits_k(a_{k,1}{}^*a_{k,1}+b_{k,1}{}^*b_{k,1}+c_{k,1}{}^*c_{k,1}-a_{k,-1}{}^*a_{k,-1}-b_{k,-1}{}^*b_{k,-1}-c_{k,-1}{}^*c_{k,-1}).$$

The scalar part of the anomalous moment is, therefore,

$$\mu_3=-\langle|(J_{\text{pion}})_3|\rangle=\sum(-\langle|a_{k,1}{}^*a_{k,1}|\rangle-\langle|b_{k,1}{}^*b_{k,1}|\rangle -\langle|c_{k,1}{}^*c_{k,1}|\rangle+\langle|a_{k,-1}{}^*a_{k,-1}|\rangle +\langle|b_{k,-1}{}^*b_{k,-1}|\rangle+\langle|c_{k,-1}{}^*c_{k,-1}|\rangle).$$

All expectation values were calculated in (20). Thus

$$\mu_3=-\frac{f^2}{4\pi}\frac{2}{\pi\mu^2}\int\frac{k^4}{\omega_k{}^3}dk+\tfrac{1}{3}[4L_{33}(K)-3L_{31}-2L_{11}], \quad (23)$$

with

$$L_{ij}(K)=I_{ij}(2,1;K).$$

Again we use the simplified assumption (22) and obtain

$$\begin{aligned}\mu_3&=-0.52+0.17=-0.35 \quad (K=5\mu),\\ \mu_3&=-0.78+0.28=-0.50 \quad (K=6\mu).\end{aligned} \quad (24)$$

The same method is not applicable for the evaluation of μ_2. However, this quantity can be calculated in a different way. From the definition of the renormalized coupling constant, we have

$$f_0\langle|\tau_3\sigma_3|\rangle=f(u\tau_3\sigma_3u)=f,$$

so that

$$\langle|\tau_3\sigma_3|\rangle=f/f_0.$$

The relation between the renormalized and unrenormalized coupling constant was given by Chew and Low. We reproduce their derivation here for use later.

By using the relation between $T_{\mathbf{k}}(n)$ and the total cross section,

$$\frac{f_0{}^2k^2}{\omega_k\mu^2}\left\langle\left|\frac{1-\tau_3}{2}\right|\right\rangle=\sum_n\frac{f_0{}^2}{4\omega_k\mu^2}\langle|\tau_-(\boldsymbol{\sigma}\mathbf{k})|n\rangle\langle n|\tau_+(\boldsymbol{\sigma}\mathbf{k})|\rangle =\sum|T_{\mathbf{k},+}(n)|^2=\frac{k^2}{2\pi\omega_k}\int\frac{\sigma^+(l)}{\omega_l}dl.$$

Therefore

$$f_0{}^2\left\langle\left|\frac{1-\tau_3}{2}\right|\right\rangle=\frac{\mu^2}{2\pi}\int\frac{\sigma^+(k)}{\omega_k}dk. \quad (25)$$

Similarly,

$$f_0{}^2\left\langle\left|\frac{1+\tau_3}{2}\right|\right\rangle=f^2+\frac{\mu^2}{2\pi}\int\frac{\sigma^-(k)}{\omega_k}dk. \quad (26)$$

Adding these two equations:

$$f_0{}^2=f^2+\frac{\mu^2}{2\pi}\int\frac{\sigma^-(k)+\sigma^+(k)}{\omega_k}dk. \quad (27)$$

Inserting numerical values, we have

$$f_0^2/4\pi=0.19.$$

The cutoff was chosen as $K=6\mu$. The integral is, however, insensitive to the cutoff.

$$\langle|\tau_3\sigma_3|\rangle=\frac{f}{f_0}=\left(\frac{0.08}{0.19}\right)^{\frac{1}{2}}=0.65.$$

The sign was chosen to be positive for reasons of continuity.[4] The anomalous moment is, therefore,

$$\mu_2=-0.18.$$

This method of calculation is quite different from that used in Sec. 3 and the early part of this section. However, we can check the consistency of both calculations. Since $\langle|T_3|\rangle$ is equal to $\langle|\sigma_3|\rangle$ in the static model, we have, from (24),

$$\mu_3=\tfrac{1}{2}\langle|(\sigma_3-1)|\rangle=-\frac{4\pi}{f_0{}^2}\frac{\mu^2}{8\pi^2}\int\frac{\sigma^+}{\omega}dk=-0.36. \quad (28)$$

This value is consistent with (24) obtained by the different method. We note that in this expression, μ_3 is rather insensitive to the cutoff if experimental values are used for σ^+. Of course, μ_3 is cutoff-dependent if theoretical values are used instead.

6. DISCUSSION OF THE RESULTS

Summing up the results, the anomalous magnetic moments are as follows:

$$\begin{aligned}\mu_v&=\ \ 1.54, \quad (K=5\mu)\\ &=\ \ 2.12, \quad (K=6\mu),\\ \mu_s&=-0.35, \quad (K=5\mu)\\ &=-0.50, \quad (K=6\mu)\\ &=-0.36 \quad [\text{Eq. (28)}].\end{aligned}$$

Experimental values are given by

$$\mu_v=1.85, \quad \mu_s=-0.06.$$

We see that μ_v fits with experimental value if the cutoff is chosen between 5μ and 6μ. This is satisfactory since from scattering experiments Chew and Low estimated that K should be about 6μ. The scalar part, μ_s, however, is more than six times larger than the experimental value.

The calculations were done using the following approximations:

(1) Use of the static model (that is, recoil and nucleon pair creation are neglected and all integrals over momenta of the virtual pions are cut off at K).

(2) An assumption about the magnetic moment operators. They are assumed to be given by Eqs. (1), (2), and (3).

(3) Three-three states are dominant. More exactly, it is assumed that

$$I_{33}\gg I_{11},\ I_{13},\ I_{31},$$

which means that

$$\int^K\frac{\sigma_{33}(k)}{\omega_k{}^2}dk\gg\int^K\frac{\sigma_{11},\ \sigma_{13},\ \sigma_{31}}{\omega_k{}^2}dk. \quad (27)$$

[4] We argue in this way. Suppose that all quantities are the function of f and the cutoff K. When $k=0$, $\langle|\tau_3\sigma_3|\rangle=1$ and f_0 must be equal to f. When K is increased, f_0 changes continuously and changes sign only after it becomes zero or infinite. However, from the Eq. (27) we see that neither case can happen.

The calculations are otherwise exact. All higher order effects are included in the cross-section term and the renormalization of the coupling constant.

Of these, the assumption (3) was made only for ease in computation. We have detailed information about the phase shifts at low energies and more accurate evaluation of the integrals I_{ij} is, of course, possible. Although the dominance of the three-three state is not true for higher energies, the assumption (27) should be fairly good since the high-energy part is not important due to the factor $1/\omega_k^2$. The error involved in this assumption could be estimated in the following way. Since

$$3\sigma^- - \sigma^+ = \tfrac{2}{3}(2\sigma_{13}+\sigma_{11}),$$

we have a rough estimate for

$$\int \frac{\sigma_{ij}}{\omega_k^2} dk \approx \frac{1}{2}\int \frac{3\sigma^- - \sigma^+}{\omega_k^2}, \quad (ij=11, 13, 31).$$

Comparing this with $\frac{3}{2}\int \sigma^+ dk/\omega^2$,

$$\int \frac{\sigma_{ij}}{\omega^2} dk \Big/ \int \frac{\sigma_{33}}{\omega^2} dk \approx 0.19 \quad (ij=11, 13, 31).$$

This means that I_{11}, I_{13}, and I_{31} are about one-fifth of I_{33}. Therefore the neglect of these integrals causes an error of a few percent for μ_1 and an error of about 10% for μ_3.

The expression (26) for μ_3 does not involve the assumption (3), and is not critically dependent upon the cutoff. This result is six times larger than the actual value. We can conclude that Sachs' mirror condition, which requires that $\mu_s = -0.06$, is never satisfied in the simple model which assumes (1) and (2).

Turning to the assumption (2), the expression (1) for the magnetic moment due to the pion current is exact and free from ambiguity so long as nucleon recoil is neglected. This is no longer true, however, for the part coming from the nucleon current. There is some ambiguity in defining the interaction current in the static approximation, although we have neglected it completely. We also assumed that the bare-proton magnetic moment is one nuclear magneton, which is not true if nucleon pair creation and other effects are taken into account. Under these circumstances the values of μ_2 and μ_3 obtained here cannot be taken too seriously.

We should point out that the correction terms to the second-order results, given by the second term in (21) and (23), are in the right direction although not sufficiently large. When three-three states are dominant, the correction terms are both positive. It increases μ_1, decreases μ_3 and makes them nearer to the experimental values. This fact can be understood in the following way: the proton with spin up likes to form a state of $i=\frac{3}{2}$ and $j=\frac{3}{2}$ with a virtual positive pion of $m=1$. A virtual negative pion of $m=-1$ must be emitted in order to balance the total angular momentum and charge. This state makes a large contribution to μ_1 but does not contribute to μ_3. The predominance of this virtual state agrees favorably with the experimental fact that μ_3 is very small compared to μ_1. This consideration bears some relation to the model proposed by Sugawara,[5] who introduced a $\frac{3}{2}$–$\frac{3}{2}$ isobar state explicitly in order to explain the magnetic moment anomaly.

The good agreement of μ_1 with the experimental value suggests that the adoption of the static model is admissible for the pion part of the anomalous magnetic moment. The nucleon current is, however, a relativistic phenomenon and the present calculation is not sufficient for this purpose.

In this paper no interaction was assumed between pions. Holladay[6] obtained good results for nucleon anomalous moments by assuming a strong correlation between pions. The conclusions in this paper would not hold if there were strong forces between pions.

Note added in proof.—G. Sandri [Phys. Rev. **101**, 1616 (1956)] has pointed out that the conservation of strangeness allows the virtual emission of K^+ by the nucleons while the emission of K^- is forbidden. This has the consequence that the K-particles contribute to μ_s but not to μ_v. The discrepancy in the value of μ_s in this paper could be removed by introducing a pseudoscalar K-particle which would give a positive contribution to μ_s.

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[5] M. Sugawara, Progr. Theoret. Phys. (Japan) 8, 549 (1952).
[6] W. G. Holladay (to be published).