By using (3.27) to eliminate $\partial\mathbf{E}/\partial\lambda$ and $\partial\mathbf{H}/\partial\lambda$ from this, an equivalent equation is obtained which, when integrated, yields (2.17).

The acceleration field (2.18) of a nonrelativistic electron can be read off immediately from (3.6). We can replace a_i, v_i by $\mathbf{a}$, $\mathbf{v}$, the ordinary acceleration and velocity and a_4, v_4 by 0 and $-c$ (see the remarks earlier in this section). To evaluate the integral (2.20), take the z-axis along $\mathbf{a}$ at the retarded time corresponding to the moment in question. Then

$$\mathbf{R}\cdot\mathbf{a}=R|\mathbf{a}|\cos\theta,\ \mathbf{R}\cdot d\boldsymbol{\sigma}=R^3d\Omega,\ \mathbf{a}\cdot d\boldsymbol{\sigma}=R^2|\mathbf{a}|\cos\theta d\Omega,$$

and therefore

$$\int_S \mathbf{S}_{\mathrm{acc}}\cdot d\boldsymbol{\sigma}=-\frac{e^2}{16\pi^2c^3}\left[\frac{R^3|\mathbf{a}|^2}{(R^2+\lambda^2)^{\frac{3}{2}}}\int_S\left(\frac{R^2\cos^2\theta}{R^2+\lambda^2}-1\right)d\Omega+\frac{R^3\lambda^2|\mathbf{a}|^2}{(R^2+\lambda^2)^{5/2}}\int_S\cos^2 d\Omega\right]$$

$$=\frac{e^2}{16\pi^2c^3}\frac{R^3|\mathbf{a}|^2}{(R^2+\lambda^2)^{\frac{3}{2}}}\int_S\sin^2\theta d\Omega=\frac{2}{3}\frac{e^2}{4\pi c^3}a^2\left(1+\frac{\lambda^2}{R^2}\right)^{-\frac{3}{2}},$$

which is the right member of (2.20).

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Transformation of Relativistic Wave Equations*

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A unitary transformation is found which transforms the Dirac equation into two uncoupled equations. These involve higher orders of the time derivative than the first. In order $(v/c)^2$ the equations involve only the first time derivative and they are then equivalent to the Foldy-Wouthuysen transformation. While the equations are uncoupled and free of odd operators the functions satisfying them cannot be interpreted as different functions describing positive and negative energies separately, the general interpretation in the exact theory remaining in terms of the four-component wave function.

The transformation is extended to quantized fields and to relativistic two-body equations. The second-order electromagnetic mass effects in the quantized Dirac equation appear, in the nonrelativistic limit, as the time derivatives of the electric terms of the nonrelativistic Hamiltonian without the radiative corrections. These mass effects in the nonrelativistic Hamiltonian are proportional to $(1/mc)^3$.

Construction of unitary transformation operators for the *ps-ps* meson theory and for the Bethe-Salpeter equation are also discussed.

I. INTRODUCTION

It is well known that for many physical systems the application of the relativistic quantum theory meets with some mathematical difficulties. Furthermore, the description of many-body systems by relativistic methods raises some conceptual difficulties with regard to the meaning of a many-body relativistic wave function. It is, therefore, desirable to construct a general and systematic method for transitions from relativistic to nonrelativistic theories. Many methods of reductions of relativistic equations to nonrelativistic forms have been known all along, but all of these methods suffer from the lack of generality and from the required tedious procedures in their executions.

In the conventional methods of approximations the 4 components of the wave function are not treated on an equal footing, and this procedure gives rise to non-Hermitian terms in the Hamiltonian of $(v/c)^2$ approximation. However, there exist methods of approximations which easily remove the above defect. In connection with a study of the nature of nuclear interactions, Breit[1] considered the $(v/c)^2$ effects for nuclear particles. His method was, essentially, based on approximate Lorentz transformations and gave consistent results for spin-spin and similar interactions.

A different treatment of the problem was given by Foldy and Wouthuysen.[2] This method involves an infinite sequence of successive canonical transformations on the Dirac Hamiltonian, for a particle interacting with an external field. It leads to a transformed Hamiltonian in the form of an infinite series in powers of p/mc. For higher order approximations in p/mc, the Foldy and Wouthuysen development is not easy to use. Moreover, their method does not provide a simple way for the investigation of many-body problems.

It has been found possible to derive two sets of two-component equations referring to positive and negative energy states which are free of odd operators. The generalization of the method to other relativistic sys-

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[1] G. Breit, Phys. Rev. **51**, 248 (1937); **51**, 778 (1937); **53**, 153 (1937).

[2] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

tems involving more than one field is rather straightforward.

II. FREE-PARTICLE EQUATION

The four-component spinor wave function ψ_0 for a free Dirac particle satisfies the equation

$$(\gamma_\rho p_\rho+\mu)\psi_0=0, \tag{II.1}$$

where

$$p_\rho\psi_0=-i\hbar(\partial\psi_0/\partial x_\rho), \quad \rho=1, 2, 3, 4$$
$$-i\hbar(\partial/\partial t)\rightarrow cp_0\rightarrow E,$$

$x_4=ict$, $\gamma_\rho\gamma_\sigma+\gamma_\sigma\gamma_\rho=-2\delta_{\rho\sigma}$, $\gamma_4=i\beta$, $\gamma_\rho{}^\dagger=-\gamma_\rho$, and $\mu=mc$. The 4-vector p_ρ stands for the 3-dimensional vector $\mathbf{p}$ and the fourth component $p_4=ip_0$.

We write Eq. (II.1) in the form

$$(\boldsymbol{\gamma}\cdot\mathbf{p}+\mu)\psi_0=\beta p_0\psi_0 \tag{II.2}$$

and introduce the operator γ_p by

$$\gamma_p=(1/p)\boldsymbol{\gamma}\cdot\mathbf{p},$$

where

$$\gamma_p{}^2=-1. \tag{II.3}$$

By introducing the transformation

$$p=\mu\tan u, \tag{II.4}$$

and using (II.3), Eq. (II.2) can be written as

$$2\mu\exp(\gamma_p u)/[\exp(\gamma_p u)+\exp(-\gamma_p u)]\psi_0=\beta p_0\psi_0. \tag{II.5}$$

For a free particle, the relation $p_\rho{}^2=-\mu^2$ and the transformation (II.4) gives

$$2/[\exp(\gamma_p u)+\exp(-\gamma_p u)]=(p^2+\mu^2)^{\frac{1}{2}}/\mu,$$

so that Eq. (II.5) reduces to

$$(p^2+\mu^2)^{\frac{1}{2}}\exp(\gamma_p u)\psi_0=\beta p_0\psi_0. \tag{II.6}$$

We can eliminate the operator $\exp(\gamma_p u)$ from (II.6) if we introduce the following transformations:

$$\varphi_0=\exp(\gamma_p u/2)\psi_0=U_0{}^\dagger\psi_0, \tag{II.7}$$

and

$$p_0'=U_0{}^\dagger p_0 U_0=p_0,$$

where

$$U_0=\exp(-\gamma_p u/2). \tag{II.8}$$

Hence Eq. (II.6) becomes

$$i\hbar(\partial\varphi_0/\partial t)=\beta E_p\varphi_0, \tag{II.9}$$

where

$$E_p=c(p^2+\mu^2)^{\frac{1}{2}}.$$

In deriving (II.9) use was made of the operator property

$$\beta U_0=U_0{}^\dagger\beta, \tag{II.10}$$

which is a consequence of the relation

$$[\beta,\boldsymbol{\gamma}]_+=0.$$

Equation (II.9) was derived by Foldy and Wouthuysen by stipulating a canonical transformation generated by the Hermitian operator

$$S=-(i/2\mu)\boldsymbol{\gamma}\cdot\mathbf{p}W(p/\mu),$$

where the function W was determined by imposing the condition that the transformed Hamiltonian

$$H_T=\exp(iS)H\exp(-iS)$$

be free of odd operators. Equation (II.9) for the new wave function is free of odd operators and therefore it can be split up into two-component wave equations describing positive and negative energy states of a free particle. So far nothing new has been obtained in this section. The reason for giving the above detail for the free particle case lies in its extensive use in later sections of this paper.

In order to separate positive and negative energy parts of (II.9), the use of the projection operators

$$\Lambda_\pm{}^0=\tfrac{1}{2}(1\pm\beta) \tag{II.11}$$

is the usual procedure. The operators $\Lambda_\pm{}^0$ can be obtained from the Casimir projection operators by taking $\mathbf{p}=0$, but this does not mean that they are related to particles at rest. They differ from the Casimir projection operators in that they are the simplest projection operators formed from even matrices. The positive- and negative-energy wave functions are defined by

$$\varphi_0{}^\pm=\Lambda_\pm{}^0\varphi_0. \tag{II.12}$$

We may further note that the operator U_0 satisfies the equation

$$H_0U_0=U_0\beta E_p, \tag{II.13}$$

where

$$H_0=c\boldsymbol{\alpha}\cdot\mathbf{p}+\beta c\mu.$$

Actually the unitary operator U_0, as shown by Foldy and Wouthuysen, can be written as

$$U_0=[2E_p(E_p+Mc^2)]^{-\frac{1}{2}}(H_0\beta+E_p). \tag{II.14}$$

Thus U_0 consists of the usual four solutions of the free-particle Dirac equation corresponding to positive and negative energy states with "up" and "down" spin states. The form (II.14) of the solutions of the Dirac equation was used by Breit and Thaler[3] in order to analyze the origin of relativistic corrections to magnetic moments and it also made it possible to extend it to a two-body system.

III. INTERACTION WITH AN EXTERNAL ELECTROMAGNETIC FIELD

The equation for a Dirac particle interacting with an external electromagnetic field is given by

$$(\boldsymbol{\gamma}\cdot\boldsymbol{\pi}+\mu)\psi=\beta\pi_0\psi, \tag{III.1}$$

where

$$\pi_\rho=p_\rho-(e/c)A_\rho, \quad \rho=1, 2, 3, 4.$$

[3] G. Breit and R. M. Thaler, Phys. Rev. 89, 1182 (1953).

The simplest way to investigate the case of interactions is, first, to represent the unitary operator U_0 of the free-particle case in a suitable form and, then, pass to the introduction of interaction just as one does in writing down the Dirac equation for a particle interacting with an external field from the free particle equation.

In order to write the unitary operator U_0 in a form suitable for the introduction of an external field, we observe that we cannot bring in the interaction with an external field when the momentum operator in (II.8) is involved both as a vector and as a magnitude of a vector. However, the operator $\gamma_p u$ can be written as

$$\begin{aligned}\gamma_p u &= \gamma_p \tan^{-1}(p/\mu)\\ &= \gamma_p[(p/\mu)-\tfrac{1}{3}(p/\mu)^3+\tfrac{1}{5}(p/\mu)^5+\cdots]\\ &= i\tan^{-1}[-(i\boldsymbol{\gamma}\cdot\mathbf{p}/\mu)].\end{aligned}$$

Hence

$$U_0=\exp(-iS_0), \tag{III.2}$$

where

$$S_0=\tfrac{1}{2}\tan^{-1}(-i\boldsymbol{\gamma}\cdot\mathbf{p}/\mu), \tag{III.3}$$

and S_0 is a Hermitian operator. The form (III.3) of the generator of the unitary operator does not involve the magnitude of the momentum operator, as only the components of $\mathbf{p}$ appear in S_0. The unitary operator U for an interacting particle can now be inferred as

$$U=\exp(-iS), \tag{III.4}$$

where the Hermitian generator S is given by

$$S=\tfrac{1}{2}\tan^{-1}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\pi}/\mu). \tag{III.5}$$

The new wave function φ will be represented by

$$\varphi=U^\dagger\psi. \tag{III.6}$$

We may now write Eq. (II.5) for the case of interaction as

$$2\mu\beta(U^2+U^{2\dagger})^{-1}\varphi=U^\dagger\pi_0U\varphi, \tag{III.7}$$

where use is made of (III.6) and the operator property

$$\beta U=U^\dagger\beta. \tag{III.8}$$

The right-hand side of Eq. (III.7) is not free of odd operators. It will be convenient to record Eq. (III.7) in the form

$$M\varphi=R\varphi+L\varphi, \tag{III.9}$$

where

$$\begin{aligned}M&=2\mu\beta(U^2+U^{2\dagger})^{-1},\\ R&=\tfrac{1}{2}(U^\dagger\pi_0U+U\pi_0U^\dagger),\\ L&=\tfrac{1}{2}(U^\dagger\pi_0U-U\pi_0U^\dagger).\end{aligned} \tag{III.10}$$

The operators M, R, and L are Hermitian. Under a change of sign of μ the unitary operator U changes to $U^\dagger$ so that, while M and R remain unchanged, the operator L changes its sign. In order to see more explicitly the structure of M, R, and L, for small enough p/μ, we can write

$$U^\dagger\pi_0U=\pi_0+i[S,\pi_0]-(1/2!)[S,[S,\pi_0]] -(i/3!)[S,[S,[S,\pi_0]]]+\cdots. \tag{III.11}$$

Hence

$$R=\pi_0-(1/2!)[S,[S,\pi_0]]+\cdots. \tag{III.12}$$

$$L=i[S,\pi_0]-(i/3!)[S,[S,[S,\pi_0]]]+\cdots. \tag{III.13}$$

We also have

$$S=-i(\boldsymbol{\gamma}\cdot\boldsymbol{\pi})/(2\mu)-(-i\boldsymbol{\gamma}\cdot\boldsymbol{\pi})^3/(6\mu^3)+\cdots. \tag{III.14}$$

From (III.12), (III.13), and (III.14) it is easy to see that R and L consist entirely of even and odd operators, respectively. The expansion of S shows also that M is an even operator.

Now, we can use the projection operators (II.11) and split up the new wave function φ into two-component functions

$$\begin{aligned}\varphi_+&=\tfrac{1}{2}(1+\beta)\varphi,\\ \varphi_-&=\tfrac{1}{2}(1+\beta)\varphi.\end{aligned} \tag{III.15}$$

We also use the operator properties $\beta L=-L\beta$, $\beta R=R\beta$, $\beta M=M\beta$, so as to record Eq. (III.9) in the form of two coupled equations:

$$\begin{aligned}\beta M\varphi_+&=R\varphi_++L\varphi_-,\\ -\beta M\varphi_-&=R\varphi_-+L\varphi_+.\end{aligned} \tag{III.16}$$

It easily follows from (III.16) that the new wave functions φ_+ and φ_- satisfy the equations

$$(\beta M-R)\varphi_++L(\beta M+R)^{-1}L\varphi_+=0, \tag{III.17}$$

$$(\beta M+R)\varphi_-+L(\beta M-R)^{-1}L\varphi_-=0. \tag{III.18}$$

These equations are exact and are free of odd operators, since the odd operator L appears twice as a factor in the equations.

Because of the unitary property of U the expectation values of the observables in both the old and the new representation are the same, provided the definition of an observable O in the old representation is replaced in the new representation by

$$O_T=U^\dagger OU. \tag{III.19}$$

For example, the probability density $\varphi^\dagger\varphi$ is conserved with a current density given by

$$\mathbf{S}=\varphi^\dagger U^\dagger\boldsymbol{\alpha}U\varphi. \tag{III.20}$$

Thus the physical interpretation of the theory is still based on the use of a 4-component wave function. For the exact theory the functions φ_+ and φ_- by themselves cannot be regarded as different wave functions, one describing the positive energy and the other the negative energy particle, respectively. Although Eqs. (III.17) and (III.18) describe positive and negative energy states separately, we are not able to conserve probability with positive energy or negative energy particles alone, the only probability that is conserved referring to the state

$$\varphi=\varphi_{+}+\varphi_{-}. \qquad \text{(III.21)}$$

This means that the possibility of reducing the Dirac equation to two-component equations does not prevent transitions to negative energy states. The physical meaning of the functions φ_{+} and φ_{-} is now clear: (i) the sum of φ_{+} and φ_{-} is a probability amplitude of the same kind as Dirac's ψ function, (ii) in the nonrelativistic limit the function φ_{-} refers to small components and the function φ_{+} is a probability amplitude, (iii) the difference between the functions φ_{+}, φ_{-} and ψ_{+}, ψ_{-} lies in the fact that, while φ_{+} and φ_{-} are solutions of equations consisting entirely of even and Hermitian operators, the same is not true for ψ_{+}, ψ_{-} defined by

$$\psi_{\pm}=\tfrac{1}{2}(1\pm\beta)\psi.$$

IV. DERIVATION OF THE NONRELATIVISTIC EQUATION

Let us assume that the interaction is weak and expand the operators M, R and L retaining only the terms proportional to $(1/mc)^2$. We use the operator properties

$$[S,\pi_0]=-(e\hbar/2\mu c)\boldsymbol{\gamma}\cdot\boldsymbol{\mathcal{E}} \qquad \text{(IV.1)}$$

and

$$[S,[S,\pi_0]]=-(e\hbar^2/4\mu^2c)\nabla\cdot\boldsymbol{\mathcal{E}}-(e\hbar/4\mu^2c)\boldsymbol{\sigma}\cdot(\boldsymbol{\mathcal{E}}\times\boldsymbol{\pi})-(e\hbar/4\mu^2c)\boldsymbol{\sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{\mathcal{E}}), \qquad \text{(IV.2)}$$

where

$$\boldsymbol{\mathcal{E}}=-\nabla\varphi-(1/c)(\partial\mathbf{A}/\partial t).$$

From (IV.1), (IV.2), (III.12), and (III.13) we obtain

$$R=\pi_0+(e\hbar^2/8\mu^2c)\nabla\cdot\boldsymbol{\mathcal{E}}+(e\hbar/8\mu^2c)[\boldsymbol{\sigma}\cdot(\boldsymbol{\mathcal{E}}\times\boldsymbol{\pi})+\boldsymbol{\sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{\mathcal{E}})],$$

$$L=-(ie\hbar/2\mu c)\boldsymbol{\gamma}\cdot\boldsymbol{\mathcal{E}},$$

$$\beta M=\mu+(1/2\mu)(\mathbf{p}-e\mathbf{A}/c)^2-(e\hbar/2\mu c)\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{H}}.$$

Hence, Eq. (III.17) becomes

$$\begin{aligned}i\hbar(\partial\varphi_{+}/\partial t)=[&mc^2+e\varphi+(1/2m)(\mathbf{p}-(e/c)\mathbf{A})^2\\&-(e\hbar/2mc)\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{H}}-(e\hbar^2/8m^2c^2)\nabla\cdot\boldsymbol{\mathcal{E}}\\&-(e\hbar/8m^2c^2)\boldsymbol{\sigma}\cdot(\boldsymbol{\mathcal{E}}\times\boldsymbol{\pi})\\&-(e\hbar/8m^2c^2)\boldsymbol{\sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{\mathcal{E}})]\varphi_{+}.\end{aligned} \qquad \text{(IV.3)}$$

For a time-independent field the last two terms give the usual spin-orbit coupling. Equation (IV.3) is in agreement with the result obtained by Foldy and Wouthuysen.

V. TRANSFORMATION OF THE QUANTIZED DIRAC EQUATION

The wave function for a quantized Dirac field is defined as the matrix element of the quantized Heisenberg operator ψ with respect to the interacting vacuum state Ψ_0 and an arbitrary one-particle state Ψ, interacting with its proper radiation field and an external field, by

$$\chi(1)=(\Psi_0,\psi(1)\Psi). \qquad \text{(V.1)}$$

Schwinger[4] has shown that the function $\chi(1)$ satisfies the equation

$$\gamma_\rho\pi_\rho^{\,e}\chi(1)+\int M(11')\chi(1')d(1')=0, \qquad \text{(V.2)}$$

where

$$\pi_\rho^{\,e}=p_\rho-eA_\rho^{\,e}/c,$$

and

$$M(11')=mc\delta(11')+\mathfrak{M}(11')$$

is Schwinger's mass operator consisting of unrenormalized radiative corrections to the particle's motion. We would like to remark that there are difficulties in regarding a function defined by (V.1) as a wave function of a relativistic system interacting with its proper radiation field. These difficulties lie in an incomplete analysis of the effect of vacuum fluctuations. Thus it has not yet been shown whether one can derive a conservation law for $|\chi(1)|^2$. After the renormalization of mass and charge, to the second order, a conservation law does exist. For higher order radiative corrections a general proof is needed.

Formally however, our transformation can be extended to quantized fields. For the sake of illustration of the ideas involved, let us start by considering the equation

$$[\gamma_\rho\pi_\rho^{\,e}+\mu-\mu'(\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{H}}+\boldsymbol{\alpha}\cdot\boldsymbol{\mathcal{E}})]\chi=0. \qquad \text{(V.3)}$$

This equation was derived by Schwinger[5] from Eq. (V.2), where the last two terms represent the second-order electromagnetic mass effects, producing a spin magnetic moment of $\alpha/2\pi$ magnetons.

The required canonical transformation, in this case, is generated by

$$S=\tfrac{1}{2}\tan^{-1}[-i\boldsymbol{\gamma}\cdot\boldsymbol{\pi}^e/\mu-\mu'\boldsymbol{\alpha}\cdot\boldsymbol{\mathcal{E}}/\mu], \qquad \text{(V.4)}$$

where

$$\mu'=(\alpha/2\pi)(e\hbar/2\mu),$$

so that the largest contribution to the Hamiltonian arising from the second-order anomalous magnetic moment is given by

$$-(\mu'/8\mu^2c)\hbar^2(\partial/\partial t)\nabla\cdot\boldsymbol{\mathcal{E}}-(\hbar\mu'/8\mu^2c)\boldsymbol{\sigma}\cdot[(\partial\boldsymbol{\mathcal{E}}/\partial t)\times\boldsymbol{\pi}^e]-(\mu'\hbar/8\mu^2c)\boldsymbol{\sigma}\cdot[\boldsymbol{\pi}^e\times(\partial\boldsymbol{\mathcal{E}}/\partial t)]. \qquad \text{(V.5)}$$

This correction to the nonrelativistic Hamiltonian arising from the radiative corrections to the motion of an electron is μ' times the time derivatives of the electric terms in (IV.3), which fact shows that in the nonrelativistic limit also the radiative corrections are associated with the high-frequency components of the interaction.

The most general canonical transformation for a quantized Dirac equation is generated by

$$S=\tfrac{1}{2}\tan^{-1}[-i\boldsymbol{\gamma}\cdot\boldsymbol{\pi}^e/\mu-i\mathfrak{M}_0/\mu], \qquad \text{(V.6)}$$

where $\mathfrak{M}_0$ is the odd part of the renormalized mass operator.

[4] J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452 (1951).
[5] J. Schwinger, Phys. Rev. **82**, 664 (1951).

VI. APPLICATION TO MESON THEORY

We shall consider only the unquantized neutral *ps-ps* meson theory. The quantized theory can be treated in the same manner as in Sec. V. The Dirac equation for an interacting meson-nucleon system is given by

$$(\boldsymbol{\gamma}\cdot\mathbf{p}-G\gamma_5\varphi+\mu)\psi=\beta p_0\psi, \tag{VI.1}$$

where

$$\gamma_5=i\gamma_1\gamma_2\gamma_3\gamma_4,$$

and

$$\gamma_5{}^2=-1.$$

The generator of the transformation for this case is

$$S=\tfrac{1}{2}\tan^{-1}[-i\boldsymbol{\gamma}\cdot\mathbf{p}/\mu+iG\gamma_5\varphi/\mu].$$

For simplicity we confine ourselves to a static interaction, in which case the operators R, M, and L of Sec. III are given by

$$R=p_0,\quad L=0,$$
$$\beta M=\mu+(p^2/2\mu)+(\hbar G/2\mu)\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\varphi. \tag{VI.2}$$

The transformed Hamiltonian is therefore

$$H=Mc^2+(p^2/2M)+(\hbar G/2M)\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\varphi. \tag{VI.3}$$

This is the usual meson Hamiltonian obtained by conventional methods in the $(v/c)^2$ approximation. It does not contain any terms proportional to $(1/mc)^2$, and nothing like a spin-orbit coupling shows up in this approximation. At this point a few remarks are necessary: a spin-orbit coupling term has recently been obtained by Klein[6] from the quantized meson theory in the fourth-order approximation. This two-nucleon spin-orbit coupling energy has recently been analyzed by Araki[7] in connection with the calculation of the fine structure of O^{17}. Araki's investigation is based on a phenomenological cutoff of the meson theoretical spin-orbit potential and therefore its success cannot be unambiguously attributed to the soundness of the meson theory.

VII. TRANSFORMATION OF THE RELATIVISTIC TWO-BODY EQUATION

The reduction of relativistic two-body equations to approximate forms has been studied by Chraplyvy[8] by a generalization of the Foldy and Wouthuysen one-particle method.

In this paper we extend the one-particle transformation discussed in Sec. III to two-body relativistic equations by using the current field-theoretical methods.[9]

[6] A. Klein, Phys. Rev. **90**, 1101 (1953).
[7] G. Araki, 1954 Glasgow Conference on Nuclear and Meson Physics (Pergamon Press, London, 1955).
[8] Z. V. Chraplyvy, Phys. Rev. **91**, 388 (1953); **92**, 1310 (1953).
[9] All the arguments in this section about the two-body transformation are intended only as a sketch of the problem.

The Bethe-Salpeter equation[10] for a two-body electrodynamic system, interacting by an exchange of a photon only, is given by

$$(\gamma_\rho p_\rho-\mu)_1(\gamma_\rho p_\rho-\mu)_2\chi(12) \\ =-e^2\gamma_{1\rho}\gamma_{2\rho}D_F(12)\chi(12), \tag{VII.1}$$

where the wave function $\chi(12)$ is defined by

$$\chi(12)=(\Psi_0,T(\psi_1(1)\psi_2(2))\Psi). \tag{VII.2}$$

As in Sec. V, Ψ_0 is the true vacuum state[11] for two interacting particles and Ψ is a two-particle state (any two-particle system formed from particles and antiparticles). The remarks made in Sec. V about the one-particle wave function apply in this case also, except that here the difficulties are not only with the radiative corrections. The well known difficulties of interpretation of $\chi(12)$ as a probability amplitude connected with the relativistic features of the problem will be disregarded in the present paper particularly because no one has made use of the Bethe-Salpeter equation in its covariant form; all the calculations with it have been carried out with an equal-time formalism or with the so-called instantaneous interactions. Formally, therefore, the employment of the equation is just like that of Breit's two-particle theory using the wave function with single time.

It is hoped that a two-particle transformation may throw some light on these problems. We shall deal with two forms of the Bethe-Salpeter equation, the second form of which is obtained by using a transformation of the wave function given by

$$\chi(12)=\int S_{F1}(11')\beta_2\varphi(1'2)d1' \\ +\int S_{F2}(22')\beta_1\varphi(12')d2'. \tag{VII.3}$$

This transformation can be made use of to obtain an equation free of the spurious plane-wave solutions of the Bethe-Salpeter equation and brings it into a more familiar form.[12] The function $\varphi(12)$ may turn out to have a normalization not corresponding to

$$N=\int\varphi^*(12)\varphi(12)d1d2.$$

If so, then correction terms to what is obtained below would have to be added. In the case of a second-order instantaneous interaction of the particles, the functions φ and χ are the same.

Equation (VII.1) with the transformation (VII.2) in

[10] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[11] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).
[12] B. Kurşunoğlu, Phys. Rev. **96**, 1697 (1954).

momentum space in the center-of-mass system becomes

$$[H_1(\mathbf{p})+H_2(-\mathbf{p})-E]\varphi(p_\mu)$$
$$=-[e^2/(2\pi)^4]\gamma_{1\rho}\gamma_{2\rho}\int D_F(p_\mu-p_\mu')$$
$$\times\{[H_1(\mathbf{p}')-\tfrac{1}{2}E-cp_0']^{-1}+[H_2(-\mathbf{p}')-\tfrac{1}{2}E+cp_0']^{-1}\}\varphi(p_\mu')d^4p', \quad \text{(VII.4)}$$

where

$$H_j=(\pm c\boldsymbol{\alpha}\cdot\mathbf{p}+Mc^2\beta)_j, \quad (j=1,2).$$

In this form the equation is similar to Breit's equation, but differs from it in respect to retarded interactions. This form of the equation is suitable for carrying out the reduction to approximate forms.

In analogy to the definition (VII.2) of the wave function, the unitary transformation operator will be defined by

$$U_{12}=T\exp(-iS_1)\exp(-iS_2), \quad \text{(VII.5)}$$

where T is Wick's[12] covariant chronological ordering operator and

$$S_j=[\tfrac{1}{2}\tan^{-1}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\pi}/\mu)]_j, \quad (j=1,2). \quad \text{(VII.6)}$$

The electromagnetic field is described by the quantized Heisenberg operators A_ρ and in the generators, only the vector part of the potential being involved. The operator U_{12} can also be used for the reduction of Eq. (VII.4).

The unitary operator U_{12} is now a 16×16 matrix and depends on two space-time points. The new wave function is given by

$$\chi_T=U_{12}^\dagger\chi. \quad \text{(VII.7)}$$

The appropriate projection operators in this case are an obvious generalization of one-particle operators given by

$$\Lambda_{\pm\pm}=\tfrac{1}{4}(1\pm\beta_1)(1\pm\beta_2). \quad \text{(VII.8)}$$

The evaluation of the unitary operator defined by (VII.5) and application of the results to various electrodynamic systems will be the subject of another paper.

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