Model for Multiple Meson Production*

E. M. HENLEY, *Department of Physics, University of Washington, Seattle, Washington*

AND

T. D. LEE, *Department of Physics, Columbia University, New York, New York*

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A model is presented for the multiple production of mesons. It is similar to that of Lewis, Oppenheimer, and Wouthuysen, but treats the spins and isotopic spins of the colliding nucleons as quantum mechanical operators. To illustrate the method, detailed calculations are carried out for a symmetrical scalar meson theory.

I. INTRODUCTION

THE recent availability of machine energies of several Bev has given new impetus to both the experimental[1] and theoretical studies of meson production. Thus, Fermi[2] has proposed a model which assumes that all mesons are emitted from a small volume in which all the energy is concentrated. Fermi's theory does not make use of any specific meson theory; instead, it assumes that in this small volume, apart from some strict conservation laws, various states of mesons and nucleons are all in statistical equilibrium with each other. The whole problem is then completely determined by the total energy, the angular momentum, and the volume of the system.

More specific models that depend on particular forms of meson theory have been discussed by Lewis, Oppenheimer and Wouthuysen[3] and by others.[4] Lewis *et al.*, make use of a Bloch-Nordsieck[5] type of treatment, which pictures the physical nucleon as a composite system composed of a core (or bare nucleon) surrounded by a meson cloud. Furthermore, the collision of two nucleons is assumed to consist of a sudden exchange of spin, isotopic spin and momentum between the cores of these two nucleons; the latter, in turn, shake off part of the surrounding meson clouds as radiation. Except for the above-assumed mechanism of a sudden collision and the neglect of certain recoil effects of the emitted radiation, the Bloch-Nordsieck type of calculation is completely rigorous and can be used for arbitrarily large values of coupling constants. However, in the treatment of Lewis, Oppenheimer, and Wouthuysen, further approximations are made, which consist of treating both the spin and isotopic spin of the nucleons as classical vectors. Consequently, the emitted mesons are almost uninhibited in their choice of charge; this makes it impossible to compare these results with the observed charge spectrum of mesons produced by nucleon collisions at cosmotron energies.

The purpose of this paper is to discuss a similar model, but one in which the spin and isotopic spin of the nucleons are treated as rigorous quantum mechanical operators. The model to be discussed is closely related to the fixed extended source theory,[6] which has been used recently to explain the scattering experiments of mesons by nucleons up to about 200-Mev incident meson energy. In these relatively low-energy scattering phenomena the nucleon is regarded essentially as at rest; thus, it can be represented by a fixed core of finite size surrounded by a meson cloud. Both the meson distribution in a physical nucleon state and the scattering state of a physical nucleon together with an additional meson have been explicitly calculated[7] using an intermediate coupling method.[8,9]

In the problem of multiple meson production, one deals with nucleons of very fast velocities. Nevertheless, it is shown that the meson distributions for these rapidly moving nucleons can be obtained directly through a Lorentz transformation from the corresponding distribution for a fixed nucleon. In a similar way, one may obtain the state functions for a physical nucleon with any number of free mesons. Identical results can be obtained by working directly with the Bloch-Nordsieck type of transformation as an alternative procedure.

The collision process between two physical nucleons is assumed to be instantaneous and only to result in a transfer of spin and isotopic spin between the two cores of the nucleons. The collision is then characterized by an $\mathcal{S}$-matrix which acts only on the spin and isotopic spin of the cores. Thus, $\mathcal{S}$ is a 4×4 matrix if it involves only spin exchange or only isotopic spin exchange; but it is (16×16)-dimensional if both of these quantities are involved during a collision. The matrix elements for the production of N mesons is then given by the $\mathcal{S}$-matrix, evaluated between the initial state of two

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physical nucleons and the final state of two physical nucleons and N free mesons.

In the present paper, we shall illustrate our method in detail for the symmetrical scalar theory. In this theory, the production ratios of various number and charge distribution of mesons are uniquely determined by the coupling constant and the source size. It is interesting to note that from this calculation we find that for large coupling constants (a) each nucleon tends to emit mesons in an isotopic spin $I=\frac{3}{2}$ state and (b) in two-meson production each nucleon prefers to emit only a single meson. The more complicated case of the symmetrical pseudoscalar meson theory and its comparison with experiments will be presented in a later publication. The case of a neutral scalar theory can be treated rigorously and is discussed in Appendix III.

II. HAMILTONIAN

We begin with a description of the system of a nucleon at rest. The nucleon is pictured as a bare core of finite extent surrounded by a cloud of mesons. The interaction of such a nucleons at rest with a symmetrical scalar meson field is given by a Hamiltonian ($\hbar=c=1$)[10]

$$H_0=m_0+H_\pi+g\int \tau_i\phi_i U(r)d^3r, \tag{1}$$

with

$$H_\pi=\frac{1}{2}\int[\pi_i^2+\mu^2\phi_i^2+(\nabla\phi_i)^2]d^3r.$$

Here m_0 is the mechanical mass of the nucleon, $U(r)$ is the normalized source function for the nucleon assumed to be at rest at the origin of the coordinate system, ϕ_i and π_i ($i=1, 2, 3$) are the canonical meson field variables, τ_i are the usual isotopic spin operators, μ is the meson mass, and g is the coupling constant. The eigenstates of this Hamiltonian comprise both the nucleon bound states and the meson-nucleon scattering states for a fixed nucleon.

The corresponding states for a nucleon in motion with velocity v may be obtained kinematically by means of a Lorentz transformation to the moving coordinate system. Under such a transformation, the 4-dimensional space-time coordinate x_μ becomes x_μ' ($\mu=1-4$), the canonical variables of the scalar meson fields undergo an unitary transformation

$$\phi_i(x_\mu)\rightarrow \mathfrak{L}\phi_i\mathfrak{L}^{-1}=\phi_i(x_\mu') \tag{2}$$

and

$$\pi_i(x_\mu)\rightarrow \mathfrak{L}\pi_i\mathfrak{L}^{-1}=\pi_i(x_\mu'),$$

where $\mathfrak{L}$ is a unitary operator. In the interaction representation, the space-time dependence of the field variables are given by

$$\phi_i=\int(16\pi^3\omega)^{-\frac{1}{2}}[a_i(\mathbf{k})e^{ik_\mu x_\mu}+a_i^\dagger(\mathbf{k})e^{-ik_\mu x_\mu}]d^3k$$

and

$$\pi_i=-i\int(16\pi^3/\omega)^{-\frac{1}{2}}[a_i(\mathbf{k})e^{ik_\mu x_\mu}-a_i^\dagger(\mathbf{k})e^{-ik_\mu x_\mu}]d^3k, \tag{3}$$

where $a_i^\dagger(\mathbf{k})$ and $a_i(\mathbf{k})$ are meson creation and annihilation operators, respectively, x_μ and k_μ are the four-dimensional space and momentum vectors, and $\omega=(k^2+\mu^2)^{\frac{1}{2}}$. The invariance of ϕ_i and π_i, Eq. (2), demonstrates that under a Lorentz transformation $a_i^\dagger(\mathbf{k})$ and $a_i(\mathbf{k})$ transform as

$$\begin{aligned} a_i(\mathbf{k})&\rightarrow \mathfrak{L}a_i(\mathbf{k})\mathfrak{L}^{-1}=a_i(\mathbf{k}')(\omega'/\omega)^{\frac{1}{2}},\\ a_i^\dagger(\mathbf{k})&\rightarrow \mathfrak{L}a_i^\dagger(\mathbf{k})\mathfrak{L}^{-1}=a_i^\dagger(\mathbf{k}')(\omega'/\omega)^{\frac{1}{2}},\end{aligned} \tag{4}$$

with $k_x'=k_x$, $k_y'=k_y$,

$$\begin{aligned} k_z'&=\gamma(k_z+v\omega),\\ \omega'&=\gamma(\omega+vk_z),\end{aligned} \tag{5}$$

and

$$\gamma=(1-v^2)^{-\frac{1}{2}}.$$

The z-axis is chosen to be along the direction of the velocity. In terms of $\mathfrak{L}$ any state function Ψ of the Hamiltonian (1) is related to a corresponding state for a moving nucleon by

$$\Psi_v=\mathfrak{L}\Psi, \tag{6}$$

while the Hamiltonian itself becomes H_v under a Lorentz transformation. Thus, we write

$$\mathfrak{L}H\mathfrak{L}^{-1}=\gamma(H_v-\mathbf{P}\cdot\mathbf{v}), \tag{7}$$

where $\mathbf{P}$ is the momentum of the nucleon in the moving system. On using Eq. (5), the Hamiltonian H_v in the moving system becomes

$$H_v=m_0\gamma^{-1}+\mathbf{P}\cdot\mathbf{v}+\int(\omega-\mathbf{k}\cdot\mathbf{v})a_i^\dagger(\mathbf{k})a_i(\mathbf{k})d^3k$$
$$+g\gamma^{-1}\int(16\pi^3\omega)^{-\frac{1}{2}}u_v(\mathbf{k})\tau_i[a_i(\mathbf{k})+a_i^\dagger(\mathbf{k})]d^3k, \tag{8}$$

where $u_v(\mathbf{k})$ is related to the source function $U(r)$ by

$$u_v(\mathbf{k})=\int U(r)\exp[i\gamma(k_z-v\omega)z+ik_xx+ik_yy]d^3r. \tag{9}$$

In Eq. (8), the zero-point energy of the meson field has been omitted.

It should be remarked that Eq. (8) is only approximately correct, since we use Eq. (4) for the expression of $\mathfrak{L}$, and this is valid only if the meson field ϕ_i obeys the free-field equation, $\Box^2\phi_i-\mu^2\phi_i=0$, as required by Eq. (3). Nevertheless, it is of interest to note that this approximation is identical with that obtained by the Bloch-Nordsieck[5] type treatment. The Block-Nordsieck

[10] In Eq. (1), as well as in the following, we shall use the contraction convention with respect to the indices i and ρ with i denoting the various charge states of mesons and ρ the isotopic spin states of the nucleon. A sum over i and ρ is required whenever these indices appear twice ($i=1, 2, 3$ and $\rho=1, 2$).

procedure is to start with a relativistic Hamiltonian

$$H=\int\psi^\dagger(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m_0)\psi d^3r+H_\pi+g\int\psi^\dagger\beta\tau_i\psi\phi_i d^3r. \quad (10)$$

By substituting for the Dirac matrices their expectation value among positive energy states of the free-nucleon spinors, the relativistic Hamiltonian is replaced by

$$H_{\mathrm{B.N.}}=\mathbf{p}\cdot\mathbf{v}+m_0\gamma^{-1}+\int\omega a_i^\dagger(\mathbf{k})a_i(\mathbf{k})d^3k$$
$$+g\gamma^{-1}\int(16\pi^3\omega)^{-\frac{1}{2}}u_v(\mathbf{k})$$
$$\times[a_i(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}+a_i^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}]\tau_i d^3k. \quad (11)$$

It is easy to verify that by a canonical transformation T,

$$T=\exp\left[i\int a_i^\dagger(\mathbf{k})a_i(\mathbf{k})\mathbf{k}\cdot\mathbf{r}d^3k\right].$$

The Block-Nordsieck Hamiltonian $H_{\mathrm{B.N.}}$ is directly related to H_v, given in Eq. (8). Thus, we have

$$H_v=TH_{\mathrm{B.N.}}T^{-1}, \quad (12)$$

showing the identity of these two different approaches.

III. STATE FUNCTION FOR PHYSICAL NUCLEON IN MOTION

We shall use the Tomonaga intermediate-coupling method[8] to describe the ground states of the Hamiltonian H_v. In this method there is no limit to the total number of virtual mesons which surround the bare core, but all mesons are assumed to be in the same orbital state $F_v(\mathbf{k})$. The best functional form of $F_v(\mathbf{k})$, together with the probability amplitude for finding various numbers of mesons are determined by the variational method

$$\delta\langle N_\rho(v)|H_v-m\gamma|N_\rho(v)\rangle=0, \quad (13)$$

with m as the rest mass of the physical nucleon and $|N_\rho(v)\rangle$ as the state vector of the physical nucleon with velocity v ($\rho=1,2$ denoting its two isotopic spin states).

As in all calculations with the Tomonaga method, it is only necessary to consider[11] a reduced Hamiltonian $\mathcal{H}_v$ which can be obtained by first commuting all the $a_i(\mathbf{k})$ to the right of the $a_i^\dagger(\mathbf{k})$ in H_v and then replacing them by $a_iF_v(\mathbf{k})$ and $a_i^\dagger F_v(\mathbf{k})$ respectively, with

$$\int F_v^2(\mathbf{k})d^3k=1, \quad (14)$$

and

$$a_i=\int a_i(\mathbf{k})F_v(\mathbf{k})d^3k,$$

$$a_i^\dagger=\int a_i^\dagger(\mathbf{k})F_v(\mathbf{k})d^3k.$$

Thus, the reduced Hamiltonian $\mathcal{H}_v$ becomes

$$\mathcal{H}_v=\mathbf{P}\cdot\mathbf{v}+\gamma^{-1}\mathcal{H}_0, \quad (15)$$

where[10]

$$\mathcal{H}_0=m_0+\Omega a_i^\dagger a_i+G\tau_i(a_i+a_i^\dagger), \quad (16)$$

with

$$\Omega=\int\gamma(\omega-\mathbf{k}\cdot\mathbf{v})F_v^2(\mathbf{k})d^3k$$

and

$$G=g\int(16\pi^3\omega)^{-\frac{1}{2}}u_v(\mathbf{k})F_v(\mathbf{k})d^3k.$$

The variational problem Eq. (13), then, reduces to one of finding the ground-state wave function $|\mathfrak{N}_\rho\rangle$ of $\mathcal{H}_0$,[12]

$$\mathcal{H}_0|\mathfrak{N}_\rho\rangle=m|\mathfrak{N}_\rho\rangle. \quad (17)$$

If we set the total momentum $P=m\gamma v$, the state vector $|\mathfrak{N}_\rho\rangle$ becomes an eigenvector of $\mathcal{H}_v$ with energy $m\gamma$. Minimizing the energy with respect to an arbitrary functional form $F_v(\mathbf{k})$, we have

$$F_v(\mathbf{k})=-\frac{gu_v(\mathbf{k})\langle\mathfrak{N}_\rho|\tau_ia_i|\mathfrak{N}_\rho\rangle}{(16\pi^3\omega)^{\frac{1}{2}}[\gamma(\omega-\mathbf{k}\cdot\mathbf{v})+\lambda]\langle\mathfrak{N}_\rho|a_i^\dagger a_i|\mathfrak{N}_\rho\rangle}, \quad (18)$$

where λ is the normalization factor and is independent of the velocity v. It may be readily verified that Ω and G are also independent of v. Thus, as a physical nucleon is set in motion, the distribution of various *number* of mesons, as given by $|\mathfrak{N}_\rho\rangle$, is unchanged, while the orbital states of the mesons are transformed according to Eq. (18).

If the angular variables are separated out, Eq. (17) can be reduced to two coupled differential equations with one radial variable. Although these equations can only be solved numerically, we shall give a simple analytic approximation in Sec. VII that gives rigorous solutions at both the weak-coupling and strong-coupling limits, and yields fairly accurate results for an intermediate range of coupling constants.

IV. COLLISION PROCESSES

In the Bloch-Nordsieck type treatment of the collision of two nucleons, it is assumed that only the cores interact. The latter, in the symmetrical scalar case, are characterized by an isotopic spin and velocity. It is the change of these two quantities which leads to the emission of mesons. We note that the velocity change Δv is related to the corresponding momentum change by $\Delta v=(1-v^2)^{\frac{3}{2}}\Delta p/m$. Thus, especially at high energy, Δv would be quite small even though Δp might not be neglected. In the following, we shall neglect the effect of Δv on meson production.

In order to make the whole collision process charge-independent, it is necessary to have the interaction

[11] T. D. Lee and D. Pines, Phys. Rev. **92**, 883 (1953).

[12] Equation (17) may be obtained directly by applying the Tomonaga method to the original Hamiltonian H_0, Eq. (1), for a nucleon at rest. The state vector $|N_\rho(v)\rangle$ can then be formed by a Lorentz transformation from $|N_\rho(v=0)\rangle$.

between the two cores conserve the total isotopic spin, I, for the two nucleons. In this case, the collision between the two cores can be expressed in terms of two real phase shifts, δ_1 for the triplet states and δ_0 for the singlet state, since the isotopic spin of each core is $\frac{1}{2}$. The scattering matrix $\mathcal{S}$ is then

$$\mathcal{S}=S_0+S\boldsymbol{\tau}^a\cdot\boldsymbol{\tau}^b, \tag{19}$$

where a and b denote the two colliding nucleons and $\boldsymbol{\tau}^a,\boldsymbol{\tau}^b$ their respective Pauli isotopic spin matrices. The complex constants S_0 and S can be expressed in terms of the phase shifts δ_0 and δ_1 as

$$4S_0=3e^{2i\delta_1}+e^{2i\delta_0} \tag{20}$$

and

$$4S=e^{2i\delta_1}-e^{2i\delta_0}.$$

The various rates of production of mesons are then determined by the corresponding matrix elements of the form

$$\langle\text{final states}|\,\mathcal{S}\,|\text{initial states}\rangle.$$

Actually, in the expression for $\mathcal{S}$, only the matrix elements of the second term, $S\boldsymbol{\tau}^a\cdot\boldsymbol{\tau}^b$ are relevant for various production processes. The first term S_0, is independent of the isotopic spin. Consequently, as long as the effect due to a change of velocity is unimportant, it cannot contribute to the emission of mesons.

V. STATES OF A PHYSICAL NUCLEON AND SEVERAL ADDITIONAL MESONS

In order to obtain the matrix elements of $\mathcal{S}$ for the production of one or more mesons, we need not only the state of a physical nucleon, as given in Sec. III, but also the description of a physical nucleon and several mesons. These states are given by the scattering states of H_v. As discussed by Lee and Christian,[9] these states may be obtained by a variational method. We shall summarize their method in a form that will be most convenient for our calculation.

We consider first the states of a physical nucleon with velocity v and one additional meson with momentum $\mathbf{k}_0$ in a total isotopic spin state I. The state function is assumed to be[10]

$$\Psi^{(1)}(I,I_z;\,\mathbf{k}_0,v)=C_i{}^\rho(I,I_z)\int\chi_I{}^v(\mathbf{k},\mathbf{k}_0)a_i{}^\dagger(\mathbf{k})d^3k\,|N_\rho(v)\rangle$$
$$+D^\rho(I,I_z)\,|N_\rho(v)\rangle, \tag{21}$$

where $|N_\rho(v)\rangle$ is the state vector of a physical nucleon with velocity v in the isotopic spin state ρ ($\rho=1,2$). The $C_i{}^\rho(I,I_z)$ are the appropriate Clebsch-Gordan coefficients for constructing a state of total isotopic angular momentum I and z-component I_z from a nucleon in the isotopic spin state ρ and a meson in the state i ($i=1, 2, 3$).

The superscript (1) in $\Psi^{(1)}(I,I_z;\,\mathbf{k}_0,v)$ indicates it is a state of a physical nucleon with one additional meson. The scattering function $\chi_I{}^v(\mathbf{k},\mathbf{k}_0)$ and the constant $D^\rho(I,I_z)$ are determined by the variational method

$$\delta\langle\Psi^{(1)}(I,I_z;\,\mathbf{k}_0,v)\,|H_v-m\gamma-\omega_0|\,\Psi^{(1)}(I,I_z;\,\mathbf{k}_0,v)\rangle=0, \tag{22}$$

where m is the rest mass of the physical nucleon and ω_0 the energy of the additional meson. In the expression of H_v, Eq. (8), we set the total momentum of the system to be

$$\mathbf{P}=m\gamma\mathbf{v}+\mathbf{k}_0. \tag{23}$$

Thus, the variational principle gives

$$D^\rho(I,I_z)=-\left\langle N_\rho(v)\middle|C_i{}^{\rho'}(I,I_z)\times\int\chi_I{}^v(\mathbf{k},\mathbf{k}_0)a_i{}^\dagger(\mathbf{k})d^3k\middle|N_{\rho'}(v)\right\rangle, \tag{24}$$

which is the same as the requirement that the scattering states be orthogonal to the ground states. The scattering function $\chi_I{}^v(\mathbf{k},\mathbf{k}_0)$ is given by

$$\gamma[(\omega-\mathbf{k}\cdot\mathbf{v})-(\omega_0-\mathbf{k}_0\cdot\mathbf{v})]\chi_I{}^v(\mathbf{k},\mathbf{k}_0)=\int K_I{}^v(\mathbf{k},\mathbf{k}')\chi_I{}^v(\mathbf{k}',\mathbf{k}_0)d^3k' \tag{25}$$

where

$$K_I{}^v(\mathbf{k},\mathbf{k}')=F_v(\mathbf{k})F_v(\mathbf{k}')[U_I\gamma(\omega-\mathbf{k}\cdot\mathbf{v}+\omega'-\mathbf{k}'\cdot\mathbf{v})+V_I\gamma(\omega_0-\mathbf{k}_0\cdot\mathbf{v})+W_I].$$

The U_I, V_I, and W_I are constants depend only on the structure of the physical nucleon at rest. They are most conveniently given in terms of the matrix elements K, L, M, N, O which in turn are defined as follows:

$$\begin{aligned}\langle\mathfrak{N}_\rho|\,a_i{}^\dagger a_j\,|\mathfrak{N}_{\rho'}\rangle&=K\delta_{ij}\delta_{\rho\rho'}+iL\epsilon_{ijk}\langle n_\rho|\,\tau_k\,|n_{\rho'}\rangle,\\ \langle\mathfrak{N}_\rho|\,a_i{}^\dagger\tau_j\,|\mathfrak{N}_{\rho'}\rangle&=M\delta_{ij}\delta_{\rho\rho'}+iN\epsilon_{ijk}\langle n_\rho|\,\tau_k\,|n_{\rho'}\rangle,\end{aligned} \tag{26}$$

and

$$\langle\mathfrak{N}_\rho|\,a_i{}^\dagger\,|\mathfrak{N}_{\rho'}\rangle=O\langle n_\rho|\,\tau_i\,|n_{\rho'}\rangle,$$

where $|n_\rho\rangle$ is the bare-nucleon isotopic spinor and $|\mathfrak{N}_\rho\rangle$ is the corresponding physical nucleon in the reduced space given by Eq. (17). The δ_{ij}, ϵ_{ijk} are the two usual symmetric isotopic tensors. The U_I, V_I, W_I can then be expressed for various $I=\frac{1}{2}$ and $\frac{3}{2}$ states as

$$\begin{aligned}U_{\frac{3}{2}}&=\tfrac{1}{2}[-(K+L)+KM^{-1}(M+N)],\\ V_{\frac{3}{2}}&=K+L,\\ W_{\frac{3}{2}}&=KM^{-1}(M+N)\lambda,\end{aligned} \tag{27}$$

and

$$\begin{aligned}U_{\frac{1}{2}}&=\tfrac{1}{2}[-(K-2L)+KM^{-1}(M-2N)],\\ V_{\frac{1}{2}}&=K-2L-3O^2,\\ W_{\frac{1}{2}}&=KM^{-1}(M-2N)\lambda,\end{aligned}$$

where λ is defined in Eq. (18).

We notice from Eq. (25) that while the state vector $\Psi^{(1)}(I,I_z;\mathbf{k}_0,v)$ obeys the ordinary orthonormal condition

$$\langle\Psi^{(1)}(I,I_z;\mathbf{k}_0,v)|\Psi^{(1)}(I',I_z';\mathbf{k}_0',v)\rangle =\delta_{II'}\delta_{I_zI_z'}\delta^3(\mathbf{k}_0-\mathbf{k}_0'), \quad (28)$$

the $\chi_I^v(k,k_0)$ satisfies a complicated orthogonal relation

$$\int \chi_I^{v*}(\mathbf{k},\mathbf{k}_0)\langle\mathbf{k}|\mathcal{V}_I|\mathbf{k}'\rangle\chi_I^v(\mathbf{k}',\mathbf{k}_0')d^3kd^3k' =\delta^3(\mathbf{k}_0-\mathbf{k}_0'), \quad (29)$$

where the matrix $\mathcal{V}_I$ is

$$\langle\mathbf{k}|\mathcal{V}_I|\mathbf{k}'\rangle=\delta^3(\mathbf{k}-\mathbf{k}')+V_IF_v(\mathbf{k})F_v(\mathbf{k}'), \quad (30)$$

with V_I given by Eq. (27). Although $\chi_I^v(\mathbf{k},\mathbf{k}_0)$ can be solved readily from Eq. (25), for pure computational reasons it is much more convenient to transform $\chi_I^v(\mathbf{k},\mathbf{k}_0)$ to $\chi_I^v(\mathbf{k},\mathbf{k}_0)$ by

$$\chi_I^v(\mathbf{k},\mathbf{k}_0)=\int\langle\mathbf{k}|\mathcal{V}_I^{-\frac{1}{2}}|\mathbf{k}'\rangle\chi_I^v(\mathbf{k}',\mathbf{k}_0)d^3k' \quad (31)$$

such that

$$\int\chi_I^v(\mathbf{k},\mathbf{k}_0)\chi_I^v(\mathbf{k},\mathbf{k}_0')d^3k=\delta^3(\mathbf{k}_0-\mathbf{k}_0'). \quad (32)$$

From Eq. (30), it can be verified directly that[13]

$$\langle\mathbf{k}|\mathcal{V}_I^{-\frac{1}{2}}|\mathbf{k}'\rangle=\delta^3(\mathbf{k}-\mathbf{k}') +[-1+(1+V_I)^{-\frac{1}{2}}]F_v(\mathbf{k})F_v(\mathbf{k}'). \quad (33)$$

In terms of the transformed $\chi_I^v(\mathbf{k},\mathbf{k}_0)$ the matrix elements of $\mathcal{S}$ can be naturally separated into two parts: one that is dependent on momentum, and the other part that is independent of momentum and can be performed in the reduced space alone. To show this, we define a state vector $\psi^{(1)}(I,I_z)$ for a physical nucleon together with a meson in the *reduced space*[10]

$$|\psi^{(1)}(I,I_z)\rangle=(1+V_I)^{-\frac{1}{2}} \times[C_i^\rho(I,I_z)a_i^\dagger|\mathfrak{N}_\rho\rangle+\mathfrak{D}^\rho(I,I_z)|\mathfrak{N}_\rho\rangle, \quad (34)$$

where $C_i^\rho(I,I_z)$ are the same Clebsch-Gordon coefficients used in Eq. (21), and

$$\mathfrak{D}^\rho(I,I_z)=-\langle\mathfrak{N}_\rho|C_i^{\rho'}(I,I_z)a_i^\dagger|\mathfrak{N}_{\rho'}\rangle. \quad (35)$$

The $|\mathfrak{N}_\rho\rangle$ and $a_i^\dagger$ are the physical nucleon state vector and the creation operator of a meson in the reduced space. It is important to notice that $\psi^{(1)}(I,I_z)$ is independent of both k_0 and v. Furthermore, it satisfies the orthonormal relations

$$\langle\psi^{(1)}(I,I_z)|\mathfrak{N}_\rho\rangle=0$$

and

$$\langle\psi^{(1)}(I,I_z)|\psi^{(1)}(I',I_z')\rangle=\delta_{II'}\delta_{I_zI_z'}. \quad (36)$$

The matrix elements for production of mesons can now be put into a product form. By using Eqs. (21), (31), (33) and (34) we have

$$\langle\Psi^{(1)}(I,I_z;\mathbf{k}_0,v)|\tau_i|N_\rho(v)\rangle =G_I^v(\mathbf{k}_0)\langle\psi^{(1)}(I,I_z)|\tau_i|\mathfrak{N}_\rho\rangle, \quad (37)$$

where

$$G_I^v(\mathbf{k}_0)=\int\chi_I^v(\mathbf{k}',\mathbf{k}_0)F_v(\mathbf{k}')d^3k'.$$

Equation (37) expresses the main advantage of introducing $\chi_I^v(\mathbf{k},\mathbf{k}_0)$ and $\psi^{(1)}(I,I_z)$. The first term, $G_I^v(\mathbf{k}_0)$, gives the momentum distribution of the emitted meson, whereas the second term, $\langle\psi^{(1)}(I,I_z)|\tau_i|\mathfrak{N}_\rho\rangle$, is a matrix element which can be calculated completely in the reduced space. The explicit functional form of $G_I^v(\mathbf{k}_0)$ is given in Appendix I. We remark here that if the scattering amplitude, given by $\chi_I^v(\mathbf{k},\mathbf{k}_0)$, has a resonance-like behavior at a certain energy, then the effect of $G_I^V(\mathbf{k}_0)$ is to have mesons emitted predominantly at energies close to the resonant energy. On the other hand, if the scattering amplitude of $\chi_I^v(\mathbf{k},\mathbf{k}_0)$ is quite small, then we have[14]

$$\chi_I^v(\mathbf{k},\mathbf{k}_0)\cong\delta^3(\mathbf{k}-\mathbf{k}_0),$$

which causes the mesons to be emitted with a momentum distribution identical to $F_v(\mathbf{k})$. The matrix element for production then becomes

$$\langle\Psi^{(1)}(I,I_z;\mathbf{k}_0,v)|\tau_i|N_\rho(v)\rangle \cong F_v(\mathbf{k}_0)\langle\psi^{(1)}(I,I_z)|\tau_i|\mathfrak{N}_\rho\rangle. \quad (38)$$

The above variational procedure can easily be generalized to determine the state vectors involving a physical nucleon together with several additional mesons. To simplify our calculations, we shall include the effect of rescattering only for state vectors containing a physical nucleon with one meson. Thus, for calculations of matrix elements involving one nucleon and two additional mesons, it is only necessary to construct an orthonormal set of functions $\psi^{(2)}(I,I_z)$ in the reduced space.

$$\begin{aligned}|\psi^{(2)}(I,I_z)\rangle=&[E_{ij}^\rho(I,I_z)a_i^\dagger a_j^\dagger|\mathfrak{N}_\rho\rangle\\ &-\langle\psi^{(1)}(I,I_z)|E_{ij}^\rho(I,I_z)a_i^\dagger a_j^\dagger|\mathfrak{N}_\rho\rangle|\psi^{(1)}(I,I_z)\rangle\\ &-\langle\mathfrak{N}_{\rho'}|E_{ij}^\rho(I,I_z)a_i^\dagger a_j^\dagger|\mathfrak{N}_\rho\rangle|\mathfrak{N}_{\rho'}\rangle]\\ &\times\text{normalization constant,}\end{aligned} \quad (39)$$

where $|\psi^{(1)}(I,I_z)\rangle$ and $|\mathfrak{N}_\rho\rangle$ are given by Eqs. (34) and and (17). The constants $E_{ij}^\rho(I,I_z)$ are again the appropriate Clebsch-Gordan coefficients[15] for constructing a

[13] From the definition of V_I, it follows that $(1+V_I)$ is always a real and positive quantity.

[14] We wish to point out that the condition for small scattering amplitude $\chi_I^v(\mathbf{k},\mathbf{k}_0)\cong\delta^3(\mathbf{k}-\mathbf{k}_0)$, is not equivalent to $\chi_I^v(\mathbf{k},\mathbf{k}_0)\cong\delta^3(\mathbf{k}-\mathbf{k}_0)$, since the former equation maintains the orthonormal relations of the state vectors $\Psi^{(1)}(I,I_z;\mathbf{k},v)$, Eq. (28), while the latter one is not compatible with Eq. (28) except in the weak coupling limit.

[15] The coefficients $E_{ij}^\rho(I,I_z)$ and $E_{ji}^\rho(I,I_z)$ are equal. Actually there exists another way, antisymmetric in the meson indices i and j to form a total isotopic spin I, I_z from one nucleon and two mesons. However, in the approximation where we neglect the rescattering of mesons in a two-meson state, these two mesons must be emitted with the same momentum distribution $F_v(\mathbf{k})$. Consequently, only the state symmetric with respect to i and j can contribute in the production process.

state of total isotopic angular momentum I and its z-component I_z from a nucleon in the isotopic spin state ρ and two mesons in the states i and j, respectively. These state vectors satisfy the following orthonormal relations:

$$\langle\psi^{(2)}(I,I_z)|\psi^{(2)}(I',I_z')\rangle=\delta_{II'}\delta_{I_zI_z'},$$
$$\langle\psi^{(2)}(I,I_z)|\psi^{(1)}(I',I_z')\rangle=0,$$

and

$$\langle\psi^{(2)}(I,I_z)|\mathfrak{N}_\rho\rangle=0. \qquad (40)$$

The superscript (1) or (2) refers to states involving one or two additional mesons. Similar to Eq. (38), we have for the matrix elements

$$\langle\Psi^{(2)}(I,I_z;\mathbf{k}_1,\mathbf{k}_2,v)|\tau_i|N_\rho(v)\rangle = F_v(\mathbf{k}_1)F_v(\mathbf{k}_2)\langle\psi^{(2)}(I,I_z)|\tau_i|\mathfrak{N}_\rho\rangle, \qquad (41)$$

where $\Psi^{(2)}(I,I_z;\mathbf{k}_1,\mathbf{k}_2,v)$ denotes the state vector representing a physical nucleon with velocity v together with two mesons of momentum $\mathbf{k}_1$, $\mathbf{k}_2$ in a total isotopic spin state I and I_z. The effects due to rescattering of mesons are, of course, neglected in Eq. (41).

The following simple identities concerning the matrix elements $\langle\psi^{(n)}(I,I_z)|\tau_i|\mathfrak{N}_\rho\rangle$ $(n=1,2)$ in the reduced space are very useful in our later calculations. We list them as

$$\begin{aligned}
\langle\psi^{(1)}(\tfrac{3}{2},\tfrac{3}{2})|\tau_+|\mathcal{P}\rangle&=-\tfrac{1}{2}\sqrt{3}\langle\psi^{(1)}(\tfrac{3}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle,\\
\langle\psi^{(1)}(\tfrac{3}{2},-\tfrac{1}{2})|\tau_-|\mathcal{P}\rangle&=\tfrac{1}{2}\langle\psi^{(1)}(\tfrac{3}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle,\\
\langle\psi^{(1)}(\tfrac{1}{2},-\tfrac{1}{2})|\tau_-|\mathcal{P}\rangle&=\langle\psi^{(1)}(\tfrac{1}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle,\\
\langle\psi^{(2)}(\tfrac{3}{2},\tfrac{3}{2})|\tau_+|\mathcal{P}\rangle&=-\tfrac{1}{2}\sqrt{3}\langle\psi^{(2)}(\tfrac{3}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle,
\end{aligned} \qquad (42)$$

and

$$\langle\mathfrak{N}|\tau_-|\mathcal{P}\rangle=\langle\mathcal{P}|\tau_3|\mathcal{P}\rangle,$$

where we represent

$$|\mathfrak{N}_{\rho=1}\rangle=|\mathcal{P}\rangle,\quad |\mathfrak{N}_{\rho=2}\rangle=|\mathfrak{N}\rangle$$

and

$$\tau_\pm=\tfrac{1}{2}(\tau_1\pm i\tau_2).$$

VI. MATRIX ELEMENTS AND CROSS SECTIONS

For purposes of illustration we first consider the collision of two protons, in the center-of-mass system, with the emission of a single π^+ meson of momentum $\mathbf{k}$. This reaction can be written as

$$P+P\rightarrow P+N+\pi^+(\mathbf{k}). \qquad (\mathrm{I})$$

If $+v$ and $-v$ are the initial velocities of the two nucleons, then there are two *different* final states for this reaction, corresponding to whether the proton or neutron moves forward after the collision. In our approximation, these two final states do not interfere since the two nucleons have quite different momenta in the initial state, and for these two states to interfere would require an exchange of momentum equal to the difference of their initial momenta. As we assume that $(\Delta v/v)\ll 1$, we imply that such a large momentum exchange is unlikely.

Each of these two final states can be reached in several ways, depending on which nucleon emits the meson. The square of the matrix element, M_{I}, for reaction (I) can then be expressed as

$$\begin{aligned}
|M_{\mathrm{I}}|^2=&|\langle P+\pi^+(\mathbf{k}),N|\mathcal{S}|P,P\rangle\\
&+\langle P,N+\pi^+(\mathbf{k})|\mathcal{S}|P,P\rangle|^2+|\langle N+\pi^+(\mathbf{k}),P|\mathcal{S}|P,P\rangle\\
&+\langle N,P+\pi^+(\mathbf{k})|\mathcal{S}|P,P\rangle|^2,
\end{aligned} \qquad (43)$$

where the state vectors always have the order of the nucleons arranged such that the one to the left is moving forward, while the one to the right is moving backwards in the center-of-mass system. Thus, for example, $\langle P+\pi^+(\mathbf{k}),N|\mathcal{S}|P,P\rangle$ represents the matrix element for a final state of a proton that moves forward and emits a positive meson with momentum $\mathbf{k}$ while the neutron moves in the backward direction without emitting any meson. Similarly, the term $\langle P,N+\pi^+(\mathbf{k})|\mathcal{S}|P,P\rangle$ represents a final state with a neutron that moves backward and emits a meson, while the proton moves forward without any emission. From Eq. (19), these matrix elements can be written as

$$\begin{aligned}
&\langle P+\pi^+(\mathbf{k}),N|\mathcal{S}|P,P\rangle\\
&\qquad=2S\langle P+\pi^+(\mathbf{k})|\tau_+|P\rangle_{+v}\langle N|\tau_-|P\rangle_{-v},\\
&\langle P,N+\pi^+(\mathbf{k})|\mathcal{S}|P,P\rangle\\
&\qquad=S\langle P|\tau_3|P\rangle_{+v}\langle N+\pi^+(\mathbf{k})|\tau_3|P\rangle_{-v},\\
&\langle N+\pi^+(\mathbf{k}),P|\mathcal{S}|P,P\rangle\\
&\qquad=S\langle N+\pi^+(\mathbf{k})|\tau_3|P\rangle_{+v}\langle P|\tau_3|P\rangle_{-v},\\
&\langle N,P+\pi^+(\mathbf{k})|\mathcal{S}|P,P\rangle\\
&\qquad=2S\langle N|\tau_-|P\rangle_{+v}\langle P+\pi^+(\mathbf{k})|\tau_+|P\rangle_{-v}.
\end{aligned} \qquad (44)$$

In Eqs. (44), we find it more instructive to represent the state vectors as

$$\begin{aligned}
|P\rangle_v&=|N_{\rho=1}(v)\rangle,\\
|N\rangle_v&=|N_{\rho=2}(v)\rangle,\\
|P+\pi^+(\mathbf{k})\rangle_v&=|\Psi^{(1)}(\tfrac{3}{2},\tfrac{3}{2};\mathbf{k},v)\rangle,
\end{aligned} \qquad (45)$$

and

$$|N+\pi^+(\mathbf{k})\rangle_v=(\tfrac{1}{3})^{\frac{1}{2}}|\Psi^{(1)}(\tfrac{3}{2},\tfrac{1}{2};\mathbf{k},v)\rangle + (\tfrac{2}{3})^{\frac{1}{2}}|\Psi^{(1)}(\tfrac{1}{2},\tfrac{1}{2};\mathbf{k},v)\rangle.$$

The subscript $\pm v$ in $\langle\ \rangle_{\pm v}$ means that the matrix elements must be evaluated for states containing a nucleon of that velocity.

It is convenient to express the momentum dependence of these matrix elements explicitly. With the use of Eq. (37), Eq. (42) and Eq. (43), the matrix element for reaction (I) can be written as

$$\begin{aligned}
|M_{\mathrm{I}}|^2=&|S\langle\mathcal{P}|\tau_3|\mathcal{P}\rangle|^2\{\langle\psi^{(1)}(\tfrac{3}{2},\tfrac{3}{2})|\tau_3|\mathcal{P}\rangle\\
&\times[-\sqrt{3}G_{\frac{3}{2}}^{v}(\mathbf{k})+(\tfrac{1}{3})^{\frac{1}{2}}G_{\frac{3}{2}}^{-v}(\mathbf{k})]\\
&+(\tfrac{2}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{1}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{1}{2}}^{-v}(\mathbf{k})\}^2
\end{aligned} \qquad (46)$$

+identical terms but with the superscripts $\pm v$ replaced by $\mp v$.

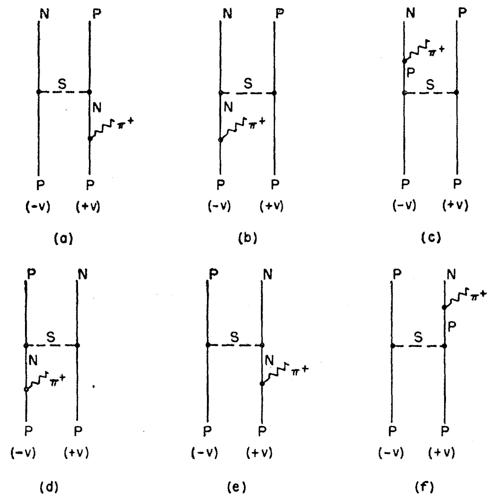

FIG. 1. Feynman diagrams for reaction $P+P\rightarrow P+N+\pi^+$.

The expression for the differential cross section $d\sigma_{\mathrm{I}}$ for reaction (I) is given by

$$d\sigma_{\mathrm{I}}=(2\pi)^{-5}v_r^{-1}|M_{\mathrm{I}}|^2\delta^3(\mathbf{p}_1+\mathbf{p}_2+\mathbf{k})\times\delta(\omega_k+\epsilon_1+\epsilon_2-E)d^3p_1d^3p_2d^3k,$$

where v_r is the relative velocity of the two initial protons; $\mathbf{p}_1$, $\mathbf{p}_2$ and ϵ_1, ϵ_2 are the final momenta and energies of the two nucleons; E is the total energy of the system. In evaluating the phase space integral, we make the following approximation suggested by Fermi.[2] We assume that the meson momentum $\mathbf{k}$ is much smaller than $\mathbf{p}_1$ or $\mathbf{p}_2$ since the nucleons, being heavier, tend to have large momenta favored by the phase-space integral. We therefore neglect the meson momentum in the delta function for momentum conservation. Thus, we can integrate $d\sigma_{\mathrm{I}}$ with respect to the nucleon momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ and the resulting differential cross section in terms of the meson momentum k can be written as

$$d\sigma_I=(2\pi)^{-5}(8v_r)^{-1}\left(\int d\Omega_{p1}\right)|M_{\mathrm{I}}|^2\times[(E-\omega)^2-4m^2]^{\frac{1}{2}}[E-\omega]d^3k. \quad (47)$$

The term $\int d\Omega_{p1}$, which is the integral extended over the direction of final nucleon momentum, is merely a multiplicative factor, and thus will not influence the relative cross sections for various production processes.

Some insight as to the meaning of these matrix elements, Eq. (43), may be obtained by a comparison with the usual Feynman diagrams. We shall discuss this relationship for small values of the coupling constants. In the weak-coupling limit, the state vectors $|N+\pi^+(\mathbf{k})\rangle_v$ and $|P+\pi^+(\mathbf{k})\rangle_v$, introduced in Eq. (45), become

$$|N+\pi^+(\mathbf{k})\rangle_v\rightarrow\alpha_{\mathbf{k}}^+|N\rangle_v-\langle P|\alpha_{\mathbf{k}}^+|N\rangle_v|P\rangle_v$$

and

$$|P+\pi^+(\mathbf{k})\rangle_v\rightarrow\alpha_{\mathbf{k}}^+|P\rangle_v,$$

where $\alpha_{\mathbf{k}}^\dagger=-[a_1^\dagger(\mathbf{k})+ia_2^\dagger(\mathbf{k})]/\sqrt{2}$ represents the creation operator for a positive meson of momentum $\mathbf{k}$. A direct substitution of these terms into Eqs. (44) yields

$$\langle P+\pi^+(\mathbf{k}),N|\mathcal{S}|P,P\rangle\rightarrow 2S\langle P|\alpha_{\mathbf{k}}\tau_+|P\rangle_{+v}\langle N|\tau_-|P\rangle_{-v}, \quad (48)$$

and

$$\langle P,N+\pi^+(\mathbf{k})|\mathcal{S}|P,P\rangle\rightarrow S\langle P|\tau_3|P\rangle_{+v}\langle N|\alpha_{\mathbf{k}}\tau_3|P\rangle_{-v} -S\langle P|\tau_3|P\rangle_{+v}\langle P|\tau_3|P\rangle_{-v}\langle N|\alpha_{\mathbf{k}}|P\rangle_{-v}. \quad (49)$$

By using the weak-coupling-limit expression for $|P\rangle_{\pm v}$ and $|N\rangle_{\pm v}$ one can easily identify the term in Eq. (48) with the Feynman diagram (a) in Fig. 1 and the two terms in Eq. (49) with the Feynman diagrams (b) and (c), respectively. Similarly, by exchanging $\pm v$ with $\mp v$, contributions due to $\langle N,P+\pi^+(\mathbf{k})|\mathcal{S}|P,P\rangle$ and $\langle N+\pi^+(\mathbf{k}),P|\mathcal{S}|P,P\rangle$ can be identified with diagrams (d), (e) and (f) in Fig. 1.

The expressions of matrix elements and cross sections for other production processes may be written in a similar way. In the following, we shall list these expressions for all single and double meson production processes for proton-proton collision. They are, besides reaction (I),

$$P+P\rightarrow P+P+\pi^0(\mathbf{k}), \quad (\mathrm{II})$$

$$P+P\rightarrow P+N+\pi^+(\mathbf{k}_1)+\pi^0(\mathbf{k}_2), \quad (\mathrm{III})$$

$$P+P\rightarrow P+P+\pi^0(\mathbf{k}_1)+\pi^0(\mathbf{k}_2), \quad (\mathrm{IV})$$

$$P+P\rightarrow P+P+\pi^+(\mathbf{k}_1)+\pi^-(\mathbf{k}_2), \quad (\mathrm{V})$$

and

$$P+P\rightarrow N+N+\pi^+(\mathbf{k}_1)+\pi^+(\mathbf{k}_2). \quad (\mathrm{VI})$$

The corresponding matrix elements for these reactions may be written, respectively, as

$$|M_{\mathrm{II}}|^2=|\langle P+\pi^0(\mathbf{k}),P|\mathcal{S}|P,P\rangle +\langle P,P+\pi^0(\mathbf{k})|\mathcal{S}|P,P\rangle|^2, \quad (50)$$

$$\begin{aligned}|M_{\mathrm{III}}|^2=|&\langle P+\pi^+(\mathbf{k}_1),N+\pi^0(\mathbf{k}_2)|\mathcal{S}|P,P\rangle\\&+\langle P+\pi^0(\mathbf{k}_2),N+\pi^+(\mathbf{k}_1)|\mathcal{S}|P,P\rangle\\&+\langle P+\pi^+(\mathbf{k}_1)+\pi^0(\mathbf{k}_2),N|\mathcal{S}|P,P\rangle\\&+\langle P,N+\pi^+(\mathbf{k}_1)+\pi^0(\mathbf{k}_2)|\mathcal{S}|P,P\rangle|^2\\&+\text{identical terms, but with the order of}\\&P\text{ and }N\text{ interchanged in the final states;}\end{aligned} \quad (51)$$

$$\begin{aligned}|M_{\mathrm{IV}}|^2=\tfrac{1}{2}|&\langle P+\pi^0(\mathbf{k}_1),P+\pi^0(\mathbf{k}_2)|\mathcal{S}|P,P\rangle\\&+\langle P+\pi^0(\mathbf{k}_2),P+\pi^0(\mathbf{k}_1)|\mathcal{S}|P,P\rangle\\&+\sqrt{2}\langle P+\pi^0(\mathbf{k}_1)+\pi^0(\mathbf{k}_2),P|\mathcal{S}|P,P\rangle\\&+\sqrt{2}\langle P,P+\pi^0(\mathbf{k}_1)+\pi^0(\mathbf{k}_2)|\mathcal{S}|P,P\rangle|^2;\end{aligned} \quad (52)$$

$$\begin{aligned}|M_{\mathrm{V}}|^2=|&\langle P+\pi^+(\mathbf{k}_1),P+\pi^-(\mathbf{k}_2)|\mathcal{S}|P,P\rangle\\&+\langle P+\pi^-(\mathbf{k}_2),P+\pi^+(\mathbf{k}_1)|\mathcal{S}|P,P\rangle\\&+\langle P+\pi^+(\mathbf{k}_1)+\pi^-(\mathbf{k}_2),P|\mathcal{S}|P,P\rangle\\&+\langle P,P+\pi^+(\mathbf{k}_1)+\pi^-(\mathbf{k}_2)|\mathcal{S}|P,P\rangle|^2;\end{aligned} \quad (53)$$

and

$$|M_{\mathrm{VI}}|^2=\tfrac{1}{2}|\langle N+\pi^+(\mathbf{k}_1), N+\pi^+(\mathbf{k}_2)|\mathcal{S}|P, P\rangle$$
$$+\langle N+\pi^+(\mathbf{k}_2), N+\pi^+(\mathbf{k}_1)|\mathcal{S}|P, P\rangle$$
$$+\sqrt{2}\langle N+\pi^+(\mathbf{k}_1)+\pi^+(\mathbf{k}_2), N|\mathcal{S}|P,P\rangle$$
$$+\sqrt{2}\langle N, N+\pi^+(\mathbf{k}_1)+\pi^+(\mathbf{k}_2)|\mathcal{S}|P, P\rangle|^2, \quad (54)$$

where the meaning of these matrix elements is similar to that of the matrix elements given in Eq. (43). The factors $\frac{1}{2}$ and $\sqrt{2}$ in Eq. (52) and Eq. (54) are due to our convention of letting both $\mathbf{k}_1$ and $\mathbf{k}_2$ vary through the entire k-domain independently, even though they may represent momenta of two identical particles. Thus, the normalizations of the states $|N+\pi^+(\mathbf{k}_1)+\pi^+(\mathbf{k}_2)\rangle$ and $|P+\pi^0(\mathbf{k}_1)+\pi^0(\mathbf{k}_2)\rangle$ are given as

$$\int\langle N+\pi^+(\mathbf{k}_1)+\pi^+(\mathbf{k}_2)|N+\pi^+(\mathbf{k}_1')+\pi^+(\mathbf{k}_2')\rangle \times d^3k_1d^3k_2=1,$$

and

$$\int\langle P+\pi^0(\mathbf{k}_1)+\pi^0(\mathbf{k}_2)|P+\pi^0(\mathbf{k}_1')+\pi^0(\mathbf{k}_2')\rangle \times d^3k_1d^3k_2=1,$$

with $\mathbf{k}_1$, $\mathbf{k}_2$ each varying through the entire $\mathbf{k}$-space. The explicit $\mathbf{k}$-dependence of these matrix elements can be obtained by using Eqs. (19), (37), and (41). They are all tabulated in Appendix II. The cross sections for these reactions can be obtained in a similar way as for the reaction (I). After integration over the nucleon momenta, the expression of $d\sigma$ for reaction (II) is identical with Eq. (47) except for the replacement of the subscript I by II. The corresponding expression for any two-meson emission process [reactions (III)–(VI)] is given by

$$d\sigma=(2\pi)^{-5}(8v_r)^{-1}\left(\int d\Omega_{p1}\right)|M|^2[(E-\omega_1-\omega_2)^2-4m^2]^{\frac{1}{2}} \times[E-\omega_1-\omega_2]d^3k_1d^3k_2. \quad (55)$$

The variables $\mathbf{k}_1$ and $\mathbf{k}_2$, again, vary over the entire $\mathbf{k}$-space independently, even in cases where they represent two identical particles.

VII. NUMERICAL RESULTS AND CONCLUSIONS

In order to see the variation of these matrix elements and cross sections with the coupling constant g, it is necessary to perform a numerical calculation by first solving for the state of a physical nucleon $|\mathfrak{N}_\rho\rangle$ in the reduced space. One can then use this state function to calculate the various matrix elements in the reduced space together with the momentum distribution functions $G_{\frac{3}{2}}{}^v(\mathbf{k})$, $G_{\frac{1}{2}}{}^v(\mathbf{k})$, and $F_v(\mathbf{k})$. As the purpose of the present calculations on the symmetrical scalar case is merely to illustrate our method, instead of doing laborious but exact numerical calculations, we shall give here a simple analytic approximate form of $|\mathfrak{N}_\rho\rangle$ which, however, does give rigorous solutions at both strong and weak limits.

According to Eq. (17), $|\mathfrak{N}_\rho\rangle$ is determined by

$$\mathcal{H}_0|\mathfrak{N}_\rho\rangle=(E+m_0)|\mathfrak{N}_\rho\rangle,$$

with $E=m-m_0$. From consideration of invariance, the state vector $|\mathfrak{N}_\rho\rangle$ must be of the form

$$|\mathfrak{N}_\rho\rangle=\sum_{m=0}^{\infty} C_{2m}(a_i{}^\dagger a_i{}^\dagger)^m+C_{2m+1}(a_i{}^\dagger a_i{}^\dagger)^m\tau_j a_j{}^\dagger]|n_\rho\rangle, \quad (56)$$

where $|n_\rho\rangle$ is the state vector for a bare nucleon in the isotopic spin state ρ. The constants C_{2m} and C_{2m+1} obey the following difference equations:

$$\begin{aligned}(2m\Omega-E)C_{2m}+GC_{2m-1}+G(2m+3)C_{2m+1}&=0,\\ [(2m+1)\Omega-E]C_{2m+1}+GC_{2m}+G(2m+2)C_{2m+2}&=0,\end{aligned} \quad (57)$$

where G and Ω have been introduced in Eq. (16).

These equations have simple solutions for very small and very large values of G. For $G\ll 1$, we obtain

$$C_0=1,\quad C_1=-G/\Omega,\quad \text{and}\quad E=-3G^2\Omega, \quad (58)$$

and for $G\gg 1$ we have

$$C_n=[-G/\Omega]^n/n! \quad \text{with} \quad E=-G^2\Omega. \quad (59)$$

For intermediate values of G, Eq. (57) can be solved numerically. However, in order to gain some insight, we shall use a variational procedure and choose as trial function

$$C_n=x^{n/2}/n! \quad (n=0, 1, \cdots, \infty), \quad (60)$$

which agrees with the forms of the rigorous solutions at strong and weak coupling limits. The parameter x is determined in a variational way. By substituting (60) into (56) and using the definition of E, we have for the expression of E in terms of x,

$$E=\Omega\{(x^2+6x+3)\cosh x+(x^2+5x+6)\sinh x-3 +2(G/\Omega)x^{\frac{1}{2}}[(x+3)\cosh x+(x+2)\sinh x]\} \times\{(1+x)\cosh x+(2+x)\sinh x\}^{-1}. \quad (61)$$

Upon minimizing E with respect to x, we have

$$(x^2+2x-3)\cosh^2x-(3x^2+8x+4)\sinh^2x -(2x^2+6x+8)\cosh x\sinh x+(G/\Omega)x^{-\frac{1}{2}} \times[(3x^2+8x-3)\cosh x-(5x+12x+4)\sinh^2x -(2x^2+4x+8)\sinh x\cosh x]=0, \quad (62)$$

from which x can be readily obtained as a function of G and Ω. This relation is tabulated in Table I.

To obtain the explicit numerical values for the various matrix elements, we choose for the source function (for $v=0$)

$$u_0(\mathbf{k})=1 \quad \text{for} \quad \omega\leq 6.21\mu$$

and

$$u_0(\mathbf{k})=0 \quad \text{for} \quad \omega>6.21\mu. \quad (63)$$

The upper cutoff in momentum is the same as the one used in the symmetrical pseudoscalar case[7] in order to

TABLE I. Various matrix elements in the reduced space.[a]

x	G/Ω	$g^2/4\pi$	$\langle \mathcal{P} \| \tau_3 \| \mathcal{P} \rangle$	$\langle \psi^{(1)}(\frac{3}{2},\frac{1}{2}) \| \tau_3 \| \mathcal{P} \rangle$	$\langle \psi^{(1)}(\frac{1}{2},\frac{1}{2}) \| \tau_3 \| \mathcal{P} \rangle$	$\langle \psi^{(2)}(\frac{3}{2},\frac{1}{2}) \| \tau_3 \| \mathcal{P} \rangle$	$\langle \psi^{(2)}(\frac{1}{2},\frac{1}{2}) \| \tau_3 \| \mathcal{P} \rangle$
$x \to 0$	$-x^{\frac{1}{2}}$	$2.06x$	1	$2(2x/3)^{\frac{1}{2}}$	$4(x/3)^{\frac{1}{2}}$	$-2(10x^2/3)^{\frac{1}{2}}$	$-4(2x^2/3)^{\frac{1}{2}}$
0.02	−0.148	0.045	0.925	0.214	0.301	−0.067	−0.061
0.05	−0.247	0.126	0.826	0.304	0.426	−0.145	−0.141
0.1	−0.382	0.301	0.615	0.367	0.453	−0.244	−0.240
0.2	−0.627	0.811	0.515	0.402	0.475	−0.341	−0.351
0.4	−1.019	2.139	0.332	0.395	0.311	−0.351	−0.391
0.6	−1.234	3.137	0.255	0.379	0.166	−0.317	−0.331
0.8	−1.340	3.699	0.225	0.371	0.072	−0.242	−0.218
1	−1.417	4.136	0.215	0.368	0.018	−0.203	−0.136
3	−2.019	8.397	0.259	0.405	−0.029	−0.134	+0.015
$x \to \infty$	$-x^{\frac{1}{2}}$	$2.06x$	$\frac{1}{3}$	$\sqrt{2}/3$	$-1/(3x^{\frac{3}{2}})$	$-\sqrt{2}/(3x)$	$\sqrt{2}/(3x^2)$

[a] Other matrix elements may be obtained by using the following identities:

$$\langle \psi^{(1)}(\tfrac{3}{2},\tfrac{3}{2}) | \tfrac{1}{2}(\tau_1+i\tau_2) | \mathcal{P} \rangle = -(\sqrt{3}/2)\langle \psi^{(1)}(\tfrac{3}{2},\tfrac{1}{2}) | \tau_3 | \mathcal{P} \rangle,$$
$$\langle \psi^{(1)}(\tfrac{3}{2},-\tfrac{1}{2}) | \tfrac{1}{2}(\tau_1-i\tau_2) | \mathcal{P} \rangle = \tfrac{1}{2}\langle \psi^{(1)}(\tfrac{3}{2},\tfrac{1}{2}) | \tau_3 | \mathcal{P} \rangle,$$
$$\langle \psi^{(1)}(\tfrac{1}{2},-\tfrac{1}{2}) | \tfrac{1}{2}(\tau_1-i\tau_2) | \mathcal{P} \rangle = \langle \psi^{(1)}(\tfrac{1}{2},\tfrac{1}{2}) | \tau_3 | \mathcal{P} \rangle,$$
$$\langle \psi^{(2)}(\tfrac{3}{2},\tfrac{3}{2}) | \tfrac{1}{2}(\tau_1+i\tau_2) | \mathcal{P} \rangle = -(\sqrt{3}/2)\langle \psi^{(2)}(\tfrac{3}{2},\tfrac{1}{2}) | \tau_3 | \mathcal{P} \rangle$$

and

$$\langle \mathfrak{N} | \tfrac{1}{2}(\tau_1-i\tau_2) | \mathcal{P} \rangle = \langle \mathcal{P} | \tau_3 | \mathcal{P} \rangle.$$

fit the experimental scattering data of π mesons. Furthermore, as a matter of expediency, we shall neglect the term λ in Eq. (18) for $F_v(\mathbf{k})$. The function $F_v(\mathbf{k})$, then, becomes

$$F_v(\mathbf{k}) \propto u_v(\mathbf{k})/[\omega^{\frac{1}{2}}\gamma(\omega-\mathbf{k}\cdot\mathbf{v})], \tag{64}$$

with the proportionality constant so chosen as to make $F_v(\mathbf{k})$ still normalized according to Eq. (14). By using (63) and (64) and the definition of G and Ω, we find

$$g^2/4\pi = (2.06)(G/\Omega)^2. \tag{65}$$

The detailed values of various matrix elements for various values of x, hence also of $g^2/4\pi$, are listed in Table I. From Table I, we see that the relative magnitudes of these matrix elements differ violently for large values of coupling constant as compared to their values from the weak-coupling formulas. In particular, we notice that for very large values of coupling constants, only matrix elements of the type $\langle \mathfrak{N}_\rho | \tau_i | \mathfrak{N}_{\rho'} \rangle$ and $\langle \psi^{(1)}(\frac{3}{2},I_z) | \tau_i | \mathfrak{N}_\rho \rangle$ approach finite constants, while the rest all become extremely small and approach zero in the limit. This means, physically, that for large values of the coupling constant the influence of these matrix elements is to have each nucleon prefer the emission of zero or one meson, with the meson emitted in the $I=\frac{3}{2}$ states.[16] Another interesting feature is that for weak coupling the probability amplitude for two-meson production is proportional to g^2 while the probability amplitude for one-meson production is proporportional to g. However, for large coupling constant both these probabilities approach constant values as limits.

Next, we consider the effects due to the momentum distribution functions of the emitted mesons. Using the explicit expression of $\chi_I{}^v(\mathbf{k},\mathbf{k}')$, derived in Appendix I, we notice that at

$$g^2/4\pi = 4.7, \tag{66}$$

the scattering state of π mesons for $I=\frac{3}{2}$ has a 90° phase shift at zero incident kinetic energy, $\omega=\mu$. This means the nucleon-meson system has a bound isobaric state of $I=\frac{3}{2}$ for $g^2/4\pi \geq 4.7$.[17] We calculate the functional form $G_{\frac{3}{2}}{}^0(\mathbf{k})$ and $G_{\frac{1}{2}}{}^0(\mathbf{k})$ in a system where $v=0$. These functions for $g^2/4\pi=4.1$ are plotted in Fig. 2 together with $F_0(\mathbf{k})$. The corresponding functions for $v\neq 0$ can be obtained through a Lorentz transformation by

$$G_I{}^v(\mathbf{k}')(\omega')^{\frac{1}{2}} = G_I{}^0(\mathbf{k})\omega^{\frac{1}{2}} \quad (I=\tfrac{3}{2},\tfrac{1}{2}), \tag{67}$$

and

$$F_v(\mathbf{k}')(\omega')^{\frac{1}{2}} = F_0(\mathbf{k})(\omega)^{\frac{1}{2}}.$$

In the weak-coupling limit, both $G_{\frac{3}{2}}{}^v(\mathbf{k})$ and $G_{\frac{1}{2}}{}^v(\mathbf{k})$ become identical with $F_v(\mathbf{k})$. However for large coupling constants as shown by Fig. (2), $G_{\frac{3}{2}}{}^v(\mathbf{k})$ tends to make the energy of the mesons emitted in the $I=\frac{3}{2}$ state lie close to the low resonance energy, while no such effect is shown by $G_{\frac{1}{2}}{}^v(\mathbf{k})$ for the $I=\frac{1}{2}$ state. Furthermore, for $\omega<4$, $G_{\frac{1}{2}}{}^0(\mathbf{k})$ is always smaller than $G_{\frac{3}{2}}{}^0(\mathbf{k})$. Because of the over-all energy conservation, mesons with very

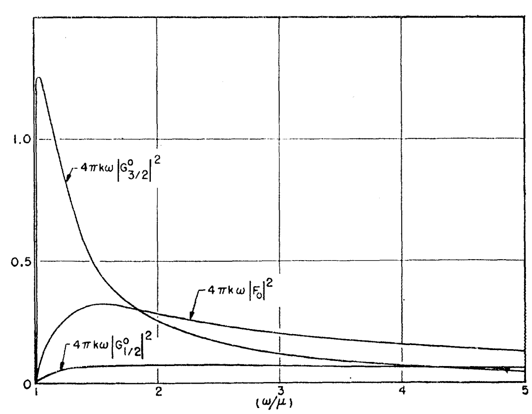

FIG. 2. The functions $4\pi k\omega |G_{\frac{3}{2}}{}^0(\omega)|^2$, $4\pi k\omega |G_{\frac{1}{2}}{}^0(\omega)|^2$ and $4\pi k\omega |F_0(\omega)|^2$ *vs* ω for $g^2/4\pi=4.1$.

[16] We limit ourselves in the present discussion only to single and double meson production processes.

[17] For $g^2/4\pi \geq 4.7$, the state functions discussed in Sec. V have to be generalized to contain the isobar state.

high values of kinetic energy are not allowed. Consequently, for large values of the coupling constant this effect of $G_{\frac{3}{2}}^{v}(\mathbf{k})$, together with the phase space integral is to enhance even more the probability for emitting mesons into a state with $I=\frac{3}{2}$ as compared to $I=\frac{1}{2}$.

In conclusion, we remark that although a test of the validity of our model can only be given by applying it to the symmetrical pseudoscalar theory, the results found above for a symmetrical scalar case seem to indicate that for large values of coupling constant the behavior of this model for multiple meson production does resemble, in a general way, those observed experimentally.[1]

ACKNOWLEDGMENTS

The authors wish to thank Professor R. Serber for several stimulating discussions.

APPENDIX I

We discuss in this appendix the detailed solution of the scattering integral equation. It is most convenient to discuss these functions in the system where $v=0$; then the corresponding functions for $v\neq 0$ are given by

$$\begin{aligned}\chi_I^v(\mathbf{k}_1',\mathbf{k}_2')(\omega_1'\omega_2')^{\frac12}&=\chi_I^0(\mathbf{k}_1,\mathbf{k}_2)(\omega_1\omega_2)^{\frac12},\\ F_v(\mathbf{k}')(\omega')^{\frac12}&=F_0(\mathbf{k})\omega^{\frac12},\\ \chi_I^v(\mathbf{k}_1',\mathbf{k}_2')(\omega_1'\omega_2')^{\frac12}&=\chi_I^0(\mathbf{k}_1\mathbf{k}_2)(\omega_1\omega_2)^{\frac12},\end{aligned}\tag{A.1}$$

where $\mathbf{k},\omega$ are related to $\mathbf{k}',\omega'$ by a Lorentz transformation. All functions with superscript or subscript 0 are referred to the system with $v=0$.

The integral equation for $\chi_I^0(\mathbf{k},\mathbf{k}_0)$ may be written as

$$(\omega_k-\omega_0)\chi_I^0(\mathbf{k},\mathbf{k}_0)=\int K_I^0(\mathbf{k},\mathbf{k}')\chi_I^0(\mathbf{k}',\mathbf{k}_0)d^3k',\tag{A.2}$$

where

$$K_I^0(\mathbf{k},\mathbf{k}')=\tfrac12 A_IF_0(\mathbf{k})F_0(\mathbf{k}')(\omega+\omega')+B_IF_0(\mathbf{k})F_0(\mathbf{k}').$$

The constants A_I, B_I, are related to U_I, V_I, W_I given in Eq. (27), by

$$\begin{aligned}A_I&=2U_I,\\ B_I&=V_I\omega_0+W_I.\end{aligned}\tag{A.3}$$

Upon using Eqs. (25), (31), and (33) we have the integral equation for $\chi_I^0(\mathbf{k},\mathbf{k}_0)$

$$(\omega_k-\omega_0)\chi_I^0(\mathbf{k},\mathbf{k}_0)=\int \mathcal{K}_I^0(\mathbf{k},\mathbf{k}')\chi_I^0(\mathbf{k}',\mathbf{k}_0)d^3k',\tag{A.4}$$

with

$$\mathcal{K}_I^0(\mathbf{k},\mathbf{k}')=\tfrac12\mathcal{A}_IF_0(\mathbf{k})F_0(\mathbf{k}')(\omega+\omega')+\mathcal{B}_IF_0(\mathbf{k})F_0(\mathbf{k}').$$

The constants $\mathcal{A}_I$ and $\mathcal{B}_I$ are given as

$$\begin{aligned}\mathcal{A}_I&=2[1-(1-U_I)(1+V_I)^{-\frac12}],\\ \mathcal{B}_I&=-\Omega+2\Omega(1-U_I)(1+V_I)^{-\frac12}\\ &\quad+(2U_I\Omega-\Omega+W_I)(1+V_I)^{-1}.\end{aligned}\tag{A.5}$$

The integral equation (A.4) can now be solved directly. Its solution is

$$\begin{aligned}\chi_I^0(\mathbf{k},\mathbf{k}_0)=\delta^3(\mathbf{k}-\mathbf{k}_0)+(\omega-\omega_0+i\epsilon)^{-1}&\\ \times[\xi_I(\mathbf{k}_0)F_0(\mathbf{k})\omega+\eta_I(\mathbf{k}_0)F_0(\mathbf{k})],&\end{aligned}\tag{A.6}$$

with

$$\begin{aligned}\xi_I(\mathbf{k}_0)&=\tfrac12[D_I^0(\mathbf{k}_0)]^{-1}\mathcal{A}_IF_0(\mathbf{k}_0)(1-\tfrac12\mathcal{A}_I)\\ \eta_I(\mathbf{k}_0)&=[D_I^0(\mathbf{k}_0)]^{-1}[\tfrac12\mathcal{A}_I\omega_0+\mathcal{B}_I+(\tfrac12\mathcal{A}_I)^2\Omega]F_0(\mathbf{k}_0).\end{aligned}$$

The function $D_I^0(\mathbf{k}_0)$ is

$$\begin{aligned}D_I^0(\mathbf{k}_0)=1-\mathcal{A}_I-(\mathcal{A}_I\omega_0+\mathcal{B}_I)\langle F_0^2\rangle&\\ +(\tfrac12\mathcal{A}_I)^2[1+(\omega_0-\Omega)\langle F_0^2\rangle]+i4\pi^2k_0\omega_0F_0^2(\mathbf{k}_0)&\\ \times[\mathcal{A}_I\omega_0+\mathcal{B}_I+(\tfrac12\mathcal{A}_I)^2(\Omega-\omega_0)],&\end{aligned}$$

where

$$\langle F_0^2\rangle=\int\frac{F_0^2(\mathbf{k})}{\omega-\omega_0}d^3k.\tag{A.7}$$

We choose $\chi_I^0(\mathbf{k},\mathbf{k}_0)$ to be the solution with an incoming wave, as it will be used in final state wave functions.

Let $G_v(\mathbf{k})$ be defined as

$$G_v(\mathbf{k})=\int\chi_I^v(\mathbf{k}',\mathbf{k})F_v(\mathbf{k}')d^3k'.\tag{A.8}$$

According to Eq. (37), $G_v(\mathbf{k})$ represents the orbital momentum distribution of the emitted meson. From Eq. (A.6), we have for $v=0$,

$$G_0(\mathbf{k})=F_0(\mathbf{k})[1-\tfrac12\mathcal{A}_I][D_I^0(\mathbf{k})]^{-1}.\tag{A.9}$$

The corresponding expression for $v\neq 0$ is

$$G_v(\mathbf{k}')=F_v(\mathbf{k}')[1-\tfrac12\mathcal{A}_I][D_I^v(\mathbf{k}')]^{-1},\tag{A.10}$$

where $D_I^v(\mathbf{k}')=D_I^0(\mathbf{k})$, with $\mathbf{k}'$ related to $\mathbf{k}$ by a Lorentz transformation.

We remark that $D_I^0(\mathbf{k})$ is actually the denominator for the scattering amplitude. Thus, if the solution has a resonance behavior at a certain meson energy $\mathbf{k}_0$ then for $\mathbf{k}$ near $\mathbf{k}_0$

$$|D_I^0(\mathbf{k})|_{\mathbf{k}\sim\mathbf{k}_0}\ll 1.$$

Consequently, the effect of $G_0(\mathbf{k})$ is to have the meson emitted with energy near the resonance energy. $D_I^0(\mathbf{k})$ may be expressed in terms of the A_I and B_I used in the original integral equation (A.2), for $\chi_I^0(\mathbf{k},\mathbf{k}_0)$. Substitution of (A.3) and (A.5) into (A.6) gives

$$\begin{aligned}D_I^0(\mathbf{k}_0)=(1+V_I)^{-1}[1-A_I-(A_I\omega_0+B_I)\langle F_0^2\rangle&\\ +(\tfrac12A_I)^2(1+\omega-\Omega)\langle F_0^2\rangle]+i4\pi^2k_0\omega_0F_0^2(\mathbf{k}_0)&\\ \times[A_I\omega_0+B_I+(\tfrac12A_I)^2(\Omega-\omega)].&\end{aligned}\tag{A.11}$$

By using the solution of $|\mathfrak{N}_\rho\rangle$ as discussed in Sec. VII, we find that if

$$g^2/4\pi=4.7,\tag{A.12}$$

then the real part of $D_I^0(\mathbf{k})=0$ for $k=0$.

This implies that for $g^2/4\pi\geq 4.7$ the system has a stable isobar state for $I=\frac32$.

The functions $G_I^0(\mathbf{k})$ for $g^2/4\pi=4.1$ are plotted in Fig. 2. We remark that for $g^2/4\pi<4.7$ there is no bound state for $I=\frac{3}{2}$. Thus the scattering function $\chi_{\frac{3}{2}}{}^0(\mathbf{k},\mathbf{k}_0)$ forms a complete set and we have

$$\int |G_{\frac{3}{2}}{}^0(\mathbf{k})|^2 d^3k=1.$$

On the other hand for $I=\frac{1}{2}$, because of the existence of the ground state of the nucleon, the scattering function $\chi_{\frac{1}{2}}{}^0(\mathbf{k},\mathbf{k}_0)$ does not form a complete set. Consequently,

$$\int |G_{\frac{1}{2}}{}^0(\mathbf{k})|^2 d^3k<1.$$

APPENDIX II

We list in this section the explicit k-dependence of various matrix elements used in Eqs. (46), (50)–(54). Those not listed can be obtained by interchanging $+v$ and $-v$ or (and) by interchanging k_1 and k_2.

$$\begin{aligned}&\langle P+\pi^+(\mathbf{k}), N|\mathcal{S}|P,P\rangle\\ &\quad=2S\langle\mathfrak{N}|\tau_-|\mathcal{P}\rangle\langle\psi^{(1)}(\tfrac{3}{2},\tfrac{3}{2})|\tau_+|\mathcal{P}\rangle G_{\frac{3}{2}}{}^v(\mathbf{k}),\end{aligned}\quad\text{(A.13)}$$

$$\begin{aligned}&\langle P, N+\pi^+(\mathbf{k})|\mathcal{S}|P, P\rangle=S\langle\mathcal{P}|\tau_3|\mathcal{P}\rangle\\ &\quad\times[(\tfrac{1}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{3}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{3}{2}}{}^{-v}(\mathbf{k})\\ &\quad+(\tfrac{2}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{1}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{1}{2}}{}^{-v}(\mathbf{k})],\end{aligned}\quad\text{(A.14)}$$

$$\begin{aligned}&\langle P+\pi^0(\mathbf{k}), P|\mathcal{S}|P,P\rangle=S\langle\mathcal{P}|\tau_3|\mathcal{P}\rangle\\ &\quad\times[(\tfrac{2}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{3}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{3}{2}}{}^{v}(\mathbf{k})\\ &\quad-(\tfrac{1}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{1}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{1}{2}}{}^{v}(\mathbf{k})],\end{aligned}\quad\text{(A.15)}$$

$$\begin{aligned}&\langle P+\pi^+(\mathbf{k}_1), N+\pi^0(\mathbf{k}_2)|\mathcal{S}|P, P\rangle\\ &\quad=2S\langle\psi^{(1)}(\tfrac{3}{2},\tfrac{3}{2})|\tau_+|\mathcal{P}\rangle G_{\frac{3}{2}}{}^v(\mathbf{k}_1)\\ &\quad\times[(\tfrac{2}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{3}{2},-\tfrac{1}{2})|\tau_-|\mathcal{P}\rangle G_{\frac{3}{2}}{}^{-v}(\mathbf{k}_2)\\ &\quad+(\tfrac{1}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{1}{2},-\tfrac{1}{2})|\tau_-|\mathcal{P}\rangle G_{\frac{1}{2}}{}^{-v}(\mathbf{k}_2)],\end{aligned}\quad\text{(A.16)}$$

$$\begin{aligned}&\langle P+\pi^0(\mathbf{k}_2), N+\pi^+(\mathbf{k}_1)|\mathcal{S}|P, P\rangle\\ &\quad=S[(\tfrac{2}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{3}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{3}{2}}{}^v(\mathbf{k}_2)\\ &\quad-(\tfrac{1}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{1}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{1}{2}}{}^v(\mathbf{k}_2)]\\ &\quad\times[(\tfrac{1}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{3}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{3}{2}}{}^{-v}(\mathbf{k}_1)\\ &\quad+(\tfrac{2}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{1}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{1}{2}}{}^{-v}(\mathbf{k}_1)],\end{aligned}\quad\text{(A.17)}$$

$$\begin{aligned}&\langle P+\pi^+(\mathbf{k}_1)+\pi^0(\mathbf{k}_2), N|\mathcal{S}|P,P\rangle\\ &\quad=2S\langle\mathfrak{N}|\tau_-|\mathcal{P}\rangle F_v(\mathbf{k}_1)F_v(\mathbf{k}_2)\\ &\quad\times(\tfrac{1}{5})^{\frac{1}{2}}\langle\psi^{(2)}(\tfrac{3}{2},\tfrac{3}{2})|\tau_+|\mathcal{P}\rangle,\end{aligned}\quad\text{(A.18)}$$

$$\begin{aligned}&\langle P, N+\pi^+(\mathbf{k}_1)+\pi^0(\mathbf{k}_2)|\mathcal{S}|P, P\rangle\\ &\quad=-S\langle\mathcal{P}|\tau_3|\mathcal{P}\rangle F_{-v}(\mathbf{k}_1)F_{-v}(\mathbf{k}_2)\\ &\quad\times(\tfrac{3}{5})^{\frac{1}{2}}\langle\psi^{(2)}(\tfrac{3}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle,\end{aligned}\quad\text{(A.19)}$$

$$\begin{aligned}&\langle P+\pi^0(\mathbf{k}_1), P+\pi^0(\mathbf{k}_2)|\mathcal{S}|P, P\rangle\\ &\quad=S[(\tfrac{2}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{3}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{3}{2}}{}^v(\mathbf{k}_1)-\\ &\quad-(\tfrac{1}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{1}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{1}{2}}{}^v(\mathbf{k}_1)]\\ &\quad\times[(\tfrac{2}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{3}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{3}{2}}{}^{-v}(\mathbf{k}_2)\\ &\quad-(\tfrac{1}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{1}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{1}{2}}{}^{-v}(\mathbf{k}_2)],\end{aligned}\quad\text{(A.20)}$$

$$\begin{aligned}&\langle P+\pi^0(\mathbf{k}_1)+\pi^0(\mathbf{k}_2), P|\mathcal{S}|P,P\rangle\\ &\quad=S\langle\mathcal{P}|\tau_3|\mathcal{P}\rangle F_v(\mathbf{k}_1)F_v(\mathbf{k}_2)[-(\tfrac{1}{3})^{\frac{1}{2}}\langle\psi^{(2)}(\tfrac{1}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle\\ &\quad+2(1/15)^{\frac{1}{2}}\langle\psi^{(2)}(\tfrac{3}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle],\end{aligned}\quad\text{(A.21)}$$

$$\begin{aligned}&\langle N+\pi^+(\mathbf{k}_1), N+\pi^+(\mathbf{k}_2)|\mathcal{S}|P, P\rangle\\ &\quad=S[(\tfrac{1}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{3}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{3}{2}}{}^v(\mathbf{k}_1)\\ &\quad-(\tfrac{2}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{1}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{1}{2}}{}^v(\mathbf{k}_1)]\\ &\quad\times[(\tfrac{1}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{3}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{3}{2}}{}^{-v}(\mathbf{k}_2)\\ &\quad-(\tfrac{2}{3})^{\frac{1}{2}}\langle\psi^{(1)}(\tfrac{1}{2},\tfrac{1}{2})|\tau_3|\mathcal{P}\rangle G_{\frac{1}{2}}{}^{-v}(\mathbf{k}_2)],\end{aligned}\quad\text{(A.22)}$$

$$\begin{aligned}&\langle N+\pi^+(\mathbf{k}_1)+\pi^+(\mathbf{k}_2), N|\mathcal{S}|P,P\rangle\\ &\quad=-2S\langle\mathfrak{N}|\tau_-|\mathcal{P}\rangle F_v(\mathbf{k}_1)F_v(\mathbf{k}_2)\\ &\quad\times(\tfrac{1}{5})^{\frac{1}{2}}\langle\psi^{(2)}(\tfrac{3}{2},\tfrac{3}{2})|\tau_+|\mathcal{P}\rangle.\end{aligned}\quad\text{(A.23)}$$

APPENDIX III

We discuss in this appendix the complete solution of multiple meson production for the case of neutral scalar theory. The Hamiltonian H_0 for a nucleon at rest is

$$H_0=m_0+\int \omega a^\dagger(\mathbf{k})a(\mathbf{k})d^3k$$
$$+g\int(16\pi^3\omega)^{-\frac{1}{2}}[a(\mathbf{k})+a^\dagger(\mathbf{k})]u_0(\mathbf{k})d^3k. \quad\text{(A.24)}$$

It is well known that this Hamiltonian can be transformed into a diagonal form by a unitary matrix U_0 given by

$$U_0=\exp\left\{g\int(16\pi^3\omega)^{-\frac{1}{2}}[a(\mathbf{k})-a^\dagger(\mathbf{k})]u_0(\mathbf{k})d^3k\right\}. \quad\text{(A.25)}$$

The state for a physical nucleon at rest then becomes

$$|N(v=0)\rangle=U_0^{-1}|n\rangle, \quad\text{(A.26)}$$

where $|n\rangle$ is the state for a bare nucleon. By using the Lorentz transformation $\mathcal{L}$, given by Eq. (4), we find that the state vector for a physical nucleon in motion with velocity $\mathbf{v}$ is

$$|N(\mathbf{v})\rangle=U_v^{-1}|n\rangle, \quad\text{(A.27)}$$

where

$$U_v=\exp\left\{g\int(16\pi^3\omega)^{-\frac{1}{2}}\gamma^{-1}(\omega-\mathbf{k}\cdot\mathbf{v})^{-1}\times[a(\mathbf{k})-a^\dagger(\mathbf{k})]d^3k\right\}.$$

Similarly, the state for a physical nucleon moving with velocity v together with m mesons of momentum $\mathbf{k}_1,\cdots\mathbf{k}_m$ is given by[18]

$$\begin{aligned}&|\Psi^{(m)}(\mathbf{k}_1\cdots\mathbf{k}_m;\mathbf{v})\rangle\\ &\quad=U_v^{-1}(m!)^{-\frac{1}{2}}a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\cdots a^\dagger(\mathbf{k}_m)|n\rangle.\end{aligned}\quad\text{(A.28)}$$

In a neutral scalar case, the physical nucleon is characterized by its velocity alone. Thus, during the

[18] The normalization of the state vector $|\Psi^{(m)}(\mathbf{k}_1\cdots\mathbf{k}_m;\mathbf{v})\rangle$ is

$$\int\langle\Psi^{(m)}(\mathbf{k}_1\cdots\mathbf{k}_m;\mathbf{v})|\Psi^{(m)}(\mathbf{k}_1'\cdots\mathbf{k}_m';\mathbf{v})\rangle d^3k_1\cdots d^3k_m=1$$

with $\mathbf{k}_1\cdots\mathbf{k}_m$ each independently varying over the entire $\mathbf{k}$-space.

collision, only the velocity of the nucleon can change from $\mathbf{v}$ to $\mathbf{v}'$. The corresponding probability for emitting m mesons with momenta $\mathbf{k}_1, \cdots \mathbf{k}_m$ due to this velocity change is

$$\mathcal{P}_m(\mathbf{k}_1\cdots\mathbf{k}_m)=|\langle\Psi^{(m)}(k_1\cdots k_m;\mathbf{v}')|N(\mathbf{v})\rangle|^2, \quad (A.29)$$

while the probability $\mathcal{P}_0$ for no-meson emission is given by

$$\mathcal{P}_0=|\langle N(\mathbf{v}')|N(\mathbf{v})\rangle|^2. \quad (A.30)$$

Upon using (A.27) and (A.28) we find

$$\mathcal{P}_0=\exp\left[-g^2\int f^2(\mathbf{k})d^3k\right], \quad (A.31)$$

and

$$\mathcal{P}_m(\mathbf{k}_1\cdots\mathbf{k}_m)=g^{2m}\mathcal{P}_0(m!)^{-1}\prod_{i=1}^{m}f^2(\mathbf{k}_i),$$

with

$$f(\mathbf{k})=(16\pi^3\omega)^{-\frac{1}{2}}[(\omega-\mathbf{k}\cdot\mathbf{v})^{-1}(1-v^2)^{\frac{1}{2}} -(\omega-\mathbf{k}\cdot\mathbf{v}')^{-1}(1-v'^2)^{\frac{1}{2}}].$$

It is easy to verify that the total probability is

$$\sum_{m=0}^{\infty}\int\cdots\int\mathcal{P}_m(\mathbf{k}_1\cdots\mathbf{k}_m)d^3k_1\cdots d^3k_m=1.$$

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Branching Ratio for Alternative Modes of Decay of Hyperons and θ^0 Mesons

GYO TAKEDA
Brookhaven National Laboratory, Upton, New York
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We study those restrictions upon the branching ratios of different decay modes of Σ^+, Λ^0, and θ^0 imposed by the invariant property of a system composed of nucleons, pions, hyperons, and heavy mesons under the Wigner time reversal. These restrictions give, in fact, upper and lower limits on the various branching ratios. Later, we use a more or less specific model of hyperons and θ mesons in order to get other possible restrictions on the branching ratios. These results are model-dependent, and subsequently provide a possible test of the model we have chosen.

I. INTRODUCTION

IT is known experimentally that the Σ^+ decays according to two different modes:

$$\Sigma^+ \begin{array}{l}\to p+\pi^0\\ \to n+\pi^+\end{array} \quad (Q\sim 116\text{ Mev}).$$

It is likewise supposed that Λ^0 and θ^0 have an alternative decay mode besides the ordinary one, although so far there is no conclusive experimental evidence for the alternate modes:

$$\Lambda^0 \begin{array}{l}\to p+\pi^-\\ \to n+\pi^0\end{array} \quad (Q\sim 37\text{ Mev}),$$

$$\theta^0 \begin{array}{l}\to \pi^++\pi^-\\ \to \pi^0+\pi^0\end{array} \quad (Q\sim 220\text{ Mev}).$$

First we shall study those restrictions[1] upon the branching ratios of different decay modes of Σ^+, Λ^0, and θ^0 imposed by the invariant property of a system composed of nucleons, pions, hyperons, and heavy mesons under the Wigner time reversal. These restrictions give, in fact, upper and lower limits on the various branching ratios. Since the limitations upon the branching ratios thus obtained prove so weak (see Tables I and II), we later adopt a more or less specific model of hyperons and θ mesons. This will give us other possible restrictions on the branching ratios of various decays. These results are model-dependent, and subsequently provide a possible test of the model we choose.

II. RESTRICTIONS ON BRANCHING RATIOS OF DECAYS IMPOSED BY THE INVARIANT PROPERTY UNDER THE WIGNER TIME REVERSAL

Let us take, for example, the decay of a Λ^0 into a proton and a π^-, or a neutron and a π^0. For a given value of the spin and parity[2] of the Λ^0, the relative angular momentum l between a pion and a nucleon after the decay is fixed. We have, then, only two different final states, namely, a (p,π^-) and an (n,π^0) state with the specified j and l. Or we can use two states with definite isotopic spin values $T=\frac{3}{2}$ and $T=\frac{1}{2}$ instead of (p,π^-) and (n,π^0).

The reaction matrix K and the scattering (or decay) matrix T are given by

$$K=V+V\left[\frac{1}{E-H_0}\right]K, \quad (1)$$

[1] This possibility was first suggested to the author by K. M. Watson. Also see K. M. Watson, Phys. Rev. 95, 228 (1954).

[2] This is the intrinsic parity of a Λ^0 relative to the intrinsic parity of a nucleon.