Shell Theory and Collective Theory for the Pb Isotopes*

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The 0.570-Mev gamma ray in Pb^{207} and the 0.803-Mev gamma ray in Pb^{206} are electric quadrupole transitions and have transition rates which are much too fast for neutron transitions. The surface tension parameters necessary to give the observed transition rates are determined by assuming that the collective motion of the core is weakly coupled to the external nucleons. The calculated surface tension parameters are 1100 Mev for Pb^{207} and 520 Mev for Pb^{206}. These two values are large compared to the hydrodynamic surface tension, as expected near closed shells, but the difference of a factor of 2 is unexpected. The quadrupole moment of Bi^{209} also indicates weak surface coupling. Also determined are the configurations which describe the low-lying energy levels of Pb^{204}, assuming short-range two-body forces. Reasonable agreement with experiment indicates that the shell model can account for the main portion of the level splitting. Many levels are predicted which have not been observed as yet. The 0.375-Mev gamma ray in Pb^{204} is anomalously slow even with weak surface coupling. A possible explanation for this slowdown is indicated by the configurations describing the initial and final states. In connection with the Pb^{204} problem, formulas are given for antisymmetrical wave functions in the j-j coupling scheme for three and four particles.

I. INTRODUCTION

THE collective model of the nucleus has had some success in explaining nuclear properties in the regions where the strong coupling approximation is valid.[1,2] Many features of the strongly coupled system, however, are insensitive to detailed assumptions of the model. In spite of less dramatic effects, a better test of the underlying hypotheses of the collective model might be obtained in a region of weak surface coupling, where one has a better chance to understand the extra-particle configurations, and where the competition between collective effects and the effects of direct particle coupling is more evident.[3]

Isotopes near Pb^{208} appear to be well suited for such a study. The very high first excited state[4] of Pb^{208}, measured electric quadrupole transition rates in other Pb isotopes[5,6] and the measured quadrupole moment[7] of Bi^{209} all give evidence of a weak surface coupling in these nuclei. The work of Alburger and Pryce[8] has shown that a very successful description of the low states of Pb^{206} can be obtained from a simple shell model perturbed only by short-range two-body forces.

If the surface coupling is sufficiently weak, it makes itself felt primarily only in electric quadrupole effects because the interaction Hamiltonian of the collective theory is itself of quadrupole form. Energy level spacings might, for example, be only slightly changed while $E2$ transition rates are greatly changed. Especially this could be the case in the isotopes of Pb, since the protons form a closed shell and cause almost the entire electric quadrupole moment to be of a collective type.

With these facts in mind, an investigation has been made of four nuclei near Pb^{208} in order (1) to find out to what extent their electric quadrupole properties can be accounted for by a given set of parameters of the collective theory, and (2) to extend and test further the shell theory of Alburger and Pryce.[8,9] From the experimentally measured $E2$ transition probabilities[6] in Pb^{206} and Pb^{207}, estimates of the required strength of surface coupling are made. The coupling strength determined from Pb^{207} is used to calculate the quadrupole moment of Bi^{209}. The result agrees favorably with the measured value of the quadrupole moment. The Pb^{206} and Pb^{207} coupling strengths, however, are not in close agreement.

The shell theory of Alburger and Pryce is extended to the four-particle system,[10] Pb^{204}, and gives a semi-quantitative description of the observed levels in this nucleus,[11] including an explanation for the long-lived 9− isomeric state. The 0.375-Mev $E2$ gamma ray in this nucleus is anomalously slow even for quite weak surface coupling. The shell model assignments of the levels involved in this transition (4+ to 2+) suggest a possible explanation for the anomaly, but still require a very weak surface coupling.

* Supported in part by a grant from the National Science Foundation.

[1] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).

[2] K. W. Ford, Phys. Rev. **90**, 29 (1953).

[3] Recent calculations, however, have also made possible detailed shell structure considerations in highly deformed nuclei. See, e.g., B. R. Mottelson and S. G. Nilsson, Phys. Rev. **99**, 1615 (1955), and reference 25.

[4] Elliott, Graham, Walker, and Wolfson, Phys. Rev. **93**, 356 (1954).

[5] N. H. Lazar and E. D. Klema, Phys. Rev. **98**, 710 (1955); D. E. Alburger and A. W. Sunyar, Phys. Rev. **98**, 276(A) (1955).

[6] P. H. Stelson and F. K. McGowan, Phys. Rev. **99**, 127 (1955).

[7] J. E. Mack, Rev. Modern Phys. **22**, 64 (1950).

[8] D. E. Alburger and M. H. L. Pryce, Phys. Rev. **95**, 1482 (1954).

[9] M. H. L. Pryce, Proc. Phys. Soc. (London) **A65**, 773 (1952).

[10] As Alburger and Pryce point out, 4 holes in a closed shell can be treated in the same manner as 4 nucleons outside of a closed shell.

[11] V. E. Krohn and S. Raboy, Phys. Rev. **97**, 1017 (1955).

II. WEAK SURFACE COUPLING

A. Theory

The interaction Hamiltonian of the collective theory has been given by several authors[1,12,13] and is of the form

$$H_{\rm int}=-\sum_i k_i(\hbar\omega/2C)^{\frac{1}{4}}\sum_m(b_m+(-1)^m b_{-m}^*)\times Y_{2m}(\theta_i,\varphi_i), \quad (1)$$

where k_i is a parameter which is of the order of magnitude of twice the kinetic energy of a particle in the the nucleus,[14] $\hbar\omega$ is the phonon energy of the collective motion, C is a measure of the surface energy due to deformation, b_m and b_m^* are the destruction and creation operators, respectively, of one phonon, and the particles are labeled by the index i.

Discussions of the theory for weak surface coupling have been given by a number of authors.[15–18] Only the important formulas needed for the following applications are summarized here. The notation follows most closely that of reference 17. $H_{\rm int}$ is considered to be a small perturbation admixing the zero-order states, $|\alpha J;\ NR;\ IM\rangle$, which are shell-model states of configuration α and angular momentum J coupled to a collective state of N phonons and angular momentum R to give a total angular momentum I. In the weak surface-coupling limit, where the particle level spacings and the energy level shifts due to the interaction can be neglected compared to the phonon energy, $\hbar\omega$, and only the one-phonon state of the collective motion is excited, perturbation theory yields the result that the wave function for a state with a given total angular momentum I can be written as

$$\psi_{I=J,M}=|\alpha J;\ 00;\ IM\rangle+\sum_{\alpha''J''}A_{I\alpha''J''}\,|\alpha''J'';\ 12;\ IM\rangle. \quad (2)$$

The coefficients $A_{I\alpha''J''}$ are given by

$$A_{I\alpha''J''}=-(\hbar\omega)^{-1}\langle\alpha J;\ 00;\ IM|H_{\rm int}|\alpha''J'';\ 12;\ IM\rangle, \quad (3)$$

and can be expressed concisely by the methods of Racah.[17,19,20] Defining $(\pi/5)^{\frac{1}{2}}Y_{2m}(\theta_i,\phi_i)$ as $\mathcal{Y}_{2m}(\theta_i,\phi_i)$ and $(5\hbar\omega/2\pi C)^{\frac{1}{2}}$ as γ, Eq. (3) becomes

$$A_{I\alpha''J''}=(k\gamma/\hbar\omega)(2I+1)^{-\frac{1}{2}}\langle\alpha J\|\sum_i\mathcal{Y}_2(\theta_i,\phi_i)\|\alpha''J''\rangle, \quad (4)$$

where the double bar reduced matrix elements are defined in references 19 and 20. The dimensionless parameter measuring the strength of coupling is $\xi=(k\gamma/\hbar\omega)$. The weak-coupling limit (one-phonon) remains valid, however, to $\xi\lesssim 1$, because the factors multiplying ξ in Eq. (4) are in general considerably less than unity. See, for example, reference 17 where, for a two-particle configuration, the weak-coupling limit is very good for $\xi\lesssim 0.8$.

Using the wave functions as given in Eq. (2), the contribution of the collective motion to the $E2$ transition rates can be determined. The collective quadrupole transition operator is

$$Q_{2m}=[3ZeR_0^2/4(5\pi)^{\frac{1}{2}}]\gamma(b_m+(-1)^m b_{-m}^*), \quad (5)$$

where R_0 is the effective radius of the nuclear charge distribution. The transition rate for $E2$ radiation is[1]

$$T=(4\pi\kappa^5/75\hbar)B_e(2), \quad (6)$$

where κ is the wave number. $B_e(2)$, the reduced transition probability, is given by

$$B_e(2)=(2I'+1)^{-1}\sum_{mMM'}|\langle IM|Q_{2m}|I'M'\rangle|^2, \quad (7)$$

where I' and I are, respectively, the angular momentum of the initial and final states. In the weak-coupling limit, $B_e(2)$ reduces to

$$B_e(2)=[3ZeR_0^2\gamma/4(5\pi)^{\frac{1}{2}}]^2(2I+1)\times|[A_{I'\alpha I}/(2I+1)^{\frac{1}{2}}]+[A_{I\alpha' I'}/(2I'+1)^{\frac{1}{2}}]|^2, \quad (8)$$

where the $A_{I\alpha J}$'s are given by Eq. (4). It is significant to note that cross terms between different particle configurations do not occur in Eq. (8) because the collective operator, Q_{2m}, is independent of the particle coordinates. Since $A_{I\alpha J}$ is proportional to $k\gamma/\hbar\omega$, $B_e(2)$ is seen to be proportional to $(k\gamma^2/\hbar\omega)^2=(5k/2\pi C)^2$. $B_e(2)$ is, therefore, independent of the mass parameter B of the collective theory and depends only on the ratio k/C of the other two independent parameters of the collective theory. This is in interesting contrast to the moments of inertia of the highly deformed nuclei which depend only on B.

The quadrupole moment is defined as the expectation value of $(16\pi/5)^{\frac{1}{2}}Q_{20}$. In the weak-coupling limit, the collective quadrupole moment is

$$Q_c=(6/5)ZeR_0^2\gamma A_{I\alpha I}[I(2I-1)/(I+1)(2I+3)]^{\frac{1}{2}}. \quad (9)$$

For a single extra particle, Eq. (9) reduces to

$$Q_c=-(3/20)ZeR_0^2(k\gamma^2/\hbar\omega)[(2I-1)/(I+1)]. \quad (10)$$

Note that Q_c, like $B_e(2)$, depends only on the ratio of k/C.

B. Applications

Lead-207

Table I shows the experimentally determined low-lying levels[5,8] of Pb^{207}. Stelson and McGowan[6] have measured the Coulomb excitation cross section of the first excited state. They determine the reduced transi-

[12] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 14 (1952).
[13] F. J. Milford, Phys. Rev. **93**, 1297 (1954).
[14] E. Feenberg and K. C. Hammack, Phys. Rev. **81**, 285 (1951).
[15] A. K. Kerman, Phys. Rev. **92**, 1176 (1953).
[16] D. C. Choudhury, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **28**, No. 4 (1954).
[17] K. W. Ford and C. Levinson, Phys. Rev. **100**, 1 (1955).
[18] P. Goldhammer and E. Feenberg, Phys. Rev. **99**, 648(A) (1955).
[19] G. Racah, Phys. Rev. **62**, 438 (1942).
[20] For a concise summary of Racah's results in reference 19, see Simon, Vander Sluis, and Biedenharn, Oak-Ridge National Laboratory Report ORNL-1679 (unpublished).

tion probability for the decay ($f_{5/2}\rightarrow p_{1/2}$) to be

$$B_e(2)=(0.096 \text{ barn})^2e^2. \quad (11)$$

This value corresponds to a half-life of 1.0×10^{-10} sec, which is consistent with the result reported by Lazar and Klema[5] that

$$0.7\times10^{-10}<t_{1/2}<4\times10^{-10} \text{ sec}. \quad (12)$$

The measured transition probability is almost as large as for a single proton transition, for which

$$[B_e(2)]_{\text{proton}}=(0.140 \text{ barn})^2e^2, \quad (13)$$

and is many orders of magnitude larger than would be expected for a pure neutron transition. [In Eq. (13), it is assumed that $\langle r^2\rangle_{\text{proton}}=\frac{3}{5}(1.17\times10^{-13}A^{1/3} \text{ cm})^2$.]

Even if the neutron in nuclear matter carries an effective charge, the transition probability cannot be accounted for in terms of a single neutron transition. Brueckner[21] has suggested an effective charge for the neutron of magnitude $e_{\text{eff}}=-0.17e$. The trend of magnetic moments of odd-neutron nuclei, however, suggests that this value is an overestimate. In particular, the magnetic moment[22] and quadrupole moment[23] of O^{17} are most easily interpreted by supposing that the effective charge of the neutron is indeed close to zero. An effective charge of $\frac{1}{6}e$, however, would still give a transition probability smaller by a factor of 17 than that observed. One is therefore forced to conclude that the major part of the $E2$ transition probability in Pb^{207} is due to the collective effects. Similar arguments apply to the other isotopes of lead. In this paper, it is assumed that the collective effects of surface coupling account for all of the transition probability.

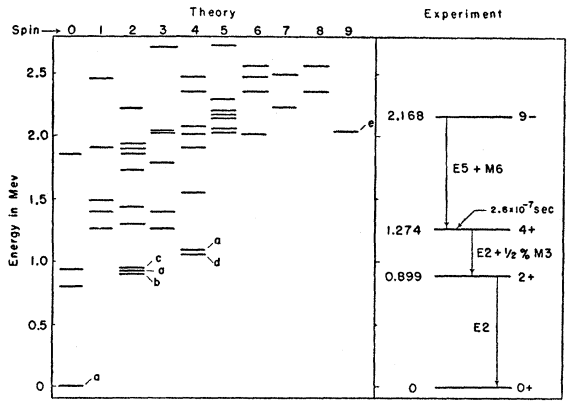

FIG. 1. Energy levels of Pb^{204}. The levels of greatest interest are labeled by small letters which designate the configurations as as follows: a—$(p_{1/2}^2, J_1=0)f_{5/2}^2$, b—$(f_{5/2}^3, J_1=5/2)p_{1/2}$, c—$(f_{5/2}^2, J_1=0)p_{1/2}p_{3/2}$, d—$(p_{1/2}^2, J_1=0)p_{3/2}f_{5/2}$, e—$(p_{1/2}^2, J_1=0)f_{5/2}i_{13/2}$. The configurations associated with the other levels may easily be found by referring to Table IV.

[21] K. A. Brueckner (private communication).
[22] F. Alder and F. C. Yu, Phys. Rev. **81**, 1067 (1951).
[23] Geschwind, Gunther-Mohr, and Silvey, Phys. Rev. **85**, 474 (1952).

As has been pointed out, $B_e(2)$ determines the ratio $(k/C)^2$. The parameter k which is determined by the first-order shift of particle energy levels is not subject to much uncertainty, and in particular should not vary markedly as one moves through the periodic table. If one takes $k=40$ Mev[1] and $R_0=1.17\times10^{-13}A^{1/3}$ cm,[24] C is determined by the transition rate to be 1100 Mev. If, in addition, the mass parameter B of the collective theory is specified (with much less certainty) by $\hbar^2/B=0.025$ Mev,[2] then the phonon energy is $\hbar\omega=[(\hbar^2/B)C]^{\frac{1}{2}}=5.2$ Mev. These values are to be compared with the "hydrodynamic" values,[12] $C=63$ Mev and $\hbar\omega=1.3$ Mev. This is in accord with the qualitative expectation that near closed shells C and therefore $\hbar\omega$ should increase, as the nucleus is not as deformable as the "hydrodynamic" model would indicate. According to Moszkowski,[25] one can expect the surface tension near closed shells to exceed the hydrodynamic value by about a factor of ten. For the parameters so determined, $\xi=(k\gamma/\hbar\omega)=0.47$, and the mixture amplitudes $A_{I\alpha''J''}$ are of the order of 0.15. Hence the one-phonon approximation is quite good for this calculation.

Lead-206

Stelson and McGowan[6] have also determined the reduced transition probability for the decay ($I'=2$ to $I=0$) from the first excited state to the ground state of Pb^{206}. They found that

$$B_e(2)=(0.158 \text{ barn})^2e^2. \quad (14)$$

As a result of the work done by Alburger and Pryce,[8] the configurations assigned to the ground state and the first excited state were respectively $p_{1/2}^2$ and $p_{1/2}f_{5/2}$.

Assuming, as in Pb^{207}, $k=40$ Mev and $R_0=1.17\times10^{-13}A^{1/3}$ cm, one gets $C=520$ Mev. This value of C is much larger than the "hydrodynamic" value but smaller by a factor of two than that found in Pb^{207}. With $\hbar^2/B=0.025$ Mev, as in Pb^{207}, $\hbar\omega=3.6$ Mev and is also different from the phonon energy determined in Pb^{207}.

In view of the approximation inherent in the limiting weak-coupling formulas used, this difference may not be a serious discrepancy. However, if the coupling strength is really as weak as suggested by the Pb^{207} result, then the perturbation formulas should not be in error by such a large factor. It is possible that this error does not arise in the perturbation formulas but in assigning particle configurations to the nuclear states. That is, with configuration interaction, the particle configurations can become mixed. In Pb^{206}, the most plausible place for configuration mixing is the ground state. One would expect that a small amount of the $f_{5/2}^2$ $J=0$ configuration would mix with the $p_{1/2}^2$ $J=0$ configuration. Calculations have shown, however, that regardless of the mixing of the $p_{1/2}^2$ and $f_{5/2}^2$ configura-

[24] K. W. Ford and D. L. Hill, Phys. Rev. **94**, 1630 (1954).
[25] S. A. Moszkowski, Phys. Rev. **99**, 803 (1955).

tions in the ground state, the values of C and $\hbar\omega$ determined from the Pb^{206} data cannot be brought into agreement with the values of C and $\hbar\omega$ determined by the Pb^{207} data. There appears, therefore, to be a real difference in the value of C for the two isotopes, and this difference does not arise from approximations made in the limiting case of weak coupling.

Bismuth-209

As a further possible check on the correctness of the weak-coupling approximation, the quadrupole moment of the ground state of Bi^{209} was calculated. The collective contribution to the quadrupole moment is given by Eq. (10). The direct particle contribution is

$$Q_p=-\tfrac{1}{2}e\langle r^2\rangle[(2I-1)/(I+1)]. \quad (15)$$

If one used $C=1100$ Mev as determined by Pb^{207}, the nuclear radius as $1.17\times10^{-13}\,A^{1/3}$ cm, and in addition, $\langle r^2\rangle$ is set equal to $\frac{3}{5}R_0^2$, one finds that

$$Q_c=-0.25 \text{ barn}, \quad Q_p=-0.21 \text{ barn},$$

and

$$Q=Q_c+Q_p=-0.46 \text{ barn}. \quad (16)$$

This value of the quadrupole moment compares favorably with the experimental value[7] of -0.4 barn. Although neither the experimental value of Q nor the theoretical value of $\langle r^2\rangle$ are sufficiently precise to make this a good test of the surface coupling strength, the result does again suggest a very weak surface coupling. A similar conclusion has been reached by Mayer and Jensen.[26]

Lead-204

The experimentally determined energy levels[11] of Pb^{204} are shown in Fig. 1. The 0.375-Mev transition (4+ to 2+) should be another case where the weak surface coupling can be tested, as the lifetime of the 4+ state is known. From the energy levels of Pb^{207} given in Table I, it is reasonable (without detailed calculations) to assign the configuration $p_{1/2}{}^2f_{5/2}{}^2$ to the 0+, 2+, and 4+ levels. Using Eqs. (6) and (8), $k=40$ Mev, $R_0=1.17\times10^{-13}A^{1/3}$ cm, and $C=1100$ Mev as determined from Pb^{207}, the lifetime of the 4+ level is 1.3×10^{-9} sec. This lifetime is much smaller than the observed value of 2.6×10^{-7} sec. A wrong choice of configurations for the energy levels is the most likely reason for this discrepancy. Therefore, the possible configurations describing the low-lying energy levels of Pb^{204} were investigated in greater detail. The results of this investigation will be discussed in Sec. IV.

It is to be noted however, that if the particle configurations for the two levels differ by more than one of the j_i of the particles (e.g., $p_{1/2}{}^2f_{5/2}{}^2$ and $p_{3/2}{}^2f_{5/2}{}^2$), the reduced transition probability will no longer be given by Eq. (8). The $A_{I\alpha''J''}$ of Eqs. (2) and (3) will now be zero as the operator H_{int} is a sum of one-particle operators and will not connect these two states in the first order calculations. Then second-order effects would have to be considered in order to calculate the contribution of the weak surface coupling to the enhancement of the transition between the two levels This point will be discussed further in Sec. IV.

III. ANTISYMMETRIC WAVE FUNCTIONS

Often it is desirable to have antisymmetric wave functions in a special form when considering certain types of matrix elements. In this paper, we shall be interested in the expectation value of the interaction energy of a group of particles when the interaction energy is a sum of two-particle interactions. That is, we shall want to evaluate the matrix element

$$V_{\text{int}}=\langle\psi_a|\sum_{i>j=1}^{n}V_{ij}|\psi_a\rangle, \quad (17)$$

where the subscript a indicates that we have an antisymmetric wave function. Due to the antisymmetry of the wave function, Eq. (17) can be rewritten as[27,28]

$$V_{\text{int}}=(n/n-2)\langle\psi_a|\sum_{i>j=1}^{n-1}V_{ij}|\psi_a\rangle. \quad (18)$$

This process can be repeated $n-2$ times and one finally obtains the matrix element const$\times\langle\psi_a|V_{12}|\psi_a\rangle$ between particles one and two.

For the evaluation of a matrix element of this type, it is convenient to have the wave function expressed in the following manner:

$$\psi_a=\sum_{j_i\alpha'J'}((\alpha'J')j_iJ\|\alpha J)|(\alpha'J')_aj_i(n)J\rangle, \quad (19)$$

where α' represents the configuration j_1, j_2, $\cdots j_{i-1}$, j_{i+1}, $\cdots j_n$ (together with other required quantum numbers) combining together to give J' which is antisymmetric with respect to the interchange of any of its $n-1$ particles, j_i is associated with the nth particle as

TABLE I. Energy levels of Pb^{207}.

Configuration	Parity	Energy (Mev)
$p_{1/2}$	−	0.00
$f_{5/2}$	−	0.570
$p_{3/2}$	−	0.87
$i_{13/2}$	+	1.634
$h_{9/2}$	−	2.35[a]

[a] *Note added in proof.*—Since the preparation of this article, it has been brought to my attention that D. E. Alburger and A. W. Sunyar [Phys. Rev. 99, 695 (1955)] have evidence that the level called $h_{9/2}$ is actually a $f_{7/2}$ level. However, this change does not affect any results in this article.

[26] M. G. Mayer and J. Hans D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley and Sons, Inc., New York, 1955), p. 115. Their surface tension parameter S is equal to $16\pi/45$ times our C.

[27] G. Racah, Phys. Rev. **76**, 1352 (1949).

[28] M. Redlich, thesis, Princeton University, January, 1954 (unpublished). M. Redlich, Phys. Rev. **99**, 1427 (1955).

indicated by the n in parentheses, and j_i is vector-coupled to J' to give an angular momentum of J. [In Eq. (19) and the following, j is shorthand for all one-particle quantum numbers, e.g., n, l, j.] If ψ_a is represented by Eq. (19), then Eq. (18) becomes

$$V_{\text{int}}=[n/(n-2)]\sum_{ji\alpha'J'}|((\alpha'J')j_iJ\|\alpha J)|^2 \times\langle\alpha'J'|\sum_{i>j=1}^{n-1}V_{ij}|\alpha'J'\rangle. \quad (20)$$

In this section, the coefficients of Eq. (19) will be derived when $n=3$ and $n=4$.

First, consider 3 particles with different j_i (i.e., inequivalent particles). It is possible to write an antisymmetric wave function in the following way[28]:

$$\psi_a=(\tfrac{1}{3})^{\frac{1}{2}}\{|[j_1(1)j_2(2)J_1]_aj_3(3)J\rangle -|[j_1(1)j_2(3)J_1]_aj_3(2)J\rangle +|[j_1(2)j_2(3)J_1]_aj_3(1)J\rangle\}, \quad (21)$$

where the configuration $[j_1(1)j_2(2)J_1]_a$ is antisymmetric with respect to interchange of particles one and two. The same notation without the subscript a will be used for nonantisymmetric vector-coupled states. The first term is already of the form of Eq. (19) while the last two terms are not. However, both of the last two terms are of the same form and can be expanded. Expanding the second term, we get

$$|[j_1(1)j_2(3)J_1]_aj_3(2)J\rangle =\sum_{J'}\{a_{J'}|[j_1(1)j_3(2)J']j_2(3)J\rangle +b_{J'}|[j_2(1)j_3(2)J']j_1(3)J\rangle\}. \quad (22)$$

$a_{J'}$ and $b_{J'}$ are given explicitly in terms of Racah coefficients[29] and are

$$a_{J'}=(\tfrac{1}{2})^{\frac{1}{2}}\langle[j_1j_3J']j_2J|[j_1j_2J_1]j_3J\rangle =[\tfrac{1}{2}(2J'+1)(2J_1+1)]^{\frac{1}{2}}W(J_1j_3j_2J';\,Jj_1), \quad (23)$$

and

$$b_{J'}=-(\tfrac{1}{2})^{\frac{1}{2}}\langle[j_2j_3J']j_1J|[j_1j_2J_1]j_3J\rangle =(-1)^{j_1+J'-J+1}[\tfrac{1}{2}(2J'+1)(2J_1+1)]^{\frac{1}{2}} \times W(j_1j_2Jj_3;\,J_1J'). \quad (24)$$

Expansion of the third term gives the same coefficients as the second term but has the particle numbers 1 and 2 reversed. Substitution of these back in Eq. (21), together with appropriate combination of terms to yield antisymmetric 2-particle wave functions, gives

$$\psi_a=(\tfrac{1}{3})^{\frac{1}{2}}|[j_1(1)j_2(2)J_1]_aj_3(3)J\rangle +\sum_{J'}\{[-(\tfrac{2}{3})^{\frac{1}{2}}a_{J'}]|[j_1(1)j_3(2)J']_aj_2(3)J\rangle +[-(\tfrac{2}{3})^{\frac{1}{2}}b_{J'}]|[j_2(1)j_3(2)J']_aj_1(3)J\rangle\}. \quad (25)$$

[29] G. Racah, Phys. Rev. **63**, 367 (1943).

Comparing Eqs. (19) and (25), we see that[30]

$$((j_1j_2J_1)_aj_3J\|\alpha J)=(\tfrac{1}{3})^{\frac{1}{2}}, \quad (26)$$

$$((j_1j_3J')_aj_2J\|\alpha J) =-[\tfrac{1}{3}(2J'+1)(2J_1+1)]^{\frac{1}{2}}W(J_1j_3j_2J';\,Jj_1), \quad (27)$$

and

$$((j_2j_3J')_aj_1J\|\alpha J)=(-1)^{j_1+J'-J}[\tfrac{1}{3}(2J'+1)(2J_1+1)]^{\frac{1}{2}} \times W(j_1j_2Jj_3;\,J_1J'). \quad (28)$$

The case when all three j_i's are the same has been treated in other places.[29,31,32]

The procedure for four particles is very similar to that for three particles and will only be outlined here. Only the cases where at least two of the j_i's are the same will be considered. One may easily extend this work to the case where all four j_i's are different. Let us consider first the case where j_1^2 couples to give J_1, j_2 and j_3 couple to give J_2, and J_1 and J_2 couple to give J. An antisymmetric wave function can be written in the following form[33]:

$$(j_1j_2m_1m_2|\,j_1j_2jm)=(-1)^{j_1+j_2-j}(j_2j_1m_2m_1|\,j_2j_1jm).$$

$$\begin{aligned}\psi_a=(\tfrac{1}{6})^{\frac{1}{2}}\{&|[j_1(1)j_1(2)J_1]_a[j_2(3)j_3(4)J_2]_aJ\rangle\\ &+|[j_1(2)j_1(3)J_1]_a[j_2(1)j_3(4)J_2]_aJ\rangle\\ &+|[j_1(3)j_1(1)J_1]_a[j_2(2)j_3(4)J_2]_aJ\rangle\}\\ &+(-1)^{J_1+J_2-J}(\tfrac{1}{6})^{\frac{1}{2}}\\ &\times\{|[j_2(1)j_3(2)J_2]_a[j_1(3)j_1(4)J_1]_aJ\rangle\\ &+|[j_2(2)j_3(3)J_2]_a[j_1(1)j_1(4)J_1]_aJ\rangle\\ &+|[j_2(3)j_3(1)J_2]_a[j_1(2)j_1(4)J_1]_aJ\rangle\}. \end{aligned} \quad (29)$$

All six terms in Eq. (29) are of the same form. The first term, when expanded, becomes

$$|[j_1(1)j_1(2)J_1]_a[j_2(3)j_3(4)J_2]_aJ\rangle =\sum_{J'}\{d_{J'}|([j_1(1)j_1(2)J_1]_aj_2(3)J')j_3(4)J\rangle +e_{J'}|([j_1(1)j_1(2)J_1]_aj_3(3)J')j_2(4)J\rangle\}, \quad (30)$$

where

$$d_{J'}=(\tfrac{1}{2})^{\frac{1}{2}}\langle[J_1j_2J']j_3J|J_1[j_2j_3J_2]J\rangle =[\tfrac{1}{2}(2J'+1)(2J_2+1)]^{\frac{1}{2}}W(J_1j_2Jj_3;\,J'J_2), \quad (31)$$

and

$$e_{J'}=-(\tfrac{1}{2})^{\frac{1}{2}}\langle[J_1j_3J']j_2J|J_1[j_2j_3J_2]J\rangle =(-1)^{j_2+j_3-J_2+1}[\tfrac{1}{2}(2J'+1)(2J_2+1)]^{\frac{1}{2}} \times W(J_1j_3Jj_2;\,J'J_2). \quad (32)$$

All six terms can be expanded in this manner. Upon expanding all six terms, substituting in Eq. (29), recom-

[30] For the special case when $j_1=j_2$, multiply Eq. (27) by $\sqrt{2}$ and ignore Eq. (28).

[31] A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. (London) **A214**, 515 (1952).

[32] S. Meshkov, Phys. Rev. **91**, 871 (1953).

[33] Where use has been made of the property of the Clebsch-Gordan coefficients that

TABLE II. Zero-order energy levels of Pb^{204}.

Configuration	Zero order energy (Mev)	Possible I's	Parity
$p_{1/2}^2 f_{5/2}^2$	1.14	0, 2, 4	+
$p_{1/2}^2 p_{3/2} f_{5/2}$	1.44	1, 2, 3, 4	+
$f_{5/2}^3 p_{1/2}$	1.71	1, 2, 3, 4, 5	+
$p_{1/2}^2 p_{3/2}^2$	1.74	0, 2	+
$f_{5/2}^2 p_{1/2} p_{3/2}$	2.01	0, 1, 2, 3, 4, 5, 6	+
$p_{1/2}^2 f_{5/2} i_{13/2}$	2.20	4, 5, 6, 7, 8, 9	−
$f_{5/2}^4$	2.28	0, 2, 4	+
$(p_{3/2}^2, J_1=2)(p_{1/2} f_{5/2}, J_2=3)$	2.31	5[a]	+
$p_{1/2}^2 p_{3/2} i_{13/2}$	2.50	5, 6, 7, 8	−
$(f_{5/2}^3, J_1=9/2) p_{3/2}$	2.58	5,[a] 6[a]	+

[a] Only the indicated spin values were investigated.

bining and comparing with Eq. (19), one gets[34]

$$([[(j_1^2 J_1) j_2 J']_a j_3 J] \alpha J) = [\tfrac{1}{4}(2J'+1)(2J_2+1)]^{\frac{1}{2}} W(J_1 j_2 J j_3; J' J_2), \quad (33)$$

$$([[(j_1^2 J_1) j_3 J']_a j_2 J] \alpha J) = (-1)^{j_2+j_3-J_2+1} [\tfrac{1}{4}(2J'+1)(2J_2+1)]^{\frac{1}{2}} \times W(J_1 j_3 J j_2; J' J_2), \quad (34)$$

and

$$([[(j_2 j_3 J_2) j_1 J']_a j_1 J] \alpha J) = (-1)^{J_1+J_2-J} [\tfrac{1}{2}(2J'+1)(2J_1+1)]^{\frac{1}{2}} \times W(J_2 j_1 J j_1; J' J_1). \quad (35)$$

The same procedure follows when one considers three particles with equal j_i adding to give J_1 which in turn adds to one particle with a different j_i to give a total angular momentum of J. In this case, the three-particle antisymmetric wave function can be written as

$$\psi_a(j_1^3 J_1) = \sum_{J_2} \beta_{J_2} | (j_1^2 J_2)_a j_1 J_1 \rangle, \quad (36)$$

where the β_{J_2} are the fractional parentage coefficients.[29,31,32] Proceeding as before, one finds for this case that

$$\psi_a = (\tfrac{1}{4})^{\frac{1}{2}} | (j_1^3 J_1)_a j_2(4) J \rangle + \sum_{J_2 J'} f_{J_2 J'} \times | ([j_1(1) j_1(2) J_2] j_2(3) J')_a j_1(4) J \rangle, \quad (37)$$

where

$$f_{J_2 J'} = -\beta_{J_2} [\tfrac{3}{4}(2J'+1)(2J_1+1)]^{\frac{1}{2}} \times W(J_1 j_2 j_1 J'; J J_2). \quad (38)$$

In this manner, antisymmetrical wave functions were found for the possible particle configurations which could occur in Pb^{204} in order to find the interaction energies between the particles. The results of this calculation are discussed in the next section.

IV. SHELL STRUCTURE OF Pb^{204}

The weakness of the surface coupling deduced in Sec. II suggests that the energies of the low-lying levels in the Pb isotopes should be well described by a shell model with a rigid spherical core. The work of Alburger and Pryce[8] has shown, moreover, that a very simple form of the shell theory, one which works in the short-range limit and ignores configuration mixing, yields semiquantitative agreement with experiment for Pb^{206}. Accordingly, this theory has been extended in a straightforward way to describe Pb^{204}.

[34] For the special case when $j_2 = j_3$, multiply Eq. (33) by $\sqrt{2}$ and ignore Eq. (34).

TABLE III. Alburger and Pryce's interaction parameters.

Configuration		ϵ in Mev
$p_{1/2}^2$	ϵ_1	0.4
$p_{1/2} f_{5/2}$	ϵ_2	0.3
$p_{1/2} p_{3/2}$	ϵ_3	0.4
$f_{5/2}^2$	ϵ_4	0.4
$p_{3/2} f_{5/2}$	ϵ_5	0.3
$p_{1/2} i_{13/2}$	ϵ_6	0.2
$p_{3/2}^2$	ϵ_7	0.4
$f_{5/2} i_{13/2}$	ϵ_8	0.2
$p_{3/2} i_{13/2}$	ϵ_{10}	0.2

The one-particle energy levels of Pb^{207} are shown in Table I. Using these as a basis, the antisymmetric wave functions for the configurations shown in Table II were found by the methods outlined in Sec. III. The calculation of diagonal matrix elements of a short-range potential with the antisymmetric wave functions then followed exactly the method of reference 8. The results for the more important low-lying levels of Pb^{204} are given in Table IV—first in terms of Alburger and Pryce's energy parameters, ϵ_1, ϵ_2, $\cdots\epsilon_{10}$ shown in Table III; second, in Mev, using the numerical values of the ϵ's used in reference 8 to describe Pb^{206} and also listed in Table III. These interaction energies shown in Table IV were then subtracted from the zero-order energies as given in Table II and the resulting energies were normalized so that the ground state of Pb^{204} would be at zero. Figure 1 shows the predicted positions of the low-lying levels of Pb^{204}, and, for comparison, the experimentally determined levels up to 2.2 Mev.

One cannot draw quantitative conclusions about the three 2+ levels which lie close together nor about the two 4+ levels, as configuration interaction has not been considered. The ϵ's used are also uncertain and thus the order of these levels may not be correct. However, for such high energy shifts due to the diagonal matrix elements, the agreement between experiment and theory is quite good. In particular, a 9− isomeric state is predicted at 2.05 Mev, to be compared with the observed isomeric state, probably 9−,[11] at 2.19 Mev. From this state, the predicted dominant gamma ray cascade 9− to 4+ to 2+ to 0+ agrees with experiment. A considerable number of other levels is predicted below 2 Mev. It would be extremely interesting to know if a careful analysis of Pb^{204} (perhaps by exciting the nucleus by some other method) could reveal some of these other levels. It is to be noted that no free parameters were adjusted in this calculation. The parameters adjusted to fit the Pb^{206} data[8] were used without modification. It may be remarked also that

TABLE IV. Interaction energies for various configurations (Mev).

Configuration	Spin I	V_{int}, the interaction energy in terms of the ϵ's	Numerical value of V_{int}	Normalized energy
$p_{1/2}^2 f_{5/2}^2$	0	$\epsilon_1+2\epsilon_2+3\epsilon_4$	2.20	0.00
$p_{1/2}^2 f_{5/2}^2$	2	$\epsilon_1+2\epsilon_2+0.686\epsilon_4$	1.27	0.93
$p_{1/2}^2 f_{5/2}^2$	4	$\epsilon_1+2\epsilon_2+0.286\epsilon_4$	1.11	1.09
$p_{1/2}^2 p_{3/2} f_{5/2}$	1	$\epsilon_1+\epsilon_2+\epsilon_3$	1.10	1.40
$p_{1/2}^2 p_{3/2} f_{5/2}$	2	$\epsilon_1+\epsilon_2+\epsilon_3+0.343\epsilon_5$	1.20	1.30
$p_{1/2}^2 p_{3/2} f_{5/2}$	3	$\epsilon_1+\epsilon_2+\epsilon_3$	1.10	1.40
$p_{1/2}^2 p_{3/2} f_{5/2}$	4	$\epsilon_1+\epsilon_2+\epsilon_3+1.14\epsilon_5$	1.44	1.06
$(f_{5/2}^3, J_1=3/2) p_{1/2}$	1	$2\epsilon_2+1.71\epsilon_4$	1.28	1.49
$(f_{5/2}^3, J_1=3/2) p_{1/2}$	2	$1.20\epsilon_2+1.71\epsilon_4$	1.04	1.73
$(f_{5/2}^3, J_1=5/2) p_{1/2}$	2	$2.20\epsilon_2+3\epsilon_4$	1.86	0.91
$(f_{5/2}^3, J_1=5/2) p_{1/2}$	3	$\epsilon_2+3\epsilon_4$	1.50	1.27
$(f_{5/2}^3, J_1=9/2) p_{1/2}$	4	$2.60\epsilon_2+1.11\epsilon_4$	1.22	1.55
$(f_{5/2}^3, J_1=9/2) p_{1/2}$	5	$0.600\epsilon_2+1.11\epsilon_4$	0.62	2.15
$p_{1/2}^2 p_{3/2}^2$	0	$\epsilon_1+2\epsilon_3+2\epsilon_7$	2.00	0.80
$p_{1/2}^2 p_{3/2}^2$	2	$\epsilon_1+2\epsilon_3+0.400\epsilon_7$	1.36	1.44
$(f_{5/2}^2, J_1=0)(p_{1/2} p_{3/2}, J_2=1)$	1	$\epsilon_2+3\epsilon_4+\epsilon_5$	1.80	1.27
$(f_{5/2}^2, J_1=0)(p_{1/2} p_{3/2}, J_2=2)$	2	$\epsilon_2+0.800\epsilon_3+3\epsilon_4+\epsilon_5$	2.12	0.95
$(f_{5/2}^2, J_1=2)(p_{1/2} p_{3/2}, J_2=1)$	1	$0.700\epsilon_2+0.686\epsilon_4+0.386\epsilon_5$	0.60	2.47
$(f_{5/2}^2, J_1=2)(p_{1/2} p_{3/2}, J_2=1)$	2	$0.900\epsilon_2+0.686\epsilon_4+0.986\epsilon_5$	0.84	2.23
$(f_{5/2}^2, J_1=2)(p_{1/2} p_{3/2}, J_2=1)$	3	$1.20\epsilon_2+0.686\epsilon_4+1.27\epsilon_5$	1.02	2.05
$(f_{5/2}^2, J_1=2)(p_{1/2} p_{3/2}, J_2=2)$	0	$1.60\epsilon_2+0.800\epsilon_3+0.686\epsilon_4+0.457\epsilon_5$	1.21	1.86
$(f_{5/2}^2, J_1=2)(p_{1/2} p_{3/2}, J_2=2)$	1	$1.50\epsilon_2+0.800\epsilon_3+0.686\epsilon_4+0.386\epsilon_5$	1.16	1.91
$(f_{5/2}^2, J_1=2)(p_{1/2} p_{3/2}, J_2=2)$	2	$1.30\epsilon_2+0.800\epsilon_3+0.686\epsilon_4+0.638\epsilon_5$	1.17	1.90
$(f_{5/2}^2, J_1=2)(p_{1/2} p_{3/2}, J_2=2)$	3	$\epsilon_2+0.800\epsilon_3+0.686\epsilon_4+1.27\epsilon_5$	1.28	1.79
$(f_{5/2}^2, J_1=2)(p_{1/2} p_{3/2}, J_2=2)$	4	$0.600\epsilon_2+0.800\epsilon_3+0.686\epsilon_4+1.27\epsilon_5$	1.16	1.91
$(f_{5/2}^2, J_1=4)(p_{1/2} p_{3/2}, J_2=1)$	3	$0.501\epsilon_2+0.286\epsilon_4+0.327\epsilon_5$	0.36	2.71
$(f_{5/2}^2, J_1=4)(p_{1/2} p_{3/2}, J_2=1)$	4	$0.900\epsilon_2+0.286\epsilon_4+0.687\epsilon_5$	0.59	2.48
$(f_{5/2}^2, J_1=4)(p_{1/2} p_{3/2}, J_2=1)$	5	$1.40\epsilon_2+0.286\epsilon_4+1.69\epsilon_5$	1.04	2.03
$(f_{5/2}^2, J_1=4)(p_{1/2} p_{3/2}, J_2=2)$	2	$1.99\epsilon_2+0.800\epsilon_3+0.286\epsilon_4+0.327\epsilon_5$	1.13	1.94
$(f_{5/2}^2, J_1=4)(p_{1/2} p_{3/2}, J_2=2)$	3	$1.70\epsilon_2+0.800\epsilon_3+0.286\epsilon_4+0.327\epsilon_5$	1.04	2.03
$(f_{5/2}^2, J_1=4)(p_{1/2} p_{3/2}, J_2=2)$	4	$1.41\epsilon_2+0.800\epsilon_3+0.286\epsilon_4+0.555\epsilon_5$	0.99	2.08
$(f_{5/2}^2, J_1=4)(p_{1/2} p_{3/2}, J_2=2)$	5	$0.800\epsilon_2+0.800\epsilon_3+0.286\epsilon_4+1.09\epsilon_5$	1.00	2.07
$(f_{5/2}^2, J_1=4)(p_{1/2} p_{3/2}, J_2=2)$	6	$0.200\epsilon_2+0.800\epsilon_3+0.286\epsilon_4+1.86\epsilon_5$	1.05	2.02
$p_{1/2}^2 f_{5/2} i_{13/2}$	4	$\epsilon_1+\epsilon_2+\epsilon_6$	0.90	2.36
$p_{1/2}^2 f_{5/2} i_{13/2}$	5	$\epsilon_1+\epsilon_2+\epsilon_6+0.294\epsilon_8$	0.96	2.30
$p_{1/2}^2 f_{5/2} i_{13/2}$	6	$\epsilon_1+\epsilon_2+\epsilon_6$	0.90	2.36
$p_{1/2}^2 f_{5/2} i_{13/2}$	7	$\epsilon_1+\epsilon_2+\epsilon_6+0.608\epsilon_8$	1.02	2.24
$p_{1/2}^2 f_{5/2} i_{13/2}$	8	$\epsilon_1+\epsilon_2+\epsilon_6$	0.90	2.36
$p_{1/2}^2 f_{5/2} i_{13/2}$	9	$\epsilon_1+\epsilon_2+\epsilon_6+1.56\epsilon_8$	1.21	2.05
$f_{5/2}^4$	0	$6\epsilon_4$	2.40	0.94
$f_{5/2}^4$	2	$3.69\epsilon_4$	1.48	1.86
$f_{5/2}^4$	4	$3.29\epsilon_4$	1.32	2.02
$(p_{3/2}^2, J_1=2)(p_{1/2} f_{5/2}, J_2=3)$	5	$1.40\epsilon_3+1.57\epsilon_5+0.400\epsilon_7$	1.19	2.18
$p_{1/2}^2 p_{3/2} i_{13/2}$	5	$\epsilon_1+\epsilon_3+\epsilon_6+1.76\epsilon_{10}$	1.35	2.21
$p_{1/2}^2 p_{3/2} i_{13/2}$	6	$\epsilon_1+\epsilon_3+\epsilon_6$	1.00	2.56
$p_{1/2}^2 p_{3/2} i_{13/2}$	7	$\epsilon_1+\epsilon_3+\epsilon_6+0.287\epsilon_{10}$	1.06	2.50
$p_{1/2}^2 p_{3/2} i_{13/2}$	8	$\epsilon_1+\epsilon_3+\epsilon_6$	1.00	2.56
$(f_{5/2}^3, J_1=9/2) p_{3/2}$	5	$1.11\epsilon_4+1.56\epsilon_5$	0.91	2.73
$(f_{5/2}^3, J_1=9/2) p_{3/2}$	6	$1.11\epsilon_4+2.38\epsilon_5$	1.16	2.48

calculations of the energy shifts of these levels due to weak coupling to the core showed that the shifts could be neglected.

It is to be noted that the configurations of the lowest levels with a spin of 2+ and 4+ are $f_{5/2}^3 p_{1/2}$ and $p_{1/2}^2 p_{3/2} f_{5/2}$ respectively. These configurations differ by having two of the j_i's different. As mentioned in Sec. II, the contribution to the $E2$ radiation between these levels will vanish in the first-order weak-coupling calculation. Estimates indicate that second-order perturbation calculations between these two levels will yield a half-life longer than is observed experimentally. But with a mixing of the configurations, which is to be expected, the transition rate will be greater than that for the pure $p_{1/2}^2 p_{3/2} f_{5/2}$ level to the pure $f_{5/2}^3 p_{1/2}$ level.

An estimate of the mixing in the 4+ level can be made as the gyromagnetic ratio for this state is known. Krohn and Raboy[11] have found for this level that $g=(+0.054\pm0.005)$ nuclear units. If the 4+ level were a pure $p_{1/2}^2 p_{3/2} f_{5/2}$ configuration, the g-factor for this state would be -0.137. On the other hand, if it were a pure $p_{1/2}^2 f_{5/2}^2$ configuration, the g-factor would be $+0.547$. Taking the true state as a linear combination of the above two states, one finds that a mixture of 72% of the $p_{1/2}^2 p_{3/2} f_{5/2}$ state and 28% of the $p_{1/2}^2 f_{5/2}^2$ state is needed to give a gyromagnetic ratio of $+0.054$. This indicates that there is quite a lot of configuration interaction although the $p_{1/2}^2 p_{3/2} f_{5/2}$ state is still the dominant one. However, states present with very small amplitudes may have a substantial effect on the g-factor if they are connected by the magnetic moment operator to the dominant states.[35] A crude calculation including

[35] R. J. Blin-Stoyle and M. A. Perks, Proc. Phys. Soc. (London) **A67**, 885 (1954); A. Arima and H. Horie, Progr. Theoret. Phys. **11**, 509 (1954); Progr. Theoret. Phys. **12**, 623 (1954).

this kind of mixing indicates that the intensity of the $p_{1/2}{}^2p_{3/2}f_{5/2}$ state is slightly larger than 72%.

V. SUMMARY

The $E2$ transition rate in Pb^{207} gives a value for the surface tension parameter, $C=1100$ Mev. The $E2$ transition rate in Pb^{206} gives $C=520$ Mev. The quadrupole moment of Bi^{209} is consistent with $C\approx 1000$ Mev. The discrepancy between the results of Pb^{207} and Pb^{206} appears to be real and cannot be resolved at this time.

The good agreement between the predicted energy levels and the experimentally determined energy levels of Pb^{204} show that short-range two-body forces describe the interaction between the external nucleons fairly well. Also the splitting of the levels for the first two Mev are due for the most part to these external nucleons and do not seem to be affected by the collective motion of the core. Even the large values of the surface tension deduced in this article are not sufficient to account for the very long lifetime of the 4+ to 2+ transition in Pb^{204}. A further partial explanation of this long lifetime is suggested by the shell model calculation—that the dominant configurations in the 2+ and 4+ states differ in the quantum numbers of two particles.

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Perturbation Procedure for Bound States of Nuclei*

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The model of nucleons moving in a fixed harmonic oscillator well provides a convenient starting point for calculations on the structure of light nuclei. One complication, the nondiagonal energy operator associated with the motion of the center of mass, can be eliminated simply by adding the harmonic potential operator of the center-of-mass motion to the correct Hamiltonian operator of the system.

Following the construction of the correct zeroth-order linear combinations (in case of degeneracy) and the evaluation of the first-order energy matrix, the further development of the theory, in the direction of an exhaustive investigation of the consequences of assumed interaction operators, requires the evaluation of second- and higher order contributions to the energy. A simple, accurate, and powerful procedure for evaluating the second-order energy and other second-order quantities in closed form is described and explicit formulas are given for the application to the ground states of the deuteron, the triton, and the alpha particle. Numerical results for the deuteron, where a comparison with exact calculations is possible, are surprisingly good. The extension of these calculations to the first p-shell and beyond with the aid of the technique of fractional parentage coefficients appears feasible.

I. INTRODUCTION

NUMEROUS attempts have been made to correlate the observed properties of light nuclei with assumptions about the nuclear forces.[1–4] The nonvanishing quadrupole moment of the deuteron requires a tensor component in the correct combination of nuclear forces. Calculations with tensor forces are difficult and have been worked out to a satisfactory degree of accuracy only for two- and three-particle problems. Rarita and Schwinger,[5] using a combination of central and tensor forces, succeeded in fitting the binding energy and quadrupole moment of the deuteron and also the low-energy $p-n$ scattering data (which essentially determines the position of the first excited state (virtual) of the deuteron). Pease and Feshbach[6] extended the theory to the three-particle problem using a variational technique to compute the binding energy of the triton. All the experimental data can be fitted except the Coulomb energy of $_2He_1$.

Feingold[7,8] has extended the study of the tensor force to four-, five-, six-, and seven-particle systems. This study indicates that some of the effects usually ascribed to a spin-orbit force may arise, in part at least, from a

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