10 percent of the transitions from 2− to 2+ states should result from the tensor interaction which gives rise to the shape factor term $C_1^{(2)}$. Assuming that the deviation from a constant shape factor for the 1.985-Mev beta group in K^{42}, Fig. 10, is due only to the $C_1^{(2)}$ term, it is estimated that this shape-factor term cannot contribute more than two percent of the transitions.

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Perturbation Treatment of the Many-Body Problem

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In the conventional perturbation treatment of the many-body problem of interacting particles, the zero-order Hamiltonian corresponds to independent particles moving in a static over-all potential. A discussion of the effects of particle interactions or 'correlations' shows that if one starts instead with a Hamiltonian representing noninteracting particles in a *velocity-dependent* over-all potential, deeper for slow, and shallower for fast particles, then part of the correlation effect is included already in zero order. In addition, a velocity-dependent over-all potential may be called for by velocity dependent interparticle forces or by exchange forces. The degree of the improvement in the convergence of a perturbation expansion based on a Hamiltonian with a velocity-dependent over-all potential is discussed and illustrated by a simple example in which the velocity dependence of the potential gives rise to a reduced "effective mass" of the particles.

The many-body problem of a large, uniform system of interacting particles (e.g., the case of a heavy nucleus without surface effects) is formulated in detail in perturbation theory, starting with a velocity-dependent potential constant in space. The work involved in such a calculation turns out to be essentially the same as with a velocity-independent potential, the effect of the velocity dependence being to reduce each term in the perturbation expansion by a constant factor raised to a power equal to the number of "energy denominators" E_m-E_n in the term in question. A simple equation is deduced for the optimum degree of velocity dependence of the over-all potential which ensures the most rapid convergence.

1. INTRODUCTION

THERE are two principal difficulties in giving a quantitative description of the properties of nuclei: our limited knowledge concerning the fundamental interactions between nucleons, and the mathematical difficulties associated with the solution of a many-body problem. In recent years, some progress has been made in the nuclear many-body problem.[1] The present paper describes a method of dealing with the many-body problem which shows many similarities with those investigations and may turn out to be equivalent in many respects as regards the physical effects considered. The present method is, however, formulated according to conventional perturbation theory, which makes the interpretation of the results in terms of familiar concepts especially easy and makes clear the relation to earlier work along these lines.

The many-body problem of interacting particles has been treated with considerable success in the case of atomic electrons by a method in which the zero-order solution corresponds to particles moving in a common potential and the difference between the actual interaction energy and that part of it which is represented by the common potential is treated by perturbation theory. The same method has been applied, with much more limited success, to the case of nuclei.[2–4] The magnitude of the second-order term (representing correlation effects) in the perturbation expansion of the total energy indicates that, in the case of typical nuclear interactions, the convergence is rather slow. The expansion has never been carried beyond the second order.

The success of a perturbation treatment of any problem depends on how close the unperturbed system is to the exact solution, i.e., on the success of the initial choice of the zero-order Hamiltonian H_0 in the division of the exact Hamiltonian H into H_0 and the perturbation $W=H-H_0$. The method to be described in this article differs from the older perturbation expansions in the first place only in the choice of H_0. As will be shown, it is possible, by a proper choice of H_0, to include already in the zero-order solution part of the correlations normally appearing in higher orders. The result is that the second- and higher order terms in the new expansion are smaller and the convergence is improved.

[1] See, for example, the article by K. A. Brueckner and C. A. Levinson, Phys. Rev. **97**, 1344 (1955) and the series of articles by Brueckner, Watson, Levinson, Mahmoud, Eden, and Francis, referred to in the above article.

[2] H. Euler, Z. Physik **105**, 553 (1937).
[3] W. Heisenberg, Z. Physik **96**, 473 (1935).
[4] R. Huby, Proc. Phys. Soc. (London) **A62**, 62 (1949).

In order to have a good initial Hamiltonian H_0, one must include in it the principal features of the system one is trying to describe. In the case of a system of interacting particles, an important feature is the appearance of correlations among the motions of the particles, expressing the tendency of the particles to stay close together in the case of attractive forces and to avoid each other in the case of repulsions. For a cohesive system with over-all attractions, the effect of the correlations is to increase the average (negative) potential energy of any particle, since each particle tends to spend as much time as possible in configurations where the potential energy is lowest. The net result is a negative contribution to the (negative) binding energy of the total system, as illustrated, for example, by the sign of the second-order term in the above-mentioned perturbation expansions.

If one is considering a system composed of particles moving with different velocities, the correlations will affect the slower particles relatively more than the fast ones. This is simply due to the greater difficulty of deflecting a fast particle from an unperturbed motion, and would be brought out most clearly in the case of a very fast particle passing right through the system, in which case the potential felt by the particle would tend, with increasing velocity, to a limiting value representing an average uninfluenced by correlations. Similarly, if instead of comparing slow and fast particles, we follow the motion of a given particle, the correlations which affect it will be more pronounced when it happens to be moving slowly and less pronounced when its velocity is high. Hence, to the extent that the effective potential felt by a particle arises partly by virtue of correlations, its magnitude will depend on the velocity of the particle.

In addition, a velocity dependence of the over-all potential may be called for by velocity-dependent interparticle interactions, resulting in a different effective potential for slow and fast particles. Also, for exchange forces the effective interaction between two particles depends on their states of motion [through the exchange integral $\int\psi_k^*(1)\psi_m^*(2)V_{12}\psi_m(1)\psi_k(2)$]. In both cases, this leads to a velocity dependence of the effective potential even before correlation effects are considered. [Discussions of this effect in the case of exchange forces are given by Bethe and Bacher[5] and Van Vleck.[6]]

The preceding discussion suggests that the approximation in which the interactions between particles are replaced by an over-all potential will be improved if this potential is made velocity-dependent, becoming deeper for slow particles and tending to a limiting value for high velocities.

[5] A.H. Bethe and R. F. Bacher, Revs. Modern Phys. 8, 82 (1936).
[6] J. H. Van Vleck, Phys. Rev. 48, 367 (1935).

To solve a many-body problem with a Hamiltonian

$$H=\sum_i \frac{p_i^2}{2m}+\sum\sum_{i<j} V_{ij}, \tag{1}$$

it would thus be advantageous to take as the starting point a Hamiltonian H_0 given by

$$H_0=\sum_i \frac{p_i^2}{2m}+\tfrac{1}{2}\sum_i U\left(r_i, \frac{p_i^2}{2m}\right), \tag{2}$$

where the average potential U felt by the ith particle is an increasing function of p_i^2. We have written p_i^2 instead of p_i in the argument of U in order to imply a symmetric function of velocity, independent of the direction of motion. [The factor $\frac{1}{2}$ in the last term of Eq. (2) expresses the fact that interaction energies between particles, depending as they do on two particles at a time, are not additive. For k-body forces the contribution to the total potential energy from a particle experiencing a potential U is $(1/k)U$.]

The Hamiltonian (2) represents the motion of particles under the action of a velocity-dependent potential, but still without explicit interactions between them. If a solution of this problem is found, the solution of the many-body problem could then proceed with the difference,

$$W=H-H_0=\sum\sum_{i<j} V_{ij}-\tfrac{1}{2}\sum_i U(r_i,t_i),$$

treated as a perturbation. (We have put $t_i=p_i^2/2m$.)

The difference from the conventional perturbation treatment with a velocity-independent U now comes in when we consider the effect of the perturbation W in introducing correlations between the motions of pairs of particles. These interactions (or "collisions") have the effect of changing the state of motion of the interacting pair. In the language of perturbation theory this is described as a transition from the ground state of the unperturbed system to a virtual level n. As is well known, the energy difference E_n-E_0 (where E_0 refers to the original state) has a controlling effect on the probability of the occurrence of the configuration associated with the state n, the probability decreasing with increasing E_n-E_0. In the case of a velocity-independent U, the difference E_n-E_0 is essentially the difference in the kinetic energies of the two particles before and during the interaction. When U is an increasing function of the kinetic energies, the energy difference will be increased because the average potential for the faster-moving particles during the interaction is shallower than for the initial state. Physically this expresses the fact that during the interaction the particles acquire a higher velocity, which makes them less susceptible to further correlations—less, that is, than one would estimate on the basis of their original velocities. This result, which makes the correlations

themselves reduce the original estimate of their magnitudes, is an effect which, with a velocity-independent U, would only appear in higher orders of a perturbation expansion.

The increase in the energy differences E_n-E_0 "stretches" the energy spectrum of the system associated with the Hamiltonian (2) as compared with the case of a velocity-independent U. The result is that transitions to states of motion differing from the unperturbed solution of H_0 are made more difficult and the convergence of a perturbation expansion is improved.

The stretching of the energy spectrum corresponds to an effect occasionally described in terms of a reduced effective mass of the particles. This is illustrated most simply in the case when $U(r,t)$ is taken to be of the form:

$$U(r,t)=f(r)+c(p^2/2m), \tag{3}$$

when the last term can be combined with the kinetic energy in the Hamiltonian to give $p^2/2m'$, with $2m'=2m(1+c)^{-1}$. The resulting Hamiltonian describes particles moving in the velocity-independent part $f(r)$ of U, the level spacing being increased in the ratio m/m' on account of the reduced mass. With the simple dependence on p^2 assumed in Eq. (2) the effect of the velocity dependence of U is thus to improve the convergence of a perturbation expansion to the extent of multiplying each term in the expansion by a power of m'/m equal to the number of energy denominators E_m-E_n occurring in the term in question. The qualitative discussion of the reasons for the stretching of the spectrum which was given before, suggests that similar conclusions are to be expected also for a dependence of U on p^2 more general than that in Eq. (3). We shall illustrate this by a second simple idealization in which the dependence of U on p^2 is unspecified, but instead U is assumed to be independent of r. This represents the frequently discussed limiting case of a large nucleus, with surface effects neglected. The results of such calculations describe the properties of nuclear matter in a statistical way. We shall discuss this case explicitly with reference to a perturbation expansion and we therefore begin with a summary of some relevant formulas.

2. PERTURBATION THEORY FORMULAS

It is required to find the solutions of

$$H\psi_m=E_m\psi_m, \tag{4}$$

assuming the solutions of

$$H_0u_m=E_m^0u_m \tag{5}$$

to be known. Writing

$$\psi_m=\sum_k c_{km}u_k,$$

and

$$Wu_k=\sum_m W_{mk}u_m, \quad \left(W_{mk}=\int u_m^*Wu_k\right)$$

subtraction of (5) from (4) gives

$$\sum_k c_{kn}W_{mk}=c_{mn}(E_n-E_m^0). \tag{6}$$

Solving once for $E_m-E_m^0$ and once for c_{mn} and applying repeatedly the latter expression to the c_{km} appearing in the former, we find an equation of the following type:

$$E_m-E_m^0=W_{mm}+\sum_{k\neq m}\frac{W_{mk}W_{km}}{E_m-E_k^0}+\sum_{k,l\neq m}\sum\frac{W_{mk}W_{kl}W_{ln}}{(E_m-E_k^0)(E_m-E_l^0)}+\sum_{k,l\neq m}\sum\frac{W_{mk}W_{kl}}{(E_m-E_k^0)(E_m-E_l^0)}\frac{1}{c_{mm}}\sum_n c_{nm}W_{ln}. \tag{7}$$

This is exact. The perturbation formula to second order is obtained by retaining the first two terms and replacing E_m by E_m^0 in the denominator:

$$E_m-E_m^0=W_{mm}+\sum_{k\neq m}\frac{W_{mk}W_{km}}{E_m^0-E_k^0}+\cdots. \tag{8}$$

The pth term in the expansion (7) has $p-1$ energy denominators. In what follows, the ground state of the system will be denoted by $m=0$.

3. PERTURBATION THEORY WITH A VELOCITY-DEPENDENT U

In the present case, H is given by Eq. (1) and H_0 by

$$H_0=\sum_i p_i^2/2m+\tfrac{1}{2}\sum_i U(t_i), \tag{9}$$

so that

$$W=\sum_{i<j}\sum V_{ij}-\tfrac{1}{2}\sum_i U(t_i)=A-B, \text{ say.} \tag{10}$$

Since U is a constant in space, the solutions u_m of (9) are Slater determinants composed of single-particle plane waves, just as in the case of a U independent of p^2. In evaluating the matrix elements of W between two such determinants u_m and u_k, the contributions from the second term in (10) vanish for $m\neq k$, since operating with $U(p_i^2/2m)$ leaves all the plane waves in a determinant unchanged (a velocity-dependent but constant potential will not cause a particle moving under its influence to change its state of motion). All the nondiagonal matrix elements (A_{mk}) are, therefore, identical with the matrix elements occurring in a perturbation expansion with a velocity-independent U. The velocity dependence affects only the energy denominators in the sums occurring in (7) and (8). The effect on these denominators is easily exhibited by

writing

$$E_m^0=\int u_m^*\left[\sum_i \frac{p_i^2}{2m}+\tfrac{1}{2}\sum_i U(t_i)\right]u_m$$
$$=T_m+B_m.$$

Here T_m is the zero-order kinetic energy associated with the plane-wave determinant u_m, and B_m is the expectation value of $\frac{1}{2}\sum_i U(t_i)$ for this state. Hence we may write

$$E_m^0-E_k^0=T_m-T_k+B_m-B_k$$
$$=(T_m-T_k)\left(1+\frac{B_m-B_k}{T_m-T_k}\right)$$
$$=(1+b)(T_m-T_k),$$

where b, which should really be written b_{mk}, is the rate of change of the expectation value of the potential energy with increasing kinetic energy (excitation), all referring to the zero-order system u_m. For a velocity-independent U, the expectation value of U would be independent of the excitation, and $b=0$. For a velocity-dependent U, the factor $1+b$ expresses the stretching of the energy spectrum discussed in the introduction.

The close relation which exists between the dependence of the potential energy B_m on the total kinetic energy T_m and the dependence of U on t_i for the separate particles can be brought out as follows. Consider an excitation of the zero-order system consisting of increasing the kinetic energy of a particle from t_i to $t_i+\Delta t_i$, the associated change in $\frac{1}{2}U(t_i)$ being $\frac{1}{2}\Delta U_i$. If the kinetic energies of several particles are affected by the excitation the total changes are $\sum_i \Delta t_i$ and $\sum_i (\frac{1}{2}\Delta U_i)$, respectively. The rate of change of B with T is then

$$\frac{\sum_i (\frac{1}{2}\Delta U_i)}{\sum_i \Delta t_i}=\frac{\sum_i [\frac{1}{2}(dU/dt)_i\Delta t_i]}{\sum_i \Delta t_i}=\tfrac{1}{2}\langle(dU/dt)\rangle_{\text{Av}},$$

where $\frac{1}{2}\langle(dU/dt)\rangle_{\text{Av}}$ is some average value of the derivative of U. The precise value of this average will depend on the excitation in question (as defined by the set of excitations Δt_i), but for the most important excited states occurring in a perturbation theory (the removal of particles from states somewhat below the surface of the Fermi sphere to states somewhat above), $\langle(dU/dt)\rangle_{\text{Av}}$ will be approximately equal to dU/dt evaluated at a kinetic energy t_0 corresponding to the top of the Fermi sphere. Hence, the quantity b is approximately given by

$$b\sim\tfrac{1}{2}(dU/dt)_{t_0}.$$

Factors of the type $(1+b)^{-1}$ will appear in all the higher order terms of (8), their number being equal to the number of energy denominators. For positive b, this will improve the convergence of the expansion.

The first term on the right-hand side of Eq. (8) consists of the diagonal elements A_m-B_m. As discussed above, B_m will depend on the state of excitation m. The quantity A_m, which is the average value of $\sum\sum_{i<j} V_{ij}$ for the unperturbed state m, would be independent of m for ordinary, nonexchange, velocity-independent interparticle forces. However, with such more general forces present, A_m will also depend on m. To bring out explicitly the dependence of A_m and B_m on the degree of excitation T_m-T_0 (here T_0 refers to the ground state), we shall write, for the small excitations relevant to a perturbation treatment:

$$B_m=B_0+\left(\frac{B_m-B_0}{T_m-T_0}\right)(T_m-T_0)\sim B_0+b(T_m-T_0),$$

$$A_m=A_0+\left(\frac{A_m-A_0}{T_m-T_0}\right)(T_m-T_0)\sim A_0+a(T_m-T_0),$$

where $a=0$ for particle interactions independent of the state of motion of the system. B_0 is an average depth of the potential for the ground state.

We may now write Eq. (8) in the following form:

$$E_m-E_m^0=A_0-B_0+(a-b)(T_m-T_0)$$
$$+(1+b)^{-1}\sum_{k\neq m}\frac{A_{mk}A_{km}}{T_m-T_k}+(1+b)^{-2}(3)$$
$$+\cdots+(1+b)^{-p+1}(p)+\cdots, \quad (13)$$

where (p) stands for the pth term in a perturbation expansion with a velocity-independent U.

4. DETERMINATION OF THE OPTIMUM PARAMETERS OF U

Equation (13) gives the change of the energy spectrum of the zero-order system caused by the perturbing interaction W, for any arbitrary value of $b\sim\frac{1}{2}(dU/dt)$. The question arises as to what value of dU/dt one should assume so as to ensure the most rapid convergence.

If one requires that the zero-order system should be such that its energy spectrum E_m^0 in a region of values of m of greatest interest resembles as closely as possible the spectrum of the exact system (or, in practice, the estimate of this spectrum after a given number of steps in the perturbation expansion), then the condition on b is that it should minimize the right-hand side of (13) in the range of m-values considered. If we apply this optimum condition to the region of low excitations (relevant for the perturbation treatment) by requiring that $(E_m-E_m^0)_{m=0}=0$ and $[(\partial/\partial T_m)(E_m-E_m^0)]_{m=0}=0$, we find:

$$A_0-B_0+(1+b)^{-1}Q=0 \quad (14)$$

and

$$a-b+(1+b)^{-1}q=0, \quad (15)$$

where

$$Q=\sum_{k\neq 0}\frac{A_{0k}A_{k0}}{T_0-T_k}$$

is the "correlation energy" in a conventional perturbation calculation and q is the rate of change of this energy with increasing excitation T_m. Terms beyond the second approximation have been neglected in Eqs. (14) and (15).

In the above equations, the quantities A_0, a, Q, and q are constants which can be calculated for any assumed particle interactions V_{ij}. The sums for Q and q, in particular, are of a type that have evaluated explicitly for Gaussian and Yukawa two-particle potentials (references 2 and 4). The two unknowns B_0 and b refer to two adjustable parameters of the potential U: an average depth and an average degree of energy dependence, respectively. Equation (15) gives directly the optimum value of b and Eq. (14) then determines B_0.

The positive solution of the quadratic equation (15) gives

$$1+b=\tfrac{1}{2}\{(1+a)+[(1+a)^2+4q]^{\frac{1}{2}}\}.$$

This shows how the factor $1+b$, which is a measure of the degree of improvement in the convergence of a perturbation expansion, increases both with increasing a and q, the quantities which give the rate of energy dependence of the interaction energy $\sum\sum_{i<j} V_{ij}$ calculated in first and second order, respectively. In particular, if q were negligible we would have

$$1+b=1+a,$$

whereas if both a and q were small compared to unity:

$$1+b=1+a+q+\cdots.$$

It is interesting to note that the presence of exchange or velocity-dependent particle forces resulting in a positive a would, by itself, tend to suppress correlation and higher order effects, and so help to preserve the validity of the "independent-particle" zero-order approximation.

5. DISCUSSION

The velocity dependence of U in the zero-order Hamiltonian H_0 was introduced on physical grounds, as the result of a discussion of correlation effects, exchange forces, and a possible velocity dependence of the particle interactions. The final formulation of the method, with a velocity dependence of U which is automatically fixed by the approximation procedure, may be regarded as an extension of conventional perturbation theory (whose formulation has been considered in the present paper only in two simple limiting cases), whose success is to be judged by its usefulness in applications to specific problems.

The zero-order solution, based on a potential U adjusted according to Eqs. (14) and (15), represents an approximate solution of the many-body problem in which part of the correlation effects are included, although the wave function is still a simple Slater determinant. Considering the effect of the perturbation W on this zero-order system, Eq. (14) ensures that the first- and second-order corrections to the ground state energy cancel out. The next (third-order) correction is decreased by a factor $(1+b)^{-2}$ as compared with a conventional perturbation calculation. Equation (15) ensures that the same is true also for low-lying excited states of the system.

In applications to problems of nuclear structure, the accuracy of the improved expansions can be expected to depend on the type of particle forces considered. The second- and higher order terms in the present method refer to effects that cannot be represented by even a velocity-dependent over-all potential. The remaining essential correlations, representing deviations from "independent-particle" motions, will depend, among other things, on the smoothness of the assumed particle interactions, and this question will become clearer in the light of the results of specific applications. (I am indebted to Professor K. Brueckner for valuable correspondence in this connection.)

There appears to be experimental evidence for a dependence of the depth of the nuclear potential on energy which, according to the present considerations, could be due to correlation effects and exchange forces as well as to an explicit velocity dependence of nucleon interactions. Thus, for neutrons of about 1 Mev, the depth of the nuclear potential well is around 42 Mev,[7] but with increasing energy it decreases to about 10–15 Mev, around which value it appears to remain approximately constant between 150 and 400 Mev.[8,9] Information about the depth of the potential felt by *bound* particles is difficult to obtain, although it is precisely in the region of the top of the Fermi sphere (i.e., at a negative energy of around minus 8 Mev) that the dependence of U on t is of most interest in the present connection. It is hoped to discuss these questions further in the light of specific applications of the perturbation calculation.

In the description of the modified perturbation method we have made references to nuclear problems, but the procedure should be applicable in principle also, for example, to atomic systems. It would be interesting, for instance, to examine the velocity dependence of the atomic Fermi-Thomas potential resulting from taking correlations into account.

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