judged in relation to the two dashed curves. These were calculated on the basis of an electron density at a distance r from the shower axis varying as $r^{-1.0}$ for D and as $r^{-0.80}$ for E, representing structure functions which are more or less steep respectively than the average experimental value of $r^{-0.85}$. The lateral shower structure is the same in each of the three directions, within the experimental error of 4%. This places an upper limit of error on Cocconi's second calculation of approximately 2%, and shows that the effect of the earth's field on shower structure at sea level is negligible compared with Coulomb scattering.

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Analysis of the Small-Angle Elastic Scattering of High-Energy Protons by Carbon*

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The usual treatment of the high-energy elastic scattering of protons by nuclei has been extended to include relativistic Coulomb corrections and a complex nuclear spin-orbit potential. With these additions it is possible to obtain a good fit of the experimental results on the polarization of high-energy protons scattered elastically by carbon for small scattering angles. In addition, it is possible to deduce the sign of the nuclear spin-orbit potential from the high-energy data alone. The significance of the imaginary spin-orbit potential is discussed.

INTRODUCTION

THE elastic scattering and polarization of high-energy protons by nuclei has been studied previously by several authors.[1,2] In this note the small-angle polarization is examined somewhat more closely. In particular, relativistic effects arising through the Coulomb interaction are calculated and, in addition, the nuclear spin-orbit potential is generalized to be complex. The relativistic correction manifests itself as a spin-orbit potential, and it will appear that this additional potential has a noticeable effect on the polarization of the proton for small angles of scattering.[3] This, together with the generalization of the nuclear spin-orbit potential as complex, makes possible a good fit of the small-angle polarization data for carbon and a deduction of the sign of the nuclear spin-orbit potential from the high-energy data alone.[4] By limiting our considerations to small angles, we minimize model-dependent features (e.g., the shape of the potential well), which tend to be more marked at larger angles.

COULOMB SPIN ORBIT POTENTIAL

To order (v/c) the relativistic corrections to the Hamiltonian[5] arising from the Coulomb potential are given by

$$V=-\frac{i\hbar c^2}{4m^2c^4}\nabla V_c\cdot\mathbf{P}+\frac{\hbar c}{2m^2c^4}(\mu-\tfrac{1}{2})\boldsymbol{\sigma}\cdot(\nabla V_c\times\mathbf{P}), \quad (1)$$

where V_c represents the Coulomb potential and μ is the magnetic moment of the proton. The nonspin-dependent term makes a small contribution and will be ignored. The spin-dependent terms arise from the magnetic-moment interaction and the Thomas precession. The contribution of a similar term from the potential will be ignored since it can be considered as being included with the usual nuclear spin-orbit potential. Similarly we can neglect the contribution of Eq. (1) coming from the Coulomb potential inside the nucleus. The additional spin-orbit potential obtained from Eq. (1) is thus given by

$$V=(-)\frac{\hbar^2Ze^2}{2m^2c^2}\frac{(\mu-\frac{1}{2})}{r^3}\boldsymbol{\sigma}\cdot\mathbf{L};$$

$$r>R=\text{radius of charge distribution}$$

$$=0; \qquad r<R.$$

With the inclusion of this term in the Hamiltonian, the scattered amplitude will have the form,

$$f(\theta)=A_c+B_c\boldsymbol{\sigma}\cdot\mathbf{n}+C_c\boldsymbol{\sigma}\cdot\mathbf{n}+A_n+B_n\boldsymbol{\sigma}\cdot\mathbf{n}.$$

The vector $\mathbf{n}$ is the unit vector normal to the plane of scattering and is taken to be positive for scattering to the right. The amplitude $(A_c+B_c\boldsymbol{\sigma}\cdot\mathbf{n})$ represents the Coulomb scattering of a proton from a point charge; $(A_n+B_n\boldsymbol{\sigma}\cdot\mathbf{n})$ represents the nuclear scattering modified in the usual way by the presence of the charge distribution. C_c is the spin-dependent correction to the

* This work was performed under the auspices of the U. S. Atomic Energy Commission.
[1] Fernbach, Heckrotte, and Lepore, Phys. Rev. **97**, 1059 (1955).
[2] R. M. Sternheimer, Phys. Rev. **97**, 1314 (1955).
[3] This particular point was discussed with reference to neutrons by J. Schwinger, Phys. Rev. **73**, 487 (1948).
[4] The sign of the polarization has of course been deduced by the measurements of L. Marshall and J. Marshall, Phys. Rev. **98**, 1398 (1955).
[5] L. Rosenfeld, *Nuclear Forces* (Interscience Publishers, Inc., New York, 1949), p. 315.

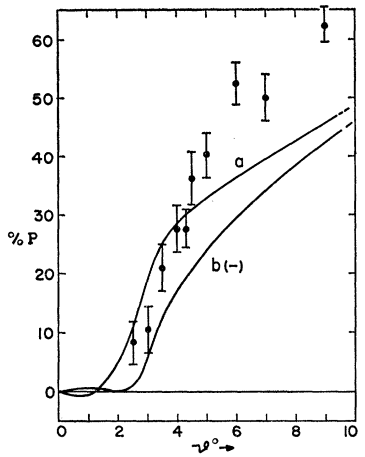

FIG. 1. Polarization of 300-Mev protons scattered elastically from carbon, (a) for the sign of the nuclear spin-orbit potential the same as the shell model assignment; (b) for the opposite sign of the nuclear spin-orbit potential. The sign of the polarization in (b) is the negative of the scale. The experimental points give the polarization of 315-Mev protons elastically scattered from carbon.[10]

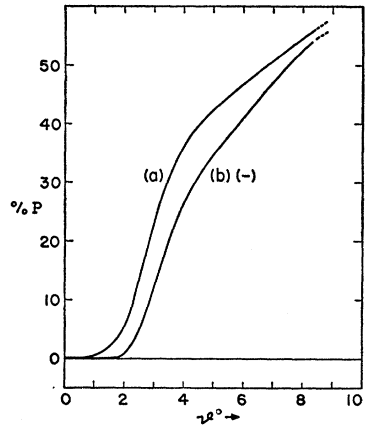

FIG. 2. Polarization of 300-Mev protons scattered elastically from aluminum, (a) for the sign of the nuclear spin-orbit potential the same as the shell model assignment; (b) for the opposite sign of the nuclear spin-orbit potential. The sign of the polarization in (b) is the negative of the scale.

Coulomb-scattering term to account for the charge distribution. A_n and B_n can be calculated in the usual manner.[1,2] To find the Coulomb-scattered amplitudes we obtain the WKB phase shifts[1] for the Coulomb central and spin-orbit potentials. These are

$$\eta_l = n\ln(l+\tfrac{1}{2}); \quad n = mZe^2/\hbar k,$$

$$\xi_l^{\pm} = n\frac{E}{mc^2}(\mu-\tfrac{1}{2})\begin{Bmatrix} l \\ -(l+1) \end{Bmatrix}\Big/(l+\tfrac{1}{2})^2, \qquad (3)$$

where η_l is the usual Coulomb phase shift, $\xi_l^{\pm}$ is the spin-dependent part of the phase shift, and E is the energy of the incident proton. Since $\xi_l^{\pm} \ll \eta_l$, we neglect the contribution of the Coulomb spin-orbit potential to A_c, which is then just the usual Coulomb scattering amplitude, and calculate B_c to first order in $\xi_l^{\pm}$. By converting the sum over l to an integral and making the small-angle approximation to $P_l(x)$,[6] one obtains for B_c,

$$B_c = \frac{(-i)}{k}\cdot 2\left(n\frac{E}{mc^2}(\mu-\tfrac{1}{2})\right)\int_0^{\infty} y^{(2in+1)-1}J_1(\theta y)dy, \qquad (4)$$

where θ is the angle of scattering. This integral can be evaluated from a formula given by Watson.[7] One readily obtains

$$B_c = \frac{n\exp[i\pi - in\ln(\theta/2)^2 + 2i\eta_0]}{2k(\theta/2)^2}\left[i\frac{E}{mc^2}(\mu-\tfrac{1}{2})\theta\right], \qquad (5)$$

where $\eta_0 = \arg\Gamma(1+in)$.

For small angle scattering such that $\sin\theta \simeq \theta$, one has then

$$A_c + B_c\boldsymbol{\sigma}\cdot\mathbf{n} = \frac{n\exp[i\pi - in\ln(\theta/2)^2 + 2i\eta_0]}{2k(\theta/2)^2} \times\left[1 + i\frac{E}{mc^2}(\mu-\tfrac{1}{2})\theta\boldsymbol{\sigma}\cdot\mathbf{n}\right]. \qquad (6)$$

This result, it may be noted, is what one obtains from the solution of the Dirac equation to the first order in n and (v/c).[8] Also it is a curious fact that, if one evaluates A_c by this method, the usual Coulomb scattering formula as given above [Eq. (6)] is obtained. Similarly, C_c is given by

$$C_c = \frac{(+i)}{k}\cdot 2\left(n\frac{E}{mc^2}(\mu-\tfrac{1}{2})\right)\int_0^{kR} y^{(2in+1)-1}J_1(\theta y)dy. \qquad (7)$$

For small-angle scattering ($kR\theta \lesssim 2$), the integral may be evaluated by expanding $J_1(\theta y)$ and using the first few terms.

It should be noted that this Coulomb spin-orbit potential can influence the small-angle scattering only. This simply follows from the uncertainty principle and the circumstance that the Coulomb spin-orbit potential is limited to the exterior of the nucleus.

CALCULATIONS

The polarization of 300-Mev protons scattered from carbon and aluminum was first calculated in the WKB approximation, assuming a parabolic-shaped central potential, and a real-gradient-type spin-orbit potential,[9] for both signs of the nuclear spin-orbit potential. The results of this calculation are shown in Figs. 1 and 2.

[6] Fernbach, Serber, and Taylor, Phys. Rev. **75**, 1352 (1949).
[7] G. N. Watson, *Bessel Functions* (University Press, Cambridge, 1944), second edition, p. 385.
[8] A. Garren, Ph.D. thesis, Carnegie Institute of Technology, 1955; Phys. Rev. **101**, 419 (1956).
[9] The parameters are given in reference 1.

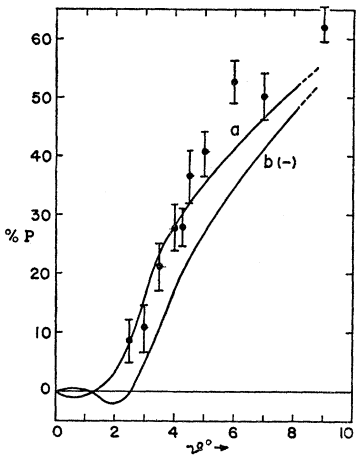

FIG. 3. Polarization (for a complex spin-orbit potential) of 300-Mev protons scattered elastically from carbon, (a) for the sign of the real part of the nuclear spin-orbit potential the same as the shell model assignment; (b) for the opposite sign of the real part of the nuclear spin-orbit potential. The sign of the polarization in (b) is the negative of the scale. The ratio of the imaginary part of the spin-orbit potential to the real part is (−0.5). The experimental points give the polarization of 315-Mev protons elastically scattered from carbon.[10]

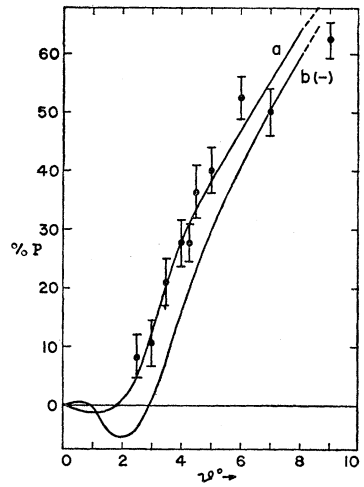

FIG. 4. Polarization (for a complex spin-orbit potential) of 300-Mev protons scattered elastically from carbon, (a) for the sign of the real part of the nuclear spin-orbit potential the same as the shell model assignment; (b) for the opposite sign of the real part of the nuclear spin-orbit potential. The sign of the polarization in (b) is the negative of the scale. The ratio of the imaginary part of the spin-orbit potential to the real part is (−1.0). The experimental points give the polarization of 315-Mev protons elastically scattered from carbon.[10]

The curve labeled (a) in each figure shows the expected polarization for the sign of the spin-orbit potential that corresponds with the choice of the shell model; curve (b), with the opposite choice. That the polarization for case (a) should be generally greater in magnitude than for case (b) follows from the sign of the Coulomb spin-orbit potential, which increases the total spin-orbit potential in case (a) and decreases it for case (b).

Accurate small-angle scattering data at this energy exist for carbon and not for aluminum, so that the calculations as they apply to carbon will be our chief concern. The experimental results[10] for the polarization of 315-Mev protons elastically scattered from carbon for angles running from 2.5° to 9° is plotted in Fig. 1. Two points stand out in the comparison of the experimental and calculated results. First, the differences in polarization between between cases (a) and (b) in the relevant range are as large as or larger than the experimental errors associated with the measurement. Second, and more important, neither curve fits the experimental points over the range of angles being considered. This discrepancy is quite marked and cannot be removed by changing the magnitude of the spin-orbit potential, which while changing the magnitude of polarization at any given point, will not materially change the shape of the curve over the considered range of angles. Neither, for instance, does any reasonable change in the charge radius effect a material difference. Also, if one refers to Sternheimer's calculations,[2] where another and markedly different form of the spin-orbit potential was also considered, one concludes that the difficulty cannot be resolved in this fashion. This leaves practically no other choice than to introduce another parameter. This was done by allowing the numerical coefficient of the spin-orbit potential to be complex. This is not unreasonable, since it merely implies that a spin dependence exists for the inelastically scattered nucleons, which has been experimentally verified.[11]

In Figs. 3 and 4 are plotted the expected polarizations for both cases (a) and (b) and on the assumption that the ratios of the imaginary to real spin-orbit potential are (−0.5) and (−1) respectively.[12] One sees in Fig. 4 that a good fit to the polarization data is obtained in case (a) for the latter ratio (−1). If the opposite sign of this ratio is assumed, the polarization is suppressed in the region beyond 4° rather than enhanced. Furthermore one notes from the behavior of the calculated polarization for case (b) in the region of 1° to 3° that one can effectively rule out this choice of sign of the spin-orbit potential.

DISCUSSION

The effect of the imaginary part of the spin-orbit potential is to make the total imaginary potential spin-

[10] Chamberlain, Segrè, Tripp, Wiegand, and Ypsilantis, Phys. Rev. **100**, 947 (1955).

[11] H. Bradner and R. Donaldson, Phys. Rev. **95**, 1701 (1954).

[12] The imaginary spin-orbit potential was treated as a first-order correction to the previous calculations (Fig. 1) in order to save a considerable amount of labor. This will cause a small uncertainty in the precise magnitude of this additional potential required to fit the experimental results, but not in the general conclusions reached.

dependent. For $j=l+\frac{1}{2}$ protons, this potential is

$$-i[w(r)-lf(r)], \tag{8a}$$

and for $j=l-\frac{1}{2}$ protons,

$$-i[w(r)+(l+1)f(r)], \tag{8b}$$

where $w(r)$ is the imaginary central potential and $f(r)$ is the imaginary spin-orbit potential. Both are assumed to be positive quantities, and the sign of the spin-orbit potential corresponds with that chosen in the previous section. Now one must insist that

$$[w(r)-lf(r)]\geq 0 \tag{9}$$

for all l. Otherwise the imaginary potential for $j=l+\frac{1}{2}$ protons would act as a source rather than a sink for these protons. Thus the form and magnitude of the imaginary spin-orbit potential are restricted by this condition. However, it is apparent that for any given $f(r)>0$ this condition is violated for l sufficiently large. Accordingly it would appear that $f(r)$ must be zero and that there can be no imaginary spin-orbit potential. This difficulty can be avoided however in several ways. The coefficient of the imaginary spin-orbit potential can be chosen to be some function of $\mathbf{l}_{op}^2$ such that the violation of Eq. (9) is avoided for large l. Or one can introduce an imaginary central potential which is some function of $\mathbf{l}_{op}^2$ such that for large l the violation of Eq. (9) is prevented. It would also be necessary in the latter case to give up the gradient form for the imaginary spin-orbit potential.

The numerical calculations that have been made are of course not consistent with the above remarks, since Eq. (9) is violated. One may ask, however, as a practical matter, of what consequence this is. We take the potentials to be of the forms (as used in the numerical calculation)

$$\begin{aligned} w(r)&=w_0\rho(r),\\ f(r)&=-\mu a^2(1/r)(d/dr)\rho(r),\\ \rho(r)&=1-r^2/R^2, \quad r<R,\\ &=0, \qquad\quad r>R, \end{aligned}$$

where w_0, μ, a^2, and R are constant. One finds then that

$$[w(r)-lf(r)]=\left[w_0(1-r^2/R^2)-l\left(\frac{2\mu a^2}{R^2}\right)\right].$$

For r sufficiently close to R this quantity will be less than zero. Thus only the outer fringes of the potential region will act as a source of $j=l+\frac{1}{2}$ protons. It is only for large l that the violation of Eq. (9) becomes of any consequence and these few high l terms may be neglected without materially affecting our calculated results or conclusions regarding the necessity of including an imaginary spin-orbit potential.

It was demonstrated in reference 1 that the gradient form of the spin-orbit potential followed from simple considerations if one interpreted the optical-model potential in terms of the individual nucleon-nucleon scattering events taking place inside the nucleus. The fact that such a radial form for the imaginary spin-orbit potential is unacceptable shows that the approximations made in that calculation are not completely justified so far as the spin-dependent scattering is concerned.

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