troughs of the associated space charge wave (which is an integral part of the electromagnetic wave). The mechanism has been discussed for simple space-charge electric waves.[2]

(4) Ion drifts should not be postulated without considering the secondary effects of the associated electric currents and electrical and mechanical fields of force.

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Growing Electric Space-Charge Waves and Haeff's Electron-Wave Tube

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A theory of "growing" electric space-charge waves in drifting, interpenetrating electron streams, or mixed ion and electron streams, has been developed by Pierce, Haeff, and others to explain the operation of the electron-wave tube and other amplifying devices and perhaps also the origin of some solar radio emission.

The theory is shown to be untenable, the growth predicted being spurious and due to misinterpretation of the dispersion equations. The waves which are thought to grow are evanescent waves being reflected back into an emitter which is moving through the gas. Only mathematically do they appear as real traveling waves with exponential growth.

An alternative mechanism to explain the operation of the amplifying tubes is briefly described.

A THEORY of "growing" electric space-charge waves in drifting, interpenetrating electron streams, or mixed ion and electron streams, has been developed by Haeff,[1–3] Pierce,[4–6] Nergaard,[7] Bohm and Gross,[8] Feinstein and Sen,[9] Rydbeck and Forsgren,[10] and others. The theory purports to show how the waves "grow" or steadily increase in amplitude as they propagate along the composite electron stream. It is believed to explain the operation of Haeff's electron-wave tube and other growing-wave tubes and perhaps also the origin of some solar radio emission.

It is the purpose of this note to show that the theory is not valid, the growth predicted being spurious and due to misinterpretation of the dispersion equations. Since the electron-wave tube and other amplifying devices demonstrably do work, an alternative mechanism is required and a likely one is briefly described.

The theory depends on a substitution analysis in which plane waves in time t and space x, of the form $\exp[i(\omega t-\Gamma x)]$, are sought of the relevant equations. The result is a dispersion equation relating ω and Γ but giving neither directly. In general both ω and Γ may be complex quantities: $\omega=\omega_r+i\omega_i$, $\Gamma=\Gamma_r+i\Gamma_i$, where ω_r, ω_i, Γ_r, Γ_i are all real and the wave has the form $\exp(\Gamma_i x-\omega_i t)\cdot\exp[i(\omega_r t-\Gamma_r x)]$. Finite values of ω_i and Γ_i are indicative of wave growth in time and space respectively. The dispersion equation for the electron-wave tube shows, within certain frequency bands, complex values of Γ for (assumed) real values of ω. This is interpreted as indicating the presence of traveling, growing waves whose energy increases steadily at the expense of the kinetic energy of the electron streams; Pierce[6] has called this an electromechanical process.

Twiss[11] has sensed a danger in this interpretation of the dispersion equation and shown that if Γ (instead of ω) is assumed real, then the equation gives complex values of ω indicating waves growing in time. Such growth is not observed experimentally so that doubt is cast on the theory. He concludes that a theory of growing waves may be developed only in relation to the boundaries which are essential in promoting growth. Neither Twiss's criticism nor emphasis on boundaries is found to be justified.

However, there is a fundamental error in the electron-wave tube theory due to misinterpretation of the dispersion equation. This is due principally to the fact fact that the frame of reference in which the wave is described is moving relative to the gas.

A space-charge wave is propagated relative to the

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electron gas in which it constitutes a perturbation and with which it may experience electromechanical effects. If the gas assumes a drift relative to the observer, the wave is carried with it and so assumes different apparent properties (described in the frame of reference of the observer). For example, a wave which is spatially attenuated but steady in time may, in certain frames of reference, appear to either grow or decay in time. It is less obvious but nevertheless true that certain waves may show spurious growth in space while remaining steady in time; an example is now given.

Consider space-charge waves in a gas whose electrons have random thermal motions but no mass drift. The well-known dispersion equation[12,13] has the form:

$$v_t^2\Gamma^2=\omega^2-\omega_0^2, \tag{1}$$

where v_t is of the order of the root-mean-square electron velocity and ω_0 is the plasma resonance frequency. Electron collisions with heavy ions and hence absorption are neglected. When $\omega>\omega_0$ the equation describes traveling, unattenuated waves moving with velocity $v_t(1-\omega_0^2/\omega^2)^{-\frac{1}{2}}$. When $\omega<\omega_0$ the waves are exponentially, spatially attenuated or evanescent waves. These are in process of being reflected back into the source of radiation (say a boundary) by the medium whose refractive index is imaginary. It is with these particular waves that we are concerned.

Now consider the same waves as seen by an observer moving with velocity U along the x axis. They have constants ω_1, Γ_1, given by the Lorentz transformation:

$$\omega=\beta(\omega_1+U\Gamma_1),\quad \Gamma=\beta[\Gamma_1+(U/c^2)\omega_1],$$

where $\beta=(1-U^2/c^2)^{-\frac{1}{2}}$. When $U\ll c$ and the wave velocity is of the order c or less, the Newtonian transformation,

$$\omega=\omega_1+U\Gamma_1,\quad \Gamma=\Gamma_1, \tag{2}$$

may be used. For simplicity this form is used here, this move being further justified by the fact that Haeff *et al.* used Newtonian mechanics. The dispersion equation is now found from Eqs. (1) and (2):

$$\Gamma_1=\frac{U\omega_1}{v_t^2-U^2}\left[1\pm\left\{1-\frac{(v_t^2-U^2)(\omega_0^2-\omega_1^2)}{U^2\omega_1^2}\right\}^{\frac{1}{2}}\right]. \tag{3}$$

When $v_t>U$ this equation gives complex values of Γ_1 for real values of ω_1 within the frequency range defined by

$$\omega_1<\omega_0(1-U^2/v_t^2)^{\frac{1}{2}}.$$

If this is interpreted in the manner of the electron-wave tube theory it indicates traveling waves, one of which grows in space.

The physical nature of these "growing waves" may easily be determined. The choice of ω_1 real means that the (steady) emitter moves (relative to the gas) with the observer as in the electron-wave tube theory; Γ_1 is then complex. Equation (2) shows that ω and Γ are both complex, the wave having the form $\exp(\Gamma_i x-\omega_i t)\cdot\exp[i(\omega_r t-\Gamma_r x)]$. There is no objection to a solution of Eq. (1) of this form; physically it means that the (steady) emitting source is moving relative to the observer (and gas) so that the signal strength changes with time. The waves are still evanescent, but during each wave period the emitter moves so that successive intensity maxima are displaced and the wave appears to travel. It is really a *traveling evanescent wave packet*, the real part of its *group* velocity being ω_i/Γ_i. When the observer also assumes this velocity, the wave has constants $\omega_1=\omega_r-\omega_i\Gamma_r/\Gamma_i$ and $\Gamma_1=\Gamma_r+i\Gamma_i$ and so appears steady in time but growing in space.

There is not the slightest reason for assuming that these waves really grow by increasing their energy at the expense of the electron kinetic energy. The spatial intensity change is due to a process of reflection and the waves appear to travel because the observer is moving relative to the gas. The presence of such spurious growing waves in this simple example suggests that they might occur in the more complex case of interpenetrating electron streams.

The dispersion equation[3] for two electron streams of velocities and densities (expressed as resonance frequencies) v_a, v_b, ω_a, ω_b is

$$\frac{\omega_a^2}{(\omega-v_a\Gamma)^2}+\frac{\omega_b^2}{(\omega-v_b\Gamma)^2}=1. \tag{4}$$

The relevant frame of reference is one in which the emitter is fixed but the gas moving. Within certain frequency bands Γ is complex for real values of ω which indicates, according to the theory, that the waves are growing in space and explains the amplifying characteristics of the electron-wave tube.

Since the observer is moving relative to the gas, Eq. (4) is analogous to Eq. (3) and might be expected to describe similar spurious growing waves. To test this, we transform to an observer sharing the "mean" gas velocity. In the case of equal density streams $(\omega_a=\omega_b=\omega_0)$, the mean velocity is obviously $\frac{1}{2}(v_a+v_b)$ and on transforming to a system with this velocity, Eq. (4) becomes

$$\frac{\omega_0^2}{(\omega_1-v\Gamma_1)^2}+\frac{\omega_0^2}{(\omega_1+v\Gamma_1)^2}=1, \tag{5}$$

where $v=\frac{1}{2}(v_b-v_a)$. If ω_1 is now assumed real, the values of Γ_1 are given by:

$$v^2\Gamma_1^2=\omega_1^2+\omega_0^2\pm(4\omega_1^2\omega_0^2+\omega_0^4)^{\frac{1}{2}}. \tag{6}$$

This equation is analogous to (1) and shows that Γ_1 may be real or imaginary but never complex so that at no frequency may growing waves ever occur in this medium. It is highly significant, however, that the frequency bands in which growing waves are predicted by a moving observer are just those, except for a

[12] J. J. Thomson and G. P. Thomson, *Conduction of Electricity in Gases* (Cambridge University Press, London, 1933), Vol. 2, p. 353.
[13] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851 (1949).

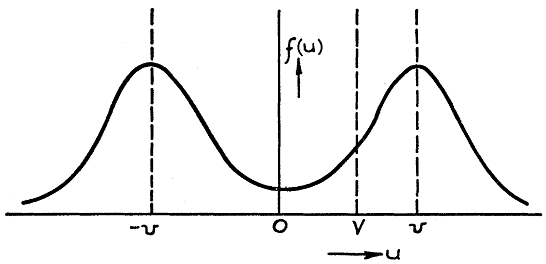

FIG. 1. A hypothetical electron velocity distribution in the double-beam electron-wave tube.

Doppler shift, in which evanescent waves are seen by a stationary observer. This same effect is seen when comparing Eqs. (1) and (3).

In the general case, $\omega_a \neq \omega_b$, it is not clear what is the physical significance of the "mean" gas velocity. However, it has been shown[9] that when it is taken as the simple arithmetic mean $\frac{1}{2}(v_a+v_b)$, then no growth is indicated. This result is sufficient for the present purposes, particularly when it is remembered that, according to Haeff, the case of equal density streams is one of particularly strong apparent growth. It is concluded that all waves which appear to grow according to the electron-wave tube and similar theories are really traveling evanescent waves. There is no real growth, the observed spatial change in intensity being due to a process of reflection and the wave travel to the fact that the gas carries the wave past the observer. It is further concluded that when wave growth is being investigated by the substitution analysis method *an observer should always be chosen stationary in the gas.*

When real and imaginary parts of a dispersion equation are considered separately, they provide only two relationships between the four variables; hence two of the variables may be chosen freely. A second rule of interpretation is that *the choice should be physically realizable and relevant* to the problem under consideration. In effect the choice amounts to assuming the method of injection of the wave into the medium. Thus in a particular equation we might assume ω real (that is we make $\omega_i=0$ and give a definite value of ω_r) and so find Γ complex. This means that the observer is assumed a fixed distance from a steady emitter so that the observed intensity is constant in time. The result that Γ is complex shows that the wave grows (or decays) as it propagates. The same equation might be interpreted (see for example reference 11) on the assumption that Γ is real and so ω complex. This means that the wave was introduced by an emitter whose intensity varied in such a way as to maintain successive wave crests of equal intensity. Subsequently the wave, uniform in space, would appear to grow (or decay) in time since successive waves passing the observer would be larger (or smaller). The two interpretations are really identical: the wave grows as it propagates through the medium.

Using different equations of motion of gas particles, Bohm and Gross[8] have derived the dispersion equation for space-charge waves in a pair of equal density electron streams having equal and opposite velocities. The equation has two roots given by

$$\omega_1^2 = -A\Gamma_1^2, \qquad (7)$$

where A is real and positive, so that now when ω_1 (Γ_1) is assumed real Γ_1 (ω_1) is imaginary. They assume Γ_1 real and conclude that the system is unstable. This means that when $t=0$ a wave of the form $\exp(-i\Gamma_r x)$ extends as a stationary and nonoscillatory wave throughout the medium. There seems no reasonable physical method of introducing such a wave, so that the assumption that Γ_1 is real appears unjustified. The alternative assumption of ω_1 real satisfies the equation equally well and has a satisfactory physical explanation: evanescent waves being reflected back into a stationary, steady emitter.

The same considerations apply to Eq. (5) when it is rearranged to give ω_1 as a function of Γ_1. One set of waves, within certain limits of Γ_1, have imaginary values of ω_1 and so are not physically realizable.

It appears that the wave growth mechanism described by the current electron-wave tube theory is spurious and an alternative theory of operation of this and other similar tubes is required. There remains one method of wave amplification and this has considerable physical intuitive appeal: growth may occur when electrons are trapped between potential troughs of a space-charge wave and lose some kinetic energy to the wave. Electrons with velocities just above and just below the wave velocity are trapped; the former on an average give energy to the wave and the latter subtract energy before being scattered back into the general velocity distribution $f(u)$. The criterion for wave growth[8] is that $\partial f(u)/\partial u$ should have a sufficiently large, positive value when $u \sim V$, the wave velocity. Figure 1 shows a hypothetical velocity distribution (about the mean velocity) for a double electron stream. Provided $V<v$, the condition for growth is clearly satisfied. The same explanation might serve for the simple "slipping" stream of electrons[3] provided the velocity distribution were asymmetrical about the mean velocity. The operation of this tube is, according to Pierce,[14] "something of a mystery." Finally the explanation may be relevant to the growing-wave tube of Pierce, although here the theory is complicated by the presence of a helix.

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[14] See reference 6, Chap. 16.