and ψ_{nLM} denotes the triplet orbital function. The values of the corrections calculated by adopting the previously evaluated approximate values of these parameters[6] are shown in Table I. They may be a little too small because the calculated intervals were smaller than observed ones, but they can indicate an order of magnitude. The observed values[1,7] of the intervals are shown in the same table. The corrections are in the range of experimental errors in the case of optical measurements while the microwave experiment will permit observation of the corrections. The previous calculation of the intervals[6] is too rough to test the correction terms even with the accurate data. We should have vastly improved orbital functions for the He excited states in order to deduce the fourth-order correction from the accurate observations.

[6] G. Araki, Proc. Phys. Math. Soc. Japan **19**, 128 (1937).
[7] Brochard, Chabbal, Chantrel, and Jacquinot, J. Phys. radium **13**, 433 (1952).

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Classical Maxwell Theory with Finite-Particle Sources

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A Lorentz-invariant finite-particle model is introduced into the Maxwell theory by extending the space from space-time to all (time-like) space-time spheres. The properties of the model are examined in the classical theory as a preliminary to the quantized case. The space-time sphere radius λ is the parameter of finiteness; it has the effect of smearing point particles into bell-shaped bounded distributions which go over into the δ-function point-particle distributions in the limit $\lambda=0$. The smeared particles give rise to fields in which the Coulomb infinity no longer exists. It is shown that the finite-particle 4-current has various indispensable formal properties: that charge is conserved; and that, in interaction with its field, momentum and energy are conserved, the integrals representing the electromagnetic self-energy and self-force being convergent for $\lambda\neq 0$. This replacement of point by finite particles results in corrections to calculations which are probably negligible where the classical theory is valid, but which might be appreciable in the quantum domain at distances comparable to λ.

1. INTRODUCTION

CLASSICAL field theories suffer from infinities due to the use of a point model of the particle sources of the fields.[1] These same infinities carry over, multiplied in number and variety, into quantum field theories,[2] (which suffer as well from other infinities of a strictly quantum-mechanical nature). What is needed to eliminate this type of infinity is a finite-particle model. Moreover, it is not unreasonable to suppose that a particle model which eliminated this kind of infinity from a classical theory would do the same in a quantized theory built from it by the correspondence principle, especially if the finite-particle model were a kinematical (i.e., geometrical) element of the theory, independent of whether the classical or quantum interpretation of the fields were used. Accordingly, the study of a finite-particle model in the classical theory should serve as a useful preliminary to its eventual introduction into the quantized theory. That is the spirit in which a finite-particle model in the classical Maxwell theory is examined in this paper.

The next question is, what sort of a model shall it be? The finite-sphere model runs into group-theoretical troubles.[3] Moreover, the idea that an elementary particle has a definite volume and boundary in 3-dimensional space seems to be interpreting the phrase "finite particle" in too literal and naive a sense. Another method of avoiding the infinities is the admixture of unphysical elements like advanced fields,[4] which, besides defying causality, leads to unphysical behavior.[5] Yet, undoubtedly, elementary particles are finite in some sense. One might demand of a finite-particle model, discarding some of the prejudices carried over from macroscopic intuition, at least the following: that there be a parameter of finiteness λ which acts analytically as a cutoff in formerly infinite expressions; that the model defined by λ be meaningful against the groups employed, i.e., (at least) Lorentz-covariant; and finally, the demand of simplicity, that λ have a natural connection with, or meaning relative to, spacetime, that it remain not forever an *ad hoc* and geometrically inexplicable element in the theory. One could add to these the stronger demand that λ admit interpretation in some end formulas as the linear dimension of a finite

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[1] L. Landau and E. Lifschitz, *The Classical Theory of Fields* (Addison-Wesley Press, Cambridge, 1951), Sec. 5–2.
[2] V. Weisskopf, Phys. Rev. **56**, 72 (1939).
[3] W. Pauli, *Die Allgemeinen Prinzipien der Wellenmechanik* (J. W. Edwards, Ann Arbor, 1947), p. 271
[4] P. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).
[5] For yet other attempts, see M. Born and L. Infeld, Proc. Roy Soc. (London) **142**, 410 (1934); **144**, 425 (1934); **147**, 522 (1934); **150**, 141 (1935); R. Feynman, Phys. Rev. **74**, 939 (1948).

particle in the naive world picture inherited from macroscopic intuition (but certainly not necessarily the *same* geometrical quantity—e.g., the radius of a sphere—in all such formulas).

A finite particle model satisfying these requirements is obtained by widening the domain of definition of physical fields in the following natural way: The point of spacetime x^m ($m=1, \cdots 4$) can be identified with the "null-sphere" (light cone) of which it is the center. Physical fields which are functions of $x^1, \cdots x^4$ can thus be said, from this viewpoint, to be defined over the null-spheres of spacetime. If now one extends the space from all the null-spheres to all the space-time spheres, characterized by their centers x^m and (directed) radii λ, this means that we take physical fields to be functions of the five coordinates $x^1, \cdots x^4, \lambda$. The radius λ then turns out to be the parameter of finiteness.

Because of its conformal form invariance, the form that the Maxwell theory, extended to this space, should take is especially clear.[6] In this paper some consequences of this finite particle model in the classical Maxwell theory are examined. The main task is to show that the particle has the correct formal properties, for example, that its charge is conserved, and that in interaction with its own field there is conservation of energy, momentum, etc. These properties are indispensable, of course, in the quantized theory as well. Besides cutting off infinities and making these questions meaningful, the presence of λ also manifests itself in systematic corrections to the calculated results of the old theory. Some of these corrections are calculated below, although it is expected that they are negligibly small for phenomena in the classical domain. On the other hand, the corrections brought by this model to quantum-mechanical calculations, which might be appreciable, could be determined just as soon as we know how the Dirac theory should be extended to this space.

This finite-particle field was first proposed in 1939 by Groenewold[7] and again in 1949 by Landé[8]; they gave the potential [our (3.5)] with the new retarded time condition involving the finiteness parameter λ. These authors simply postulated this potential by analogy with the old Liénard-Wiechert potential; with them it was not a question of first having a new set of field equations and then verifying that their ansatz is a solution. Landé recognized in addition that this field entailed, as its source in the second Maxwell set, a finite, smeared-out particle [our (2.9)]. Both were primarily interested in the dynamics of a particle, especially the self-force question, a matter excluded from consideration here. However, we may make the following remarks in passing: (1) that we believe that a quantum-theoretical treatment is indispensable in the self-force question; (2) that the dynamics of the finite particle advanced here seems to us still a subject fraught with ambiguities (e.g., it is not at all clear how the integral of the force density over the whole spatial extent of the particle, which represents in one sense the total force on it, governs the motion of the single *point* which specifies completely the position of the finite particle); (3) that these authors see various formal reasons to admix advanced fields in the dynamical problem, e.g., because they emerge along with the retarded fields from a natural, Fokker-type variation principle. These seem to us to be scant reasons for using such unphysical entities, especially when they are no longer needed to cancel infinities in the retarded fields. On the other hand, questions like the charge distribution of the finite particle, energy-momentum exchange between it and the total field, etc., untouched on by Groenewold and Landé, are treated here in detail.

Finally, the difference in viewpoint should be mentioned. Those authors seemed unaware of the fundamental connection of the theory with the sphere geometry of space-time, i.e., that the extension of the space brings with it fifth components, leading to a unified treatment of field and 4-current, etc. For example, Landé's "reciprocity" is a characteristic feature of any theory built on sphere geometry.[9]

2. SURVEY OF RESULTS

The physical effect of introducing the length λ into the Maxwell theory as a fifth variable alongside the four of space-time is to smear out particle sources, pointlike in the old theory, into charge distributions spread through all space but mainly concentrated in regions of linear dimension λ around the old point sources. The bell-shaped curve of charge density against distance becomes higher and narrower as λ decreases, going into the 3-dimensional δ-function distribution of the point source in the limit $\lambda=0$. The total charge of the distribution, independent of λ, is just e, the charge on the point source. The electromagnetic field of a particle is just that which would be calculated in the old theory from such a charge distribution, and is thus bounded, showing a finite maximum at the position of the old point charge. In the limit $\lambda=0$ it goes over into the classical unbounded field of a particle. Geometrically, λ is a Lorentz invariant and might be identified, e.g., with the Compton wavelength of the particle. For a particle at rest the charge distribution is spherically symmetrical; for a moving particle, however, an anisotropy appears, depending on the velocity, acceleration, and hyperacceleration.[10] Working in the extended space is equivalent to working with the conventional Maxwell theory in which all point sources are replaced by certain continuous distributions with effective spreads λ (and where, for a given $\lambda\neq 0$, solutions corresponding to point sources no longer exist).

[6] R. Ingraham, Proc. Natl. Acad. Sci. U. S. **41**, 165 (1955).

[7] H. J. Groenewold, Physica **6**, 115 (1939). I am indebted to the referee for this and the following reference.

[8] A. Landé, Phys. Rev. **76**, 1176 (1949); **77**, 814 (1950).

[9] See R. Ingraham, Nuovo cimento **12**, 825 (1954), p. 834, ff.

[10] By hyperacceleration we mean $d^3\mathbf{r}/dt^3$.

This effects systematic corrections to the end results calculated on the basis of point particles: λ appears as a parameter in these formulas. As an illustration, the correction to the total radiation flux from an accelerated slow electron is calculated. Incidentally, the inverse phase, the motion of these finite particles in a given field, is not treated here.

The introduction of the fifth variable λ has as a formal consequence that charge-current and field are united in a unified description by means of a 5-force $F_{\alpha\beta}$ ($\alpha, \beta=1, \cdots 5$). $F_{\alpha\beta}$ satisfies field equations formally identical with the old Maxwell laws for empty space. The 4-vector of charge-current $j_n=(J/c, -\rho)$ is described this way as

$$j_n=\left(\frac{\partial}{\partial\lambda}-\frac{1}{\lambda}\right)F_{5n}. \qquad (2.1)$$

It is the source of the electromagnetic field F_{mn} and satisfies a continuity equation. The second main purpose of this paper is to show that the right member of (2.1) may also be consistently interpreted as the 4-current when we come to energy-momentum considerations. It is shown that the decrease of the electromagnetic energy and momentum in a region of space is due not only to the loss of these quantities on the boundary but also to the work done by the field in the one case—and the force exerted by the field in the other—on a charge distribution whose 4-current is exactly the right member of (2.1). There is also a fifth conservation equation unfamiliar from the old theory, whose possible significance will be discussed later in this section.

To come to details, the 5-potential F_α of a particle of charge e and velocity v is found to be

$$\mathbf{F}\equiv\mathbf{A}=\frac{e}{4\pi}\frac{\mathbf{v}/c}{(R^2+\lambda^2)^{\frac{1}{2}}-(\mathbf{R}\cdot\mathbf{v})/c},$$

$$F_4\equiv-\varphi=\frac{-e}{4\pi}\frac{1}{(R^2+\lambda^2)^{\frac{1}{2}}-(\mathbf{R}\cdot\mathbf{v})/c}, \quad F_5=0, \qquad (2.2)$$

where $\mathbf{A}$ and φ are the vector and scalar potentials, $\mathbf{R}$ is the radius vector from the position $\mathbf{r}'$ of the charge at the retarded time

$$t_{\text{ret}}'=t-(R^2+\lambda^2)^{\frac{1}{2}}/c \qquad (2.3)$$

to the point of observation $\mathbf{r}$, and $\mathbf{v}$ is the velocity at the retarded time. The curl of F_α gives the 5-force[11] $F_{\alpha\beta}$ of which the components[11] F_{mn} form the electromagnetic field of the particle and the other components $G_n\equiv F_{5n}$ determine the particle 4-current j_n via (2.1), which can be conveniently written

$$j_n=\lambda\frac{\partial}{\partial\lambda}(\lambda^{-1}G_n). \qquad (2.4)$$

[11] General index conventions: Greek letters go from 1 to 5; Roman letters m, n, p, q, etc., go from 1 to 4; Roman letters i, j, k go from 1 to 3.

This 4-current, which is the source of the field F_{mn}, satisfies the continuity equation

$$\partial j^n/\partial x^n=0. \qquad (2.5)$$

From (2.2), (2.3), it is evident that we get back the well-known Liénard-Wiechart solution for $\lambda=0$, but that for $\lambda\neq0$ these potentials, and hence the 5-force $F_{\alpha\beta}$ and 4-current j_n, are bounded quantities. Indeed, by (2.3), $t_{\text{ret}}'<t$ everywhere and

$$(t_{\text{ret}}')_{\max}=t-\lambda/c.$$

The presence of the nonzero parameter λ thus prohibits the instantaneous action of source on field point leading to infinities in the fields. If the source be pictured as a finite particle in the macroscopic sense, which acts, as far as the effect of its field on points external to it is concerned, as if it were concentrated at its center, then this last formula says that λ is as near as we can get to the center of the source, i.e., that λ is interpretable here as the radius of the finite particle.

The electromagnetic field of the particle is computed in Sec. 3. The charge-current turns out to be

$$\mathbf{J}=\frac{e\lambda^2}{4\pi c^2D^3}\left[\mathbf{b}+\frac{3cP}{D}\mathbf{a}+\frac{cQ}{D}\mathbf{v}\right], \quad \rho=\frac{e\lambda^2Q}{4\pi cD^4}, \qquad (2.6)$$

where $\mathbf{v}$, $\mathbf{a}$, and $\mathbf{b}$ are the ordinary velocity, acceleration, and hyperacceleration[6] of the old point source, and

$$D=(R^2+\lambda^2)^{\frac{1}{2}}-\frac{\mathbf{R}\cdot\mathbf{v}}{c},$$

$$P=1-\frac{v^2}{c^2}+\frac{\mathbf{R}\cdot\mathbf{a}}{c^2}, \qquad (2.7)$$

$$Q=\frac{3cP^2}{D}+\frac{\mathbf{R}\cdot\mathbf{b}}{c^2}-\frac{3\mathbf{a}\cdot\mathbf{v}}{c^2}.$$

The charge distribution thus shows an anisotropy if the particle is in motion. For a source moving with constant velocity the charge density simplifies to

$$\rho=\frac{e^3\lambda^2}{4\pi D^5}\left(1-\frac{v^2}{c^2}\right)^2,$$

so that, e.g., the maximum value of ρ (that at the position of the old point source) exhibits a $[1-(v^2/c^2)]^2$ velocity dependence.

For a source at rest at the origin, only $\varphi=e/4\pi(r^2+\lambda^2)^{\frac{1}{2}}$ is nonzero, giving the 5-field

$$\mathbf{E}=-\nabla\varphi=\frac{e}{4\pi}\frac{\mathbf{r}}{(r^2+\lambda^2)^{\frac{3}{2}}}, \quad \mathbf{H}=0, \quad \mathbf{G}=0,$$

$$G_4=-\frac{\partial}{\partial\lambda}\varphi=\frac{e}{4\pi}\frac{\lambda}{(r^2+\lambda^2)^{\frac{3}{2}}}. \qquad (2.8)$$

This electrostatic field is that due to the charge-current derived from $\mathbf{G}, G_4$, which by (2.6), or directly from (2.4), is

$$\mathbf{J}=c\lambda\frac{\partial}{\partial\lambda}(\lambda^{-1}\mathbf{G})=0,$$

$$\rho=-\lambda\frac{\partial}{\partial\lambda}(\lambda^{-1}G_4)=\frac{e}{4\pi}\frac{3\lambda^2}{(r^2+\lambda^2)^{5/2}}. \tag{2.9}$$

This distribution into which the old point source has been smeared has the following interesting properties. The charge is mainly concentrated in a region of linear dimension $\sim\lambda$ around $r=0$, attenuating rapidly (like r^{-5}) for $r>\lambda$. The function $\rho(r)$ becomes higher and narrower for decreasing λ, whereas for any value of λ its total charge is

$$\int_{\text{all space}}\rho dv=\left(\frac{3e\lambda^2}{4\pi}\right)4\pi\int_0^\infty\frac{r^2dr}{(r^2+\lambda^2)^{5/2}}=e, \tag{2.10}$$

exactly the charge on the old point source. Thus ρ/e is an approximation for the 3-dimensional δ function:

$$\lim_{\lambda\to 0}(3\lambda^2/4\pi)(r^2+\lambda^2)^{-5/2}=\boldsymbol{\delta}(r),$$

so that we get back the point-particle $\boldsymbol{\delta}$-function distribution in the limit $\lambda=0$. One further property, which ties λ to a macroscopic finite particle interpretation, should be mentioned here. By (2.9), at $r=0$,

$$\rho=\rho_{\max}=e(4\pi\lambda^3/3)^{-1}, \tag{2.11}$$

and this maximum charge density is just what one would obtain in the naive picture in which the point source was spread uniformly throughout a sphere of radius exactly λ.

In the macroscopic Maxwell theory, one replaces a large number of particle sources by a fictitious continuous charge distribution. The solution due to the continuous spread is built up by linearity from the elementary Coulomb solution as an integral of the charge density over the appropriate volume, surface, or curve. By the same device the fields due to continuous source distributions may be built up from the elementary solution (2.2), (2.3). The presence of $\lambda\neq 0$ smears each "point particle" of the fictitious fluid over all space; the charge-current j_n differs thus in its detailed distribution from σ_n, the given continuous spread of sources, although they coincide in the limit $\lambda=0$. A calculation of the dipole radiation of a current of atomic dimensions, for example, shows a correction due to $\lambda\neq 0$. The total charge, whether given by j_n or σ_n, is the same. The details, given in Part 3, are passed over here because we are more interested in the changes which this extension brings to the exact, microscopic theory.

Energetics.—From the 5-force an energy 5-tensor $T_{\alpha\beta}$ of the usual form can be built, except that here we must use the appropriate metric, that of the extended space, to raise the indices. The energy tensor satisfies a conservation law in virtue of the field equations. In terms of the electric and magnetic fields and the 4-vector G_m, the components $\bar{T}_{\alpha\beta}=-\lambda^{-2}T_{\alpha\beta}$ turn out to be

$$\begin{aligned}
&\bar{T}_{..}=-(\mathbf{EE}+\mathbf{HH})+\tfrac{1}{2}\boldsymbol{\delta}(\mathbf{E}^2-\mathbf{H}^2)+\mathbf{GG}+\tfrac{1}{2}\boldsymbol{\delta}(G_4{}^2-\mathbf{G}^2),\\
&\bar{T}_{.4}=-(\mathbf{E}\times\mathbf{H}-G_4\mathbf{G}),\\
&\bar{T}_{44}=\tfrac{1}{2}(\mathbf{E}^2+\mathbf{H}^2+\mathbf{G}^2+G_4{}^2),\\
&\bar{T}_{.5}=-G_4\mathbf{E}+\mathbf{G}\times\mathbf{H},\\
&\bar{T}_{45}=-\mathbf{G}\cdot\mathbf{E},\\
&\bar{T}_{55}=\tfrac{1}{2}(\mathbf{E}^2-\mathbf{H}^2+\mathbf{G}^2-G_4{}^2),
\end{aligned} \tag{2.12}$$

where the dots indicate suppressed 3-vector indices, so that, e.g., $\bar{T}_{..}$ and $\bar{T}_{.4}$ are 3-dyad and 3-vector respectively, and $\boldsymbol{\delta}$ means the unit 3-dyad. The components of $\bar{T}_{mn}$ are the quantities familiar from the old theory enlarged by terms in the 4-vector G_m, and the $\bar{T}_{5\alpha}$ are new. The conservation laws written for the quantities $\bar{T}_{\alpha\beta}$ read

$$\bar{T}_\beta\equiv\frac{\partial}{\partial x^m}\bar{T}^m{}_\beta+\lambda\frac{\partial}{\partial\lambda}(\lambda^{-1}\bar{T}_{5\beta})+\lambda^{-1}\delta_\beta{}^5\bar{T}=0, \tag{2.13}$$

where $\bar{T}=\bar{T}_{11}+\bar{T}_{22}+\bar{T}_{33}-\bar{T}_{44}+\bar{T}_{55}$ and the index m is raised with the Lorentz metric. Further support for the identification (2.1) now comes from the fact that the integral conservation laws

$$\int_V\bar{T}_\beta dv=0, \tag{2.14}$$

where V is a volume of boundary S, give for $\beta=4$:

$$-\frac{\partial}{\partial t}\int_V\left(\frac{\mathbf{E}^2+\mathbf{H}^2}{2}\right)dv=c\int_S(\mathbf{E}\times\mathbf{H})\cdot d\boldsymbol{\sigma}+\int_V\mathbf{J}\cdot\mathbf{E}dv, \tag{2.15}$$

and for $\beta=1, 2, 3$

$$-\frac{\partial}{\partial t}\int_V\frac{\mathbf{E}\times\mathbf{H}}{c}dv=\int_S\mathbf{S}\cdot d\boldsymbol{\sigma}+\int_V\left(\rho\mathbf{E}+\frac{\mathbf{J}}{c}\times\mathbf{H}\right)dv, \tag{2.16}$$

where the charge ρ and current $\mathbf{J}$ are given by (2.1). (Some manipulation using the field equations to eliminate $\partial\mathbf{E}/\partial\lambda$ and $\partial\mathbf{H}/\partial\lambda$ has been performed.) Here $\mathbf{S}$ is the classical stress dyad of the field [the first two terms of $\bar{T}_{..}$ in (2.12)]. These are the classical laws expressing the conservation of the energy and momentum of the field in a region V containing charge of density ρ and current $\mathbf{J}$.

The component $\beta=5$ gives (after similar manipulation) the conservation law

$$-\frac{\partial}{\partial t}\int_V \frac{\mathbf{G}\cdot\mathbf{E}}{c^2}dv=\frac{1}{2}\int_S(-G_4\mathbf{E}+\mathbf{G}\times\mathbf{H})\cdot d\boldsymbol{\sigma} -\frac{1}{c}\int_V\left\{\mathbf{E}\cdot\left(-\nabla G_4+\frac{1}{c}\frac{\partial}{\partial t}\mathbf{G}\right) +\mathbf{H}\cdot(\nabla\times\mathbf{G})-\rho G_4-\frac{\mathbf{J}}{c}\cdot\mathbf{G}\right\}dv. \quad (2.17)$$

It suggests that a mass density $\mathbf{G}\cdot\mathbf{E}/c^2$ be ascribed to the field; then (2.17) is a conservation law for this mass in V, relating its decrease to certain volume and surface interactions with the other components of the 5-force and their derivatives, including the charge-current (2.1). $\mathbf{G}$ could be interpreted as an electric polarization; it is this polarization which gives rise to the current via $\mathbf{J}=c\lambda(\partial/\partial\lambda)(\lambda^{-1}\mathbf{G})$.

Specializing these considerations to the elementary solution (2.2), one gets for a nonrelativistic electron the acceleration field

$$\mathbf{E}_{\text{acc}}=\frac{e}{4\pi c^2}\left[\frac{\mathbf{R}\cdot\mathbf{a}}{(R^2+\lambda^2)^{\frac{3}{2}}}\mathbf{R}-\frac{1}{(R^2+\lambda^2)^{\frac{1}{2}}}\mathbf{a}\right], \quad \mathbf{H}_{\text{acc}}=\frac{e}{4\pi c^2}\frac{\mathbf{R}\times\mathbf{a}}{(R^2+\lambda^2)}, \quad (2.18)$$

hence the acceleration dependent part of the Poynting vector is

$$\mathbf{S}_{\text{acc}}\equiv c(\mathbf{E}_{\text{acc}}\times\mathbf{H}_{\text{acc}})=-\frac{e^2}{16\pi^2c^3} \times\left\{\left[\frac{(\mathbf{R}\cdot\mathbf{a})^2}{(R^2+\lambda^2)^{5/2}}-\frac{a^2}{(R^2+\lambda^2)^{\frac{3}{2}}}\right]\mathbf{R}+\frac{\lambda^2\mathbf{R}\cdot\mathbf{a}}{(R^2+\lambda^2)^{5/2}}\mathbf{a}\right\}. \quad (2.19)$$

Integrating this over a sphere of radius r around the particle as center, one gets the power loss of a slow electron by radiation:

$$\int_S \mathbf{S}_{\text{acc}}\cdot d\boldsymbol{\sigma}=\frac{2}{3}\frac{(e')^2}{c^3}a^2\Theta, \quad \Theta\equiv\left(1+\frac{\lambda^2}{r^2}\right)^{-\frac{3}{2}}, \quad (2.20)$$

where $e'=e/(4\pi)^{\frac{1}{2}}$. The correction factor Θ, which goes to unity as $r\to\infty$, involves the square of the ratio λ/r, presumably small for r of macroscopic size ($\sim$1 cm).

For the field of a particle, the integrals in (2.15) representing the energy of the particle field and the work done by this field on the smeared charge spread of the particle itself can be calculated for any given state of motion of the old point source. These integrals are finite ($\lambda\neq 0$) as contrasted with their divergent nature in the classical theory. Hence we can talk meaningfully about the electromagnetic self-energy and self-force of a particle with this model. For a resting particle the electromagnetic self-energy is

$$\mathcal{E}=\frac{1}{2}\int_{\text{all space}}E^2dv=\left(\frac{e^2}{32\pi^2}\right)4\pi\int_0^\infty\frac{r^4dr}{(r^2+\lambda^2)^3} =\left(\frac{1}{8\pi}\right)\left(\frac{3\pi}{16}\right)\left(\frac{e^2}{\lambda}\right). \quad (2.21)$$

The classical expression, for a finite particle of radius a, is $\mathcal{E}_{\text{class}}=e^2/8\pi a$. Hence from energy considerations we would be led to interpret λ as $3\pi/16$ times the radius of the particle in the naive picture.

Probable size of λ.—The corrections to results in the classical domain, for example Θ in (2.20), cannot be expected to furnish an idea of the size of λ, whether it should be taken as the classical radius, say, or the Compton wavelength, or something else again. In the quantized theory, however, the cutoff λ would yield finite results for quantities which today are evaluated by a subtraction process in a perturbation formalism. A comparison with experiment here could be expected not only to fix λ but also to answer the prior question whether field theory built on the present extension of space-time is consistent with observation.

3. DETAILS OF THE CALCULATIONS

The extended Maxwell laws are

$$\gamma^{-\frac{1}{2}}\frac{\partial}{\partial x^\alpha}(\gamma^{\frac{1}{2}}\gamma^{\alpha\delta}\gamma^{\beta\epsilon}F_{\delta\epsilon})=0, \quad \frac{\partial}{\partial x^\alpha}F_{\beta\gamma}+\frac{\partial}{\partial x^\beta}F_{\gamma\alpha}+\frac{\partial}{\partial x^\gamma}F_{\alpha\beta}=0, \quad (3.1)$$

where $F_{\alpha\beta}=-F_{\beta\alpha}$ is the 5-force, $\gamma_{\alpha\beta}$ the metric in sphere-space, defined by

$$d\theta^2=\gamma_{\alpha\beta}dx^\alpha dx^\beta=-\lambda^{-2}(dr^2-c^2dt^2+d\lambda^2), \quad (3.2)$$

giving the infinitesimal angle $d\theta$ between the neighboring spheres[12] of centers $x^m=(\mathbf{r},ct)$ and $x^m+dx^m=(\mathbf{r}+d\mathbf{r}, c(t+dt))$ and directed radii $x^5=\lambda$ and $x^5+dx^5=\lambda+d\lambda$ respectively; $\gamma^{\alpha\beta}$ are the normalized cofactors of $\gamma_{\alpha\beta}$, $\gamma=\text{Det}\gamma_{\alpha\beta}$. From (3.2), $\gamma^{\alpha\beta}=-\lambda^2g^{\alpha\beta}$ where $g^{\alpha\beta}=g_{\alpha\beta}=(+++-+)1$ on the diagonal, $g^{\alpha\beta}=g_{\alpha\beta}=0$, $\alpha\neq\beta$; and $\gamma^{\frac{1}{2}}=\lambda^{-5}$. Hence (3.1) decomposes into

$$\frac{\partial}{\partial x^m}F^m{}_n=-\left(\frac{\partial}{\partial\lambda}-\frac{1}{\lambda}\right)F_{5n}, \quad \frac{\partial}{\partial x^m}F_5{}^m=0, \quad (3.3)$$

and

$$\frac{\partial}{\partial x^m}F_{np}+\frac{\partial}{\partial x^n}F_{pm}+\frac{\partial}{\partial x^p}F_{mn}=0, \quad \frac{\partial}{\partial x^m}F_{5n}-\frac{\partial}{\partial x^n}F_{5m}=\frac{\partial}{\partial\lambda}F_{mn}, \quad (3.4)$$

[12] Only time-like spheres are considered, i.e., those whose surfaces and centers are related time-like in space-time. This choice leads to the plus sign before $d\lambda^2$ in (3.2). The inclusion of space-like spheres brings with it various unphysical features; see reference 5 or R. Ingraham, Nuovo cimento 1, 82 (1955).

in which the indices m, n, p are Lorentz indices and are raised with the Lorentz metric. For the subspace $\lambda=0$ we take $F_{5m}=0$; then (3.3) and (3.4) go into the ordinary Maxwell laws for empty space. To verify that the 5-potential (2.2), (2.3) gives a solution of these field equations, it is convenient to write it in its Lorentz invariant form,[13] thus

$$F_m=-\frac{e}{4\pi}\frac{v_m}{R^pv_p},\quad F_5=0;\quad R^pR_p=-\lambda^2, \tag{3.5}$$

where $[y^n(\omega),0]$ is the path in sphere-space of the old point source in terms of any Lorentz-invariant running parameter ω; $v^m=dy^m/d\omega$ evaluated at the retarded time (i.e., at the corresponding value of ω), indices are lowered with the Lorentz metric g_{mn}, and $R^p=x^p-y^p$ is the radius 4-vector from the point source at the retarded time to the observation point x^m. (3.5) gives the potential at the sphere (x^m,λ) and the last equation fixes the retarded ω as a function of x^m and λ. The 5-force is the curl of F_α, or

$$\begin{aligned}F_{mn}=&-\frac{e}{4\pi}\times\frac{1}{(R^pv_p)^2}\\&\times\left[R_ma_n-R_na_m-\frac{A}{(R^pv_p)}[R_mv_n-R_nv_m]\right],\\F_{5n}=&-\frac{e}{4\pi}\frac{\lambda}{(R^pv_p)^2}\left[a_n-\frac{A}{R^pv_p}v_n\right],\end{aligned} \tag{3.6}$$

where $a^m=d^2y^m/d\omega^2$, $A=R^pa_p-v^pv_p$, and indices are lowered with Lorentz metric. We have used the formulas

$$\frac{\partial\omega_{\text{ret}}}{\partial x^m}=\frac{R_m}{\mathcal{R}},\quad \frac{\partial\omega_{\text{ret}}}{\partial\lambda}=\frac{\lambda}{\mathcal{R}},\quad (\mathcal{R}\equiv R^pv_p) \tag{3.7}$$

obtained from differentiating the last equation of (3.5). From these follow the formulas:

$$\begin{aligned}&\text{(a)}\quad \frac{\partial R_n}{\partial x^m}=g_{mn}-\frac{v_nR_m}{\mathcal{R}},\quad \left(\frac{\partial R^m}{\partial x^m}=3\right),\\&\text{(b)}\quad \frac{\partial v_n}{\partial x^m}=a_n\frac{R_m}{\mathcal{R}},\quad \frac{\partial v_n}{\partial\lambda}=a_n\frac{\lambda}{\mathcal{R}},\\&\text{(c)}\quad \frac{\partial a_n}{\partial x^m}=b_n\frac{R_m}{\mathcal{R}},\quad \frac{\partial a_n}{\partial\lambda}=b_n\frac{\lambda}{\mathcal{R}},\quad \left(b^n=\frac{d^3y^n}{d\omega^3}\right),\\&\text{(d)}\quad \frac{\partial\mathcal{R}}{\partial x^m}=v_m+\frac{A}{\mathcal{R}}R_m,\quad \frac{\partial\mathcal{R}}{\partial\lambda}=\frac{A}{\mathcal{R}}\lambda,\\&\text{(e)}\quad \frac{\partial A}{\partial x^m}=a_m+\frac{(R^pb_p-3a^pv_p)}{\mathcal{R}}R_m,\\&\qquad \frac{\partial A}{\partial\lambda}=\frac{(R^pb_p-3a^pv_p)}{\mathcal{R}}\lambda.\end{aligned} \tag{3.8}$$

[13] See Landau and Lifschitz, reference 1, p. 176 for the corresponding formula for $\lambda=0$.

The set (3.4) is satisfied automatically in virtue of $F_{\alpha\beta}=\partial F_\beta/\partial x^\alpha-\partial F_\alpha/\partial x^\beta$. As for the set (3.3), we verify first that the divergence of $F_5{}^m$ vanishes. We have from (3.6) and (3.8):

$$\begin{aligned}\frac{\partial F_5{}^n}{\partial x^n}=&-\frac{e\lambda}{4\pi}\Bigg[-2\mathcal{R}^{-3}\left(v_n+\frac{A}{\mathcal{R}}R_n\right)\left(a^n-\frac{A}{\mathcal{R}}v^n\right)\\&+\mathcal{R}^{-2}\Bigg\{\frac{(Rb)}{\mathcal{R}}-\frac{A}{\mathcal{R}}\frac{(Ra)}{\mathcal{R}}+Av^n\mathcal{R}^{-2}\left(v_n+\frac{A}{\mathcal{R}}R_n\right)\\&-\frac{v_n}{\mathcal{R}}\left(a_n+[(Rb)-3(av)]\frac{R_n}{\mathcal{R}}\right)\Bigg\}\Bigg]\\=&-\frac{e}{4\pi}\frac{\lambda}{\mathcal{R}^3}\Bigg[-2(av)-2\frac{A}{\mathcal{R}}(Ra)+2\frac{A}{\mathcal{R}}(v)^2+\frac{2A^2}{\mathcal{R}}\\&+(Rb)-\frac{A}{\mathcal{R}}(Ra)+\frac{A}{\mathcal{R}}(v)^2+\frac{A^2}{\mathcal{R}}\\&-(av)-(Rb)+3(av)\Bigg],\end{aligned} \tag{3.9}$$

with the notation ( ) for the Lorentz inner product, e.g., $(Rb)=R_nb^n$. All terms except those involving A and A^2 cancel immediately and the rest give

$$\begin{aligned}\frac{\partial F_5{}^n}{\partial x^n}=&\frac{e}{4\pi}\frac{\lambda}{\mathcal{R}^3}\left[\frac{3A^2}{\mathcal{R}}+\frac{A}{\mathcal{R}}[-3(Ra)+3(v)^2]\right]\\=&-\frac{e\lambda}{4\pi\mathcal{R}^3}\left[\frac{3A^2}{\mathcal{R}}-\frac{3A^2}{\mathcal{R}}\right]=0.\end{aligned} \tag{3.10}$$

Hence our solution satisfies the last equation of the set (3.3). As for the first set of (3.3) we first compute

$$\begin{aligned}\frac{\partial}{\partial x^m}\left(a_n-\frac{A}{\mathcal{R}}v_n\right)=&b_n\frac{R_m}{\mathcal{R}}-\frac{A}{\mathcal{R}}a_n\frac{R_m}{\mathcal{R}}+\frac{Av_n}{\mathcal{R}^2}\left(v_m+\frac{A}{\mathcal{R}}R_m\right)\\&-\frac{v_n}{\mathcal{R}}\left(a_m+\left[\frac{(Rb)-3(av)}{\mathcal{R}}\right]R_m\right),\\\frac{\partial}{\partial\lambda}\left(a_n-\frac{A}{\mathcal{R}}v_n\right)=&b_n\frac{\lambda}{\mathcal{R}}-\frac{A}{\mathcal{R}}a_n\frac{\lambda}{\mathcal{R}}+\frac{A}{\mathcal{R}^2}v_n\frac{A\lambda}{\mathcal{R}}\\&-\frac{v_n}{\mathcal{R}}\left[\frac{(Rb)-3(av)}{\mathcal{R}}\right]\lambda.\end{aligned} \tag{3.11}$$

Abbreviating $\Phi_n=a_n-(A/\mathcal{R})v_n$, we get from (3.6) and (3.8):

$$\begin{aligned}\frac{\partial F^m{}_n}{\partial x^m}=&-\frac{e}{4\pi}\Bigg[-2\mathcal{R}^{-3}\left(v_m+\frac{A}{\mathcal{R}}R_m\right)(R^m\Phi_n-\Phi^mR_n)\\&+\mathcal{R}^{-2}\Bigg\{3\Phi_n-\Phi^m\left(g_{mn}-\frac{R_m}{\mathcal{R}}v_n\right)\\&+R^m\frac{\partial}{\partial x^m}\Phi_n-\frac{\partial}{\partial x^m}\Phi^mR_n\Bigg\}\Bigg].\end{aligned} \tag{3.12}$$

From (3.11),

$$R^m\frac{\partial\Phi_n}{\partial x^m}=\frac{(R)^2}{\mathcal{R}}b_n-\frac{A}{\mathcal{R}^2}(R)^2a_n+\left(\frac{-(v)^2}{\mathcal{R}}+\frac{A^2}{\mathcal{R}^3}(R)^2-\frac{(R)^2}{\mathcal{R}^2}(Rb)+\frac{3(R)^2}{\mathcal{R}^2}(av)\right)v_n,$$

$$\frac{\partial\Phi^m}{\partial x^m}=\frac{(Rb)}{\mathcal{R}}-\frac{A}{\mathcal{R}}\frac{(Ra)}{\mathcal{R}}+\frac{A}{\mathcal{R}^2}[(v)^2+A]-\frac{(av)}{\mathcal{R}}-\frac{(Rb)}{\mathcal{R}}+\frac{3(av)}{\mathcal{R}}=\frac{2(av)}{\mathcal{R}}. \tag{3.13}$$

Using (3.13) and the relations

$$(R\Phi)=(Ra)-A=(v)^2,\quad (v\Phi)=(av)-(A/\mathcal{R})(v)^2$$

in (3.12), we get

$$\frac{\partial F^m{}_n}{\partial x^m}=-\frac{e}{4\pi}\Bigg[-2\mathcal{R}^{-3}\left(\mathcal{R}+\frac{A}{\mathcal{R}}(R)^2\right)\left(a_n-\frac{A}{\mathcal{R}}v_n\right)-\left(-(av)+\frac{A}{\mathcal{R}}(v)^2-\frac{A}{\mathcal{R}}(v)^2\right)R_n$$
$$+\mathcal{R}^{-2}\Bigg\{2a_n-\frac{2A}{\mathcal{R}}v_n+\frac{(v)^2}{\mathcal{R}}v_n+\frac{(R)^2}{\mathcal{R}}b_n-\frac{A(R)^2}{\mathcal{R}^2}a_n+\left(-\frac{(v)^2}{\mathcal{R}}+\frac{A^2(R)^2}{\mathcal{R}^3}-\frac{(R)^2}{\mathcal{R}^2}(Rb)+\frac{3(R)^2}{\mathcal{R}^2}(av)\right)v_n-\frac{2(av)}{\mathcal{R}}R_n\Bigg\}\Bigg].$$

Grouping together the terms proportional to b_m, a_m, v_m, and R_m, this becomes finally

$$\frac{\partial F^m{}_n}{\partial x^m}=-\frac{e}{4\pi}\frac{(R)^2}{\mathcal{R}^3}\times\left[b_n-\frac{3A}{\mathcal{R}}a_n+\left(\frac{3A^2}{\mathcal{R}^2}-\frac{(Rb)}{\mathcal{R}}+\frac{3(av)}{\mathcal{R}}\right)v_n\right]. \tag{3.14}$$

Next we compute

$$-\left(\frac{\partial}{\partial\lambda}-\frac{1}{\lambda}\right)F_{5n}=-\lambda\frac{\partial}{\partial\lambda}(\lambda^{-1}F_{5n})$$

from (3.6), (3.8), and (3.11).

$$-\lambda\frac{\partial}{\partial\lambda}(\lambda^{-1}F_{5n})=\frac{e\lambda}{4\pi}\Bigg[-2\mathcal{R}^{-3}\frac{A\lambda}{\mathcal{R}}\left(a_n-\frac{A}{\mathcal{R}}v_n\right)+\mathcal{R}^{-2}\left\{\frac{\lambda}{\mathcal{R}}b_n-\lambda\frac{A}{\mathcal{R}^2}a_n+\left(\frac{A^2\lambda}{\mathcal{R}^3}-\frac{(Rb)}{\mathcal{R}^2}+\frac{3(av)}{\mathcal{R}^2}\right)v_n\right\}\Bigg]$$
$$=\frac{e\lambda^2}{4\pi\mathcal{R}^3}\left[b_n-\frac{3A}{\mathcal{R}}a_n+\left(\frac{3A^2}{\mathcal{R}^2}-\frac{(Rb)}{\mathcal{R}}+\frac{3(av)}{\mathcal{R}}\right)v_n\right]. \tag{3.15}$$

Finally, it follows that the right members of (3.14) and (3.15) are equal in virtue of the retarded time condition $(R)^2=-\lambda^2$ in (3.5). Hence (3.5) is demonstrated a solution of the field equations.

The statement that $[(\partial/\partial\lambda)-(1/\lambda)]F_{5n}$ is the source of the electromagnetic field F_{mn} and satisfies an equation of continuity follows from (3.3) and (3.4).

The fact that the field (3.6) is bounded for $\lambda\neq 0$ can be seen very easily from this invariant form: an infinity could occur only where R^p vanished or was orthogonal to the timelike vector v^p, in which cases we would have $R^pR_p\geq 0$. But by (3.5) R^p is time-like.

The 4-current j_n for a general state of motion of the particle is given by the negative of the right member of (3.15). If we replace the invariant parameter $d\omega$ by $dt'=c^{-1}dy^4$ [as we may do by the homogeneity of F_α in v_p, see (3.5)], the new b^4 and a^4 vanish, and the new $v^4=c$. $\mathcal{R}$ becomes $-c[(R^2+\lambda^2)^{\frac{1}{2}}-(\mathbf{R}\cdot\mathbf{v}/c)]$. Substituting these into (3.15), we get the expressions (2.6), (2.7).

In a macroscopic theory, consider the field due to the continuous space spread $\sigma_n=(\sigma\mathbf{v},-\sigma)$. Replacing e by σdv in (2.2) and integrating, one gets the solution

$$\mathbf{A}=\frac{1}{4\pi}\int\frac{\sigma(\mathbf{v}/c)dv}{(R^2+\lambda^2)^{\frac{1}{2}}-(\mathbf{R}\cdot\mathbf{v}/c)},$$
$$\varphi=\frac{1}{4\pi}\int\frac{\sigma dv}{(R^2+\lambda^2)^{\frac{1}{2}}-(\mathbf{R}\cdot\mathbf{v}/c)},\quad F_5=0, \tag{3.16}$$

where σ, $\mathbf{v}$, and $\mathbf{R}$ are the retarded quantities. For the 4-current (2.1) associated with this field $j_n\neq\sigma_n$ but it easily can be seen that

$$\lim_{\lambda\to 0}j_n(x^p,\lambda)=\sigma_n(x^p),\quad \int_{\text{all space}}\rho dv=\int_{\text{all space}}\sigma dv, \tag{3.17}$$

where the last equation refers to an electrostatic distribution. As an application, the electrostatic potential of an infinite straight filament of charge $\sigma=\text{const}$ per unit length along the z-axis is

$$\varphi=\frac{\sigma}{4\pi}\int_{-\infty}^{\infty}\frac{d\zeta}{[x^2+y^2+(z-\zeta)^2+\lambda^2]^{\frac{1}{2}}}=\frac{-\sigma}{2\pi}\log(r_1{}^2+\lambda^2)^{\frac{1}{2}}, \tag{3.18}$$

where $r_1=(x^2+y^2)^{\frac{1}{2}}$. For $\lambda=0$, one gets the familiar result that φ is the potential of a point charge σ at the origin in two dimensions. $G_4=-(\partial/\partial\lambda)\varphi$, hence the charge distribution ρ due to this field is

$$\rho=-\lambda\frac{\partial}{\partial\lambda}(\lambda^{-1}G_4)=-\frac{\sigma}{\pi}\frac{\lambda^2}{(r_1{}^2+\lambda^2)^2}. \tag{3.20}$$

ρ is cylindrically symmetrical, attenuates like $r_1{}^{-4}$ for $r_1>\lambda$, and has a total charge (considered as twodimensional distribution)

$$\int_{xy\text{-plane}} \rho da = \frac{\sigma}{\pi}\lambda^2 \times 2\pi \int_0^\infty \frac{r_1 dr_1}{(r_1^2+\lambda^2)^2} = 2\sigma\lambda^2 \times 1/(2\lambda^2) = \sigma, \quad (3.21)$$

or just the charge on the point source. Thus ρ/σ is an approximation for the 2-dimensional δ function:

$$\lim_{\lambda\to 0} \frac{1}{\pi}\frac{\lambda^2}{(r_1^2+\lambda^2)^2} = \delta(r_1).$$

We note further that at $r_1=0$

$$\rho = \rho_{\max} = \sigma(\pi\lambda^2)^{-1}, \quad (3.22)$$

which is just the density of a point charge σ smeared uniformly through a circle of radius λ.

Energetics.—The symmetric energy 5-tensor is

$$T_{\alpha\beta} = F^\gamma{}_\alpha F_{\gamma\beta} - \tfrac{1}{4}\gamma_{\alpha\beta}F^{\gamma\delta}F_{\gamma\delta}, \quad (3.23)$$

where indices are raised with the angle metric $\gamma_{\alpha\beta}$ defined by (3.2). It satisfies

$$\nabla_\alpha T^\alpha{}_\beta = 0, \quad (3.24)$$

where ∇_α is the covariant derivative with respect to the metric $\gamma_{\alpha\beta}$. This is proved by a calculation formally identical with the same one in the conventional theory,[14] using the field equations (3.1), of which the first may be written $\nabla_\alpha F^\alpha{}_\beta = 0$. Equation (3.24) expanded reads

$$\gamma^{-\frac{1}{2}}\left\{\frac{\partial}{\partial x^\alpha}(\gamma^{\frac{1}{2}}T^\alpha{}_\beta) + \tfrac{1}{2}\gamma^{\frac{1}{2}}\frac{\partial}{\partial x^\beta}\gamma^{\alpha\gamma}T_{\alpha\gamma}\right\} = 0. \quad (3.25)$$

Using $\gamma_{\alpha\beta} = -\lambda^{-2}g_{\alpha\beta}$ and $\gamma^{\frac{1}{2}} = \lambda^{-5}$, where $g_{\alpha\beta}$ are the constants defined after Eq. (3.2) and writing $T_{\alpha\beta} = -\lambda^2\bar{T}_{\alpha\beta}$, this gives the conservation laws in the form (2.13).

To derive the integral conservation laws (2.15)–(2.17), we first set $\beta=4$ in (2.13), getting

$$-\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mathbf{E}^2+\mathbf{H}^2+\mathbf{G}^2+G_4^2}{2}\right) - \nabla\cdot(\mathbf{E}\times\mathbf{H}-G_4\mathbf{G}) - \left(\frac{\partial}{\partial\lambda}-\frac{1}{\lambda}\right)(\mathbf{G}\cdot\mathbf{E}) = 0. \quad (3.26)$$

Write the second sets of field equations in (3.3) and (3.4) as

$$\text{(a)}\quad \nabla\cdot\mathbf{G} = -\frac{1}{c}\frac{\partial G_4}{\partial t},$$

$$\text{(b)}\quad \nabla G_4 - \frac{1}{c}\frac{\partial\mathbf{G}}{\partial t} = \frac{\partial\mathbf{E}}{\partial\lambda}, \quad (3.27)$$

$$\text{(c)}\quad \nabla\times\mathbf{G} = \frac{\partial\mathbf{H}}{\partial\lambda}.$$

[14] See Landau and Lifschitz, reference 1, pp. 87, 88.

Take the scalar product of (b) with $\mathbf{G}$, transform the first term by product differentiation, and use (a) to replace $\nabla\cdot\mathbf{G}$. We get

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mathbf{G}^2+G_4^2}{2}\right) - \nabla\cdot(G_4\mathbf{G}) + \mathbf{G}\cdot\frac{\partial\mathbf{E}}{\partial\lambda} = 0.$$

Adding this to (3.26), we obtain

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mathbf{E}^2+\mathbf{H}^2}{2}\right) + \nabla\cdot(\mathbf{E}\times\mathbf{H}) + \lambda\frac{\partial}{\partial\lambda}(\lambda^{-1}\mathbf{G})\cdot\mathbf{E} = 0, \quad (3.28)$$

which leads on integration over V to the Poynting Theorem (2.15). Proceeding similarly with the momentum, (2.13) for $\beta=1, 2, 3$ gives

$$\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H}-G_4\mathbf{G}) + \nabla\cdot\bar{T}.. + \left(\frac{\partial}{\partial\lambda}-\frac{1}{\lambda}\right) \times(-G_4\mathbf{E}+\mathbf{G}\times\mathbf{H}) = 0. \quad (3.29)$$

Multiply (3.27b) by G_4 and take the vector product of $\mathbf{G}$ and (3.27c). One gets

$$\tfrac{1}{2}\nabla G_4^2 - \frac{1}{c}G_4\frac{\partial\mathbf{G}}{\partial t} - G_4\frac{\partial\mathbf{E}}{\partial\lambda} = 0,$$

and

$$\mathbf{G}\times\frac{\partial\mathbf{H}}{\partial\lambda} - \tfrac{1}{2}\nabla\mathbf{G}^2 + \mathbf{G}\cdot\nabla\mathbf{G} = 0,$$

respectively. Add these, transform the last term thus

$$\mathbf{G}\cdot\nabla\mathbf{G} = \nabla\cdot(\mathbf{G}\mathbf{G}) - (\nabla\cdot\mathbf{G})\mathbf{G} = \nabla\cdot(\mathbf{G}\mathbf{G}) - \frac{1}{c}\frac{\partial G_4}{\partial t}\mathbf{G}$$

and subtract from (3.29). One obtains

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{E}\times\mathbf{H}}{c}\right) + \nabla\cdot\mathbf{S}.. - \lambda\frac{\partial}{\partial\lambda}(\lambda^{-1}G_4)\mathbf{E} + \lambda\frac{\partial}{\partial\lambda}(\lambda^{-1}\mathbf{G})\times\mathbf{H} = 0, \quad (3.30)$$

where $\mathbf{S}..$ is the classical stress dyad, which leads directly to (2.16). Equation (2.13) gives for $\beta=5$

$$\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{G}\cdot\mathbf{E}) + \nabla\cdot(-G_4\mathbf{E}+\mathbf{G}\times\mathbf{H}) + \frac{1}{2}\frac{\partial}{\partial\lambda}(\mathbf{E}^2-\mathbf{H}^2+\mathbf{G}^2-G_4^2) + \lambda^{-1}(G_4^2-\mathbf{G}^2) = 0. \quad (3.31)$$

By using (3.27) to eliminate $\partial\mathbf{E}/\partial\lambda$ and $\partial\mathbf{H}/\partial\lambda$ from this, an equivalent equation is obtained which, when integrated, yields (2.17).

The acceleration field (2.18) of a nonrelativistic electron can be read off immediately from (3.6). We can replace a_i, v_i by $\mathbf{a}$, $\mathbf{v}$, the ordinary acceleration and velocity and a_4, v_4 by 0 and $-c$ (see the remarks earlier in this section). To evaluate the integral (2.20), take the z-axis along $\mathbf{a}$ at the retarded time corresponding to the moment in question. Then

$$\mathbf{R}\cdot\mathbf{a}=R|\mathbf{a}|\cos\theta,\ \mathbf{R}\cdot d\boldsymbol{\sigma}=R^3d\Omega,\ \mathbf{a}\cdot d\boldsymbol{\sigma}=R^2|\mathbf{a}|\cos\theta d\Omega,$$

and therefore

$$\int_S\mathbf{S}_{\text{acc}}\cdot d\boldsymbol{\sigma}=-\frac{e^2}{16\pi^2c^3}\left[\frac{R^3|\mathbf{a}|^2}{(R^2+\lambda^2)^{\frac{3}{2}}}\int_S\left(\frac{R^2\cos^2\theta}{R^2+\lambda^2}-1\right)d\Omega+\frac{R^3\lambda^2|\mathbf{a}|^2}{(R^2+\lambda^2)^{5/2}}\int_S\cos^2 d\Omega\right]$$

$$=\frac{e^2}{16\pi^2c^3}\frac{R^3|\mathbf{a}|^2}{(R^2+\lambda^2)^{\frac{3}{2}}}\int_S\sin^2\theta d\Omega=\frac{2}{3}\frac{e^2}{4\pi c^3}a^2\left(1+\frac{\lambda^2}{R^2}\right)^{-\frac{3}{2}},$$

which is the right member of (2.20).

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Transformation of Relativistic Wave Equations*

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A unitary transformation is found which transforms the Dirac equation into two uncoupled equations. These involve higher orders of the time derivative than the first. In order $(v/c)^2$ the equations involve only the first time derivative and they are then equivalent to the Foldy-Wouthuysen transformation. While the equations are uncoupled and free of odd operators the functions satisfying them cannot be interpreted as different functions describing positive and negative energies separately, the general interpretation in the exact theory remaining in terms of the four-component wave function.

The transformation is extended to quantized fields and to relativistic two-body equations. The second-order electromagnetic mass effects in the quantized Dirac equation appear, in the nonrelativistic limit, as the time derivatives of the electric terms of the nonrelativistic Hamiltonian without the radiative corrections. These mass effects in the nonrelativistic Hamiltonian are proportional to $(1/mc)^3$.

Construction of unitary transformation operators for the *ps-ps* meson theory and for the Bethe-Salpeter equation are also discussed.

I. INTRODUCTION

It is well known that for many physical systems the application of the relativistic quantum theory meets with some mathematical difficulties. Furthermore, the description of many-body systems by relativistic methods raises some conceptual difficulties with regard to the meaning of a many-body relativistic wave function. It is, therefore, desirable to construct a general and systematic method for transitions from relativistic to nonrelativistic theories. Many methods of reductions of relativistic equations to nonrelativistic forms have been known all along, but all of these methods suffer from the lack of generality and from the required tedious procedures in their executions.

In the conventional methods of approximations the 4 components of the wave function are not treated on an equal footing, and this procedure gives rise to non-Hermitian terms in the Hamiltonian of $(v/c)^2$ approximation. However, there exist methods of approximations which easily remove the above defect. In connection with a study of the nature of nuclear interactions, Breit[1] considered the $(v/c)^2$ effects for nuclear particles. His method was, essentially, based on approximate Lorentz transformations and gave consistent results for spin-spin and similar interactions.

A different treatment of the problem was given by Foldy and Wouthuysen.[2] This method involves an infinite sequence of successive canonical transformations on the Dirac Hamiltonian, for a particle interacting with an external field. It leads to a transformed Hamiltonian in the form of an infinite series in powers of p/mc. For higher order approximations in p/mc, the Foldy and Wouthuysen development is not easy to use. Moreover, their method does not provide a simple way for the investigation of many-body problems.

It has been found possible to derive two sets of two-component equations referring to positive and negative energy states which are free of odd operators. The generalization of the method to other relativistic sys-

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