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Angular Distribution of Betatron Target Radiation

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An attempt is made to combine the bremsstrahlung intrinsic energy-angle distribution for radiated photons with Molière's multiple-scattering theory for the incident electrons.

The angular distribution of betatron radiation is essentially characterized by a function $I(\theta)$, which is given as a series in inverse powers of Molière's parameter B. The first term of this expansion, which is dominant for small angles, is equivalent to a combination theory developed on the basis of a Gaussian approximation for multiple scattering. This term is worked out exactly and its final expression may be compared with previous theories which depend upon mathematical approximations. The term of order B^{-1} becomes important for larger angles and gives the transition to the contribution of single-scattered electrons. Expressions for the forward radiation and energy-angle distribution are given. The angular distribution predicted by these results was found to be in good agreement with an experiment performed by Lanzl and Hanson.

INTRODUCTION

EXPRESSIONS for the intrinsic energy-angle distribution of target bremsstrahlung have been derived by Sommerfeld and Schiff on the basis of the Bethe-Heitler radiation theory.[1,2]

It has been pointed out that in the range of target thickness that is more frequently used in betatrons and synchrotrons, the energy-angle distribution of the radiation is modified by the elastic multiple scattering of the electrons in the target. On the other hand, energy loss of the electrons and absorption of the radiation in the target can be neglected.[3]

There are a number of papers in which the influence of multiple scattering is taken into account.[4–9] It is a common feature of these papers that they consider the problem on the basis of a Gaussian approximation for multiple scattering.[10] There is, however, some experimental and theoretical evidence for the necessity of a more complete and detailed theory (for example, see discussion in Sec. II-C of reference 7).

In the present paper we consider the combination of the scattering and radiation theories on the basis of Molière's complete theory of multiple scattering.[11,12] Our main results depend essentially on two parameters, B and λ. The former is identical to the parameter introduced in Molière's theory, while the latter is given by the square of the ratio of the angular width of the intrinsic radiation to the angular width of the multiple-scattering distribution. In the cases of main interest, λ and $1/B$ are small numbers (usually smaller than 0.2).

The angular distribution is essentially characterized by a function $I(\theta)$ [Eq. (7)] which is given as a series in $1/B$. The first term of this expansion is exactly the expression we would obtain for the combination theory on the basis of a Gaussian approximation for multiple scattering. This zeroth-order term,[10] which is dominant for small angles, is worked out exactly and its final expression may be compared directly with previous

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[2] L. I. Schiff, Phys. Rev. **83**, 252 (1951).

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[4] L. I. Schiff, Phys. Rev. **70**, 87 (1946).

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[7] L. Lanzl and A. Hanson, Phys. Rev. **83**, 959 (1951).

[8] Muirhead, Spicer, and Lichtblau, Proc. Phys. Soc. (London) **A65**, 59 (1952).

[9] R. M. Warner and E. F. Shrader, Rev. Sci. Instr. **25**, 663 (1954).

[10] Hereafter, the combination of the radiation theory with a Gaussian approximation for multiple scattering will be called for brevity "zeroth-order approximation" or "zeroth-order term." (This denomination will become clear later.)

[11] G. Molière, Z. Naturforsch. **3a**, 78 (1948).

[12] H. A. Bethe, Phys. Rev. **89**, 1256 (1953).

theories which made use of mathematical simplifications. Next we consider the $1/B$ term in an approximate way, retaining only the main contribution in the limit of small λ. This term becomes more and more important when the angle increases and, in the case of $\lambda < 1/(2B)$, it turns out to give the dominant contribution in the asymptotic region.

The present results were found to be in good agreement with the experimental angular distribution as determined by Lanzl and Hanson (reference 7, Fig. 12).

FORMULATION

Let $\sigma(k,\theta)\theta d\theta dk$ be the intrinsic differential cross section for the emission of a photon into the angular interval $d\theta$ and energy interval dk (θ is the angle between the directions of the photon and incident electron), $f(\theta,t)\theta d\theta$ the number of electrons in the interval $d\theta$ about an angle θ with the original beam after traversing a thickness t, and $P(k,\theta)\theta d\theta dk$ the final number of photons emerging from the target in the intervals $d\theta$ and dk. For small angles, as is obvious, the angles of the photons and scattered electrons can be represented as vectors in the plane perpendicular to the direction of the incident electron. Then the energy-angle distribution of the radiation from a layer of thickness dt at a depth t is given by the convolution of $\sigma(k,\theta)$ and $f(\theta,t)$ over that plane. Since the electrons radiate at all values of t from 0 to T (total thickness of the target), it is necessary to perform an additional integration over t. Thus, we have

$$P(k,\theta)\theta d\theta dk = N\theta d\theta dk\int_0^T dt\int\int f(\boldsymbol{\theta}-\boldsymbol{\chi},t)\sigma(\chi)dS_\chi, \quad (1)$$

where $\boldsymbol{\theta}-\boldsymbol{\chi}$ is the vector in the plane representing the direction of the electron before the radiative collision, $dS_\chi=\chi d\chi d\phi/2\pi$ (ϕ: azimuth of the vector $\boldsymbol{\chi}$ in the plane) and N is the number of scattering atoms per cm^3.

In this paper, $f(\theta,t)$ will be given by Molière's theory of multiple scattering; we can then write (see Sec. 5 of reference 11)

$$f(\theta,t)\theta d\theta=\theta d\theta\int_0^{\Omega_0^{\frac{1}{2}}/\chi_c(t)} udu J_0(\theta u) \times\exp\left[\frac{u^2\chi_c^2(t)}{4}\left(-b(t)+\ln\frac{u^2\chi_c^2(t)}{4}\right)\right], \quad (2)$$

where $\chi_c(t)$ and $b(t)$ are complicated functions defined in references 11 and 12. The quantity Ω_0, which represents the total number of elastic collisions and is usually a very large number, does not play a significant role (later on we will set $\Omega_0=\infty$).

The next step is to put Eq. (2) into a more convenient form in order to perform the operations indicated in Eq. (1). Observing Eqs. (10) and (22) of reference 12, we immediately realize that χ_c^2 is proportional to t and that $-b(t)+\ln[u^2\chi_c^2(t)/4]$ is independent of t. Using these simple facts, introducing $\tau=t/T$ and defining the parameter B through the transcendental equation

$$B(T)-\ln B(T)=b(T), \quad (2a)$$

we get

$$\frac{1}{T}\int_0^T f(\theta,t)dt=\int_0^1 d\tau\int_0^{\Omega_0^{\frac{1}{2}}/\chi_c(t)} udu J_0(\theta u) \times\exp\left[-\frac{u^2}{4}\chi_c^2(T)B\tau\left(1-\frac{1}{B}\ln\frac{u^2}{4}\chi_c^2(T)B\right)\right]. \quad (2b)$$

Following Molière, we introduce now the reduced angle

$$\vartheta=\theta/[\chi_c(T)B^{\frac{1}{2}}(T)]. \quad (2c)$$

Setting $y=u\chi_c(T)B^{\frac{1}{2}}$ and expanding the second exponential in Eq. (2b), the following result is obtained:

$$\frac{\theta d\theta}{T}\int_0^T f(\theta,t)dt=F(\vartheta)\vartheta d\vartheta =\vartheta d\vartheta[F^{(0)}(\vartheta)+B^{-1}F^{(1)}(\vartheta) +B^{-2}F^{(2)}(\vartheta)+\cdots], \quad (2d)$$

where

$$F^{(n)}(\vartheta)=\frac{1}{n!}\int_0^1 d\tau\int_0^\infty ydyJ_0(\vartheta y) \times\exp(-\tau y^2/4)\left(\tau\frac{y^2}{4}\ln\frac{y^2}{4}\right)^n. \quad (2e)$$

By restricting the series (2d) to a finite number of terms, we are allowed to set the upper limit of the integration over y equal to infinity.[13]

The next step is to express Schiff's intrinsic differential cross section σ as a function of the reduced angle ϑ:

$$\sigma(k,\theta)\theta d\theta dk=\sigma(k,\vartheta)\vartheta d\vartheta dk =\vartheta d\vartheta\frac{d\eta}{\eta}\frac{2Z^2}{137}\left(\frac{e^2}{\mu}\right)^2\left\{[\ln M(\vartheta)(2-2\eta+\eta^2)-(2-\eta)^2] \times\frac{2/\lambda}{(1+\vartheta^2/\lambda)^2}+\tfrac{2}{3}(1-\eta)[4-\ln M(\vartheta)]\frac{12\vartheta^2/\lambda^2}{(1+\vartheta^2/\lambda)^4}\right\}, \quad (3)$$

where

$$\frac{1}{M(\vartheta)}=\left[\frac{Z^{\frac{1}{3}}}{111(1+\vartheta^2/\lambda)}\right]^2+\left[\frac{\mu\eta}{2E_0(1-\eta)}\right]^2, \quad (3a)$$

$$\lambda=\mu^2/(E_0^2\chi_c^2B). \quad (3b)$$

Here μ is the rest energy of the electron, E_0 and k are the energies of the incident electron and the photon and $\eta=k/E_0$.

[13] See discussions after Eqs. (5,5) and (7,3b) in reference 11, and after Eq. 20 in reference 12.

The parameter λ will play an important role. It is essentially the square of the ratio of the angular width of the intrinsic radiation (μ/E_0) to that of the multiple scattering distribution $(\chi_c B^{\frac{1}{2}})$. Its physical meaning is simple. Roughly speaking, the value of λ measures the relative importance of the multiple scattering and radiative distributions: when $\lambda \to 0$, the former is the dominant one; the contrary occurs when $\lambda \to \infty$. In the cases of interest here, λ is a small number (usually $\lambda \lesssim 0.2$). This fact will prove to be important for maintaining simplicity in the main results.

The first term in the expression for $1/M(\vartheta)$ represents the influence of the screening of the nucleus by the outer electrons. For the moment we will not consider the slow angular dependence of $\ln M(\vartheta)$, an approximation also used in all previous calculations. Of course, this procedure does not introduce any error when the screening is neglected [i.e., when we set $Z=0$ in Eq. (3a)]. Moreover, when λ is small, the final angular distribution does not depend very sensitively on the intrinsic distribution and, therefore, we may expect that this approximation will introduce only a small error.

Equation (1) may be written now in the abbreviated form[14]

$$P(\vartheta)\vartheta d\vartheta = P(\theta)\theta d\theta = NT\vartheta d\vartheta F(\vartheta) * \sigma(\vartheta). \quad (4)$$

We obtain immediately for the combined energy-angle distribution:

$$P(\vartheta)\vartheta d\vartheta = \vartheta d\vartheta NT \frac{d\eta}{\eta} \frac{2Z^2}{137}\left(\frac{e^2}{\mu}\right)^2 \times\{[\ln M(\chi_1)(2-2\eta+\eta^2)-(2-\eta)^2]I(\vartheta,\lambda) + \tfrac{2}{3}(1-\eta)[4-\ln M(\chi_2)]J(\vartheta,\lambda)\}, \quad (5)$$

where[15]

$$I(\vartheta,\lambda) = F(\vartheta) * \frac{2}{\lambda}(1+\vartheta^2/\lambda)^{-2}, \quad (5a)$$

$$J(\vartheta,\lambda) = F(\vartheta) * \frac{12\vartheta^2}{\lambda^2}(1+\vartheta^2/\lambda)^{-4}. \quad (5b)$$

In Eq. (5), χ_1 and χ_2 are two conveniently chosen angles which will be discussed later. It is easily seen that the functions $J(\vartheta)$ and $I(\vartheta)$ are related by a simple expression:

$$J(\vartheta,\lambda) = \left[1-\left(\lambda\frac{\partial}{\partial\lambda}\right)^2\right] I(\vartheta,\lambda). \quad (5c)$$

In order to prove Eq. (5c), it is sufficient to observe that an identical relation holds between the second convolution factors of Eqs. (5a) and (5b), as may be readily verified by differentiation.

It is well known that the angular distribution of the radiation is essentially given by the function $I(\vartheta,\lambda)$ (see reference 7). In order to evaluate this function, we observe that the Bessel (Fourier) transform of the second convolution factor of Eq. (5a) may be written as follows (see Appendix A):

$$\frac{2}{\lambda}\int_0^\infty \vartheta d\vartheta J_0(y\vartheta)(1+\vartheta^2/\lambda)^{-2} = \int_0^\infty e^{-\alpha}\exp(-\lambda y^2/4\alpha)d\alpha. \quad (6)$$

Now we make use of the folding theorem: the convolution of the two functions of Eq. (5a) is given by the Bessel transform of the product of their Bessel transforms. Remembering Eqs. (2d), (2e), and (6), we immediately obtain

$$I(\vartheta,\lambda) = I^{(0)} + B^{-1}I^{(1)} + B^{-2}I^{(2)} + \cdots, \quad (7)$$

$$I^{(n)}(\vartheta,\lambda) = \frac{1}{n!}\int_0^1 d\tau \int_0^\infty e^{-\alpha}d\alpha \int_0^\infty y dy J_0(\vartheta y) \times\exp\left[-\frac{y^2}{4}(\tau+\lambda/\alpha)\right]\left(\tau\frac{y^2}{4}\ln\frac{y^2}{4}\right)^n. \quad (7a)$$

It is interesting to note that if we eliminate the integration over τ and set $\tau=1$ (i.e., $t=T$) and $\lambda=0$ in Eqs. (7) and (7a), we reobtain Molière's theory of multiple scattering in its expanded form, as should be expected from the physical interpretation of λ. In the next section the leading terms of the series (7) will be discussed in detail.

Let us observe that in the derivation of Eq. (2d), and therefore of Eq. (7), we have proceeded from the integral representation of Molière's theory rather than from its expanded form:

$$f(\vartheta,t)\vartheta d\vartheta = \vartheta d\vartheta[2\exp(-\vartheta^2) + B^{-1}(t)f^{(1)}(\vartheta) + B_{(t)}{}^{-2}f^{(2)}(\vartheta) + \cdots]. \quad (8)$$

The main difficulty in constructing the convolution factor $\int_0^T f(\vartheta,t)\vartheta d\vartheta dt$ from this series is the fact that, in Eq. (8), $B(t)$ is a rather complicated though slowly varying function of the variable of integration t. (In Eq. (8) the angle ϑ also depends on t in a complicated way, $\vartheta(t)=\theta/[\chi_c(t)B^{\frac{1}{2}}(t)]$.) Through the procedure indicated in this section, this difficulty is completely avoided because both $B(T)$ and $\vartheta(T)$ are now independent of the variables of integration.[16]

We turn now to a brief discussion of the two angles

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[14] Here, the symbol $F(\vartheta)*\sigma(\vartheta)$ means the convolution of $F(\vartheta)$ and $\sigma(\vartheta)$ in the plane of $\boldsymbol{\vartheta}$.

[15] The convolution factors in Eqs. (5a) and (5b) are already normalized to unity, i.e.,

$$\int_0^\infty F(\vartheta)\vartheta d\vartheta = (2/\lambda)\int_0^\infty (1+\vartheta^2/\lambda)^{-2}\vartheta d\vartheta = \cdots = 1.$$

[16] This exact elimination of the dependence of B on the variable t results in the fact that, for $n \geqslant 1$, the convolution factors $F^{(n)}(\vartheta)$ are not exactly equal to $\int_0^1 (d\tau/\tau)f^{(n)}(\vartheta/\tau^{\frac{1}{2}})$. (The equality of these expressions should be expected from the simple expansion (8), if the slow t dependence of B were neglected.)

χ_1 and χ_2 introduced in Eq. (5). These angles arise from the angular dependence of $\ln M(\vartheta)$ which has not been considered in the previous mathematical derivations. As was pointed out above, when the screening is neglected, $\ln M(\vartheta)$ does not depend any longer on ϑ and there is no problem at all. In general, however, χ_1 and χ_2 will depend on λ, ϑ, and η in a rather complicated way. It is not difficult to see that, when λ is small, $\ln M(\chi_i)$ is almost independent of ϑ. Therefore, it does not appreciably affect the angular dependence of the combined distribution, which is the point of main interest here. The dependence of $\ln M(\chi_i)$ on η has, however, some effect on the shape of the spectrum. As we are mainly interested in small values of λ, a way of determining approximately the χ_i is to consider the case $\lambda=0$. It is well known that in that limit the theoretical spectrum is almost independent of the angle and is essentially given by the integrated spectrum of the intrinsic distribution.[3,4,17] Taking into account this last fact and using some results of reference 2, we find the following approximations[18]:

$$\ln M(\chi_1)=\ln M(0)+2-(2/\rho)\tan^{-1}\rho, \tag{9}$$

$$\ln M(\chi_2)=\ln M(\chi_1)+[4\rho^{-2}-3\rho^{-2}\ln(1+\rho^2) -2\rho^{-3}(2-\rho^2)\tan^{-1}\rho-\tfrac{1}{3}], \tag{9a}$$

$$\rho=2E_0(1-\eta)Z^{\frac{1}{3}}/(111\eta\mu). \tag{9b}$$

Equations (5), (7), and (9) give an approximate description of the combined energy-angle distribution. In the next section the evaluation of the fundamental function $I(\vartheta)$ will be considered in detail.

EVALUATION

Let us consider the first term in Eq. (7). Performing the integration over y [see Appendix (A, 1)], we readily obtain

$$I^{(0)}(\vartheta)=2\int_0^\infty e^{-\alpha}d\alpha\int^1 \frac{d\tau\exp[-\vartheta^2/(\tau+\lambda/\alpha)]}{\tau+\lambda/\alpha}. \tag{10}$$

This expression is clearly equivalent to a theory based on a Gaussian approximation for multiple scattering. In fact, let us suppose for a moment that, instead of considering the complete theory of multiple scattering, we make use of a Gaussian law:

$$F(\vartheta)\sim 2\int_0^1 \frac{\exp(-\vartheta^2/\tau)}{\tau}d\tau. \tag{11}$$

Then, according to the general definition (5a) and taking into account the elementary integral representation

$$\frac{2}{\lambda}(1+\vartheta^2/\lambda)^{-2}=\frac{2}{\lambda}\int_0^\infty \alpha d\alpha\exp[-\alpha(1+\vartheta^2/\lambda)], \tag{11a}$$

we get

$$I\sim\int_0^\infty d\alpha e^{-\alpha}\int_0^1 d\tau\frac{2}{\tau}\exp(-\vartheta^2/\tau)*\frac{2}{\lambda}\alpha\exp(-\alpha\vartheta^2/\lambda). \tag{11b}$$

Using the well-known convolution properties of the Gaussian function, Eq. (11b) leads again to Eq. (10). This proves our assertion.

Two simple methods of working out exactly Eq. (10) will be considered. The first one gives the development of $I^{(0)}(\vartheta,\lambda)$ as a power series in ϑ and is outlined in Appendix B.

A second and more interesting, though mathematically equivalent, procedure is the following[19]: Introducing $z=\alpha\tau$ as variable of integration instead of τ and interchanging the order of integration

$$\left(\int_0^\infty d\alpha\int_0^\alpha dz\to\int_0^\infty dz\int_z^\infty d\alpha\right),$$

one gets

$$I^{(0)}(\vartheta)=2\int_0^\infty \frac{e^{-z}}{z+\lambda+\vartheta^2}\exp[-z\vartheta^2/(z+\lambda)]dz$$

$$=2\exp(\lambda-\vartheta^2)\int_\lambda^\infty e^{-\xi}\exp(\lambda\vartheta^2/\xi)\frac{d\xi}{\xi+\vartheta^2}.$$

Developing in powers the second exponential, integrating by parts and setting $x=\vartheta^2$, the following result is obtained:

$$I^{(0)}(x,\lambda)/2=-Ei(-\lambda-x) -e^{-x}\sum_{n=1}^\infty(-\lambda)^n\left[R_n(x)-\frac{e^\lambda Ei(-\lambda)}{n!}\Lambda_{n-1}(x)\right], \tag{12}$$

where

$$-Ei(-u)=\int_u^\infty \frac{e^{-t}}{t}dt, \tag{12a}$$

$$R_n(x)=\sum_{\nu=1}^\infty x^\nu\frac{(\nu-1)!}{(\nu+n)!}\left[\frac{1}{\nu!}+\frac{x}{(\nu+1)!}+\cdots+\frac{x^{n-1}}{(\nu+n-1)!}\right], \tag{12b}$$

$$\Lambda_n(x)=\sum_{\nu=0}^n\frac{x^\nu}{\nu!}. \tag{12c}$$

[17] However, Warner, and Shrader (reference 9) have found reproducible and rather large variations in the experimental angular distribution of different spectral components of betatron radiation.

[18] The parameter ρ, introduced in reference 2, measures the relative importance of the screening; $\rho=0$ means no screening while $\rho=\infty$ means complete screening. For all values of ρ, the function between brackets in Eq. (9a) is small and, for $\rho<3$, it can be neglected.

[19] This second method was developed by Professor G. Molière. The author is very grateful to Professor Molière for the communication of the essentials of this proof. The leading terms of the expansion (12) turned out to be identical with an incomplete expansion previously developed by the writer in a more complicated way. It is also mathematically equivalent to the exact development given in Appendix B.

Alternately, $R_1(x)$ and $R_2(x)$ can be simply evaluated from the closed expressions (see also Table I)[20]:

$$R_1(x)=\bar{E}i(x)-\ln(\gamma x)-\frac{(e^x-1)}{x}+1, \quad (12d)$$

$$2R_2(x)=[\bar{E}i(x)-\ln(\gamma x)](x-1)-e^x-\frac{(e^x-1-x)}{x^2}+\frac{5}{2}x+\frac{3}{2}. \quad (12e)$$

A method to reduce the $R_n(x)$ to closed expressions is given in Appendix C.

In spite of its rather complicated form, Eq. (12) has a simple interpretation. For $\lambda=0$, $I^{(0)}(\vartheta)$ reduces to $-2Ei(-x)$, which is essentially the result of Schiff's theory for the angular distribution. The first term of Eq. (12) has the advantage that it does not diverge for $\vartheta=0$. Besides, it must be pointed out that, when ϑ increases, $\lambda e^{-x}R_1(x)$ becomes more and more important and is the dominant term of $I^{(0)}$ in the asymptotic region; it behaves asymptotically as λ/ϑ^4. Due to the mathematical approximations used in previous papers, this feature was never present. The behavior of Eq. (12) can be easily understood from an intuitive point of view. When λ is very small, the angular width of the multiple scattering distribution is much larger than that of the intrinsic radiation. Therefore, for angles $\lambda<\vartheta^2\lesssim 1$, its contribution is the dominant one. On the contrary, for large angles $\vartheta^2\gg 1$, the convolution factor $(2/\lambda)[1+(\vartheta^2/\lambda)]^{-2}$ tends to zero much slower than the exponential integral term and therefore it determines the asymptotic behavior of Eq. (12). On the other hand, for small values of ϑ ($\vartheta\lesssim\lambda$), both contributions are important and their detailed combination is necessary in order to avoid logarithmic divergencies in the limit $\vartheta=0$.

Equation (12) is very useful for numerical calculations. If $\lambda\lesssim 0.2$, it is usually sufficient to consider the first and second terms in the series in λ. Remembering Eq. (5c), we readily get the corresponding approximation for the function $J(\vartheta,\lambda)$:

$$\frac{J^{(0)}(x,\lambda)}{2}=\frac{I^{(0)}}{2}+\frac{\lambda e^{-(\lambda+x)}}{(\lambda+x)}\left(1-\lambda-\frac{\lambda}{\lambda+x}\right)+e^{-x}\sum_{n=1}^{\infty}(-\lambda)^n\left\{n^2R_n(x)-\frac{\Lambda_{n-1}(x)}{n!}\times[2n+\lambda+e^{\lambda}Ei(-\lambda)(n^2+(2n+1)\lambda+\lambda^2)]\right\}. \quad (13)$$

[20] For the definition and properties of the functions $\bar{E}i(x)$ and $Ei(-x)$, see E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1945). For numerical applications, see *Tables of Sine, Cosine, and Exponential Integrals*, 2 volumes (Mathematical Tables Project, New York, 1940).

TABLE I. The functions $R_1(x)$, $R_2(x)$, and $h(x)$.

$x=\vartheta^2$	$R_1(x)$	$R_2(x)$	$h(x)$
0	0	0	0.577
0.01	0.005	0.002	0.575
0.05	0.025	0.008	0.568
0.1	0.051	0.018	0.556
0.2	0.103	0.038	0.526
0.4	0.214	0.084	0.439
0.6	0.333	0.140	0.328
0.8	0.462	0.206	0.198
1	0.596	0.282	0.054
1.5	0.998	0.523	−0.191
2	1.489	0.849	−0.739
2.5	2.107	1.274	−1.135
3	2.896	1.822	−1.511
4	5.268	3.402	−2.197
5	9.516	5.943	−2.795

In the important case of the forward radiation ($x=0$), Eqs. (12) and (13) reduce to closed functions:

$$I^{(0)}(0)/2=-e^{\lambda}Ei(-\lambda), \quad (14)$$

$$J^{(0)}(0)/2=-e^{\lambda}Ei(-\lambda)(1-\lambda-\lambda^2)+\lambda, \quad (14a)$$

which are exact and valid for all values of λ. Equations (14) and (14a) may be seen more directly from Eqs. (B, 5) and (B, 9) of the Appendix. For $\lambda\rightarrow 0$, both $I^{(0)}(0)$ and $J^{(0)}(0)$ tend to $-2\ln(\gamma\lambda)$ (where γ is Euler's constant). For $\lambda\rightarrow\infty$, $I^{(0)}(0)$ tends to zero as $2/\lambda$ while $J^{(0)}(0)$ tend to zero as $2/\lambda^2$. Inserting Eqs. (14) and (14a) into Eq. (5), we get the expression for the forward spectrum.

Equations (12), (13), and (14) complete the study of the zeroth order contribution. As the various Gaussian approximations of multiple scattering depend on the variables τ and ϑ in almost the same way [i.e., as in Eq. (11)], these results are independent of the particular Gaussian theory used in the calculation. (Of course, the width χ_c^2B is defined in different ways according to the various Gaussian theories, but its dependence on τ is essentially the same.) Therefore, the exact results of Eqs. (12) and (14) may be compared directly with previous theories that are based on Gaussian laws and depend upon mathematical approximations.

Let us consider now the contribution of order $1/B$ in Eq. (7):

$$I^{(1)}(\vartheta,\lambda)=\int_0^1\tau d\tau\int_0^{\infty}e^{-\alpha}d\alpha\int_0^{\infty}ydyJ_0(\vartheta y)\times\exp\left[-\frac{y^2}{4}(\tau+\lambda/\alpha)\right]\frac{y^2}{4}\ln\left(\frac{y^2}{4}\right). \quad (15)$$

A method of working out this integral is outlined in Appendix D. The general result [Eq. (D, 2)] is rather complicated. Fortunately, it simplifies considerably in the limit of small λ. Retaining only the main contribu-

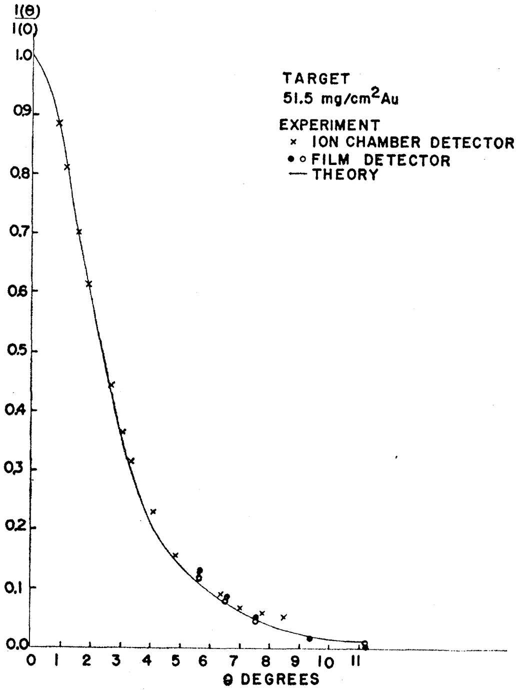

FIG. 1. Angular distribution of radiation from 51.5-mg/cm² Au target, normalized to unity for $\theta=0$. Experimental points are taken from reference 7, Figs. 12 and 13. Theoretical curve is calculated according to Eqs. (7), (12), and (15a) (see reference 22).

tion, the following result is obtained:

$$I^{(1)}/2=\ln(\gamma\lambda)\ \ln\gamma+\frac{1}{2}\left[(\ln\gamma\lambda)^2+\frac{\pi^2}{6}\right]+h(x)$$

$$+L\left(\frac{x}{x+\lambda}\right)+\ln\left(1+\frac{x}{\lambda}\right)\left[\ln(\gamma^2\lambda)+\tfrac{1}{2}\ln\left(1+\frac{x}{\lambda}\right)\right],\quad (15a)$$

where

$$h(x)=e^{-x}[\bar{E}i(x)-\ln x]-\sum_{n=1}^{\infty}\frac{(-x)^n}{n!n}\psi(n-1),\quad (15b)$$

$$L(u)=\int_0^u \ln(1-t)\frac{dt}{t}.\quad (15c)$$

The asymptotic expressions for $h(x)$ and $L(u)$ are given in Appendix D. A short table of the former is given in Table I; the latter has been tabulated by Mitchell.[21] In Eq. (15b) the function $\psi(n)$ is the logarithmic derivative of the factorial function, as defined in reference 20.

It is interesting to study the asymptotic behavior of Eq. (15) in the limit $\lambda=0$. This is most simply done by setting $\lambda=0$ and developing in series the exponential in Eq. (15). Then, making use of Eq. (C, 3′) of reference 11, we readily find:

$$\frac{I^{(1)}}{2}\rightarrow\sum_{n=1}^{\infty}\frac{n!n}{(n+1)x^{n+1}}.\quad (15d)$$

Remembering that the zeroth-order term behaves asymptotically as λ/ϑ^4, we see that Eq. (15a) will represent the dominant contribution in the asymptotic region when $\lambda<1/(2B)$.

Remembering Eq. (5c) and retaining only the main contribution in the limit of small λ, the corresponding term $B^{-1}J^{(1)}$ in the expansion of $J(x,\lambda)$ is readily obtained:

$$\frac{J^{(1)}(x,\lambda)}{2}=\frac{I^{(1)}}{2}-\frac{\lambda^2}{(x+\lambda)^2}.\quad (16)$$

DISCUSSION

The essential features of the theory of the angular distribution are contained in Eqs. (12) and (15a). The former is identical with the zeroth order approximation and is dominant for small angles. The latter is essential for larger angles, say $x\gtrsim 1$, and gives the transition to the contribution of single scattered electrons. Possible refinements include the study of the higher order terms that have been neglected in Eq. (15a) [see the exact representation (D, 2)] and the contribution of order B^{-2} that has not been considered. However, these terms appear only as corrections and their mathematical treatment would complicate the theory to a very large extent. Moreover, the omission of the contribution of order B^{-2} may be partly justified by the well-known fact that the two leading terms of Eq. (8) give usually a good approximate representation of the multiple scattering theory at any angle.[12]

A comparison between theory and experiment is made in Fig. 1. The experimental values for the angular distribution of the radiation are taken from reference 7 for the case $Z=79$, $T=51.5$ mg/cm²; the kinetic energy of the incident electrons was 16.93 Mev. As the experiment does not consider the dependence on η, we have compared that result with the normalized function $I(\vartheta)/I(0)$ as defined from Eqs. (7), (12), and (15a).[22]

A detailed calculation based on the complete energy-angle distribution, as given in Eq. (5), has shown that the normalized angular distributions for the spectral components $\eta=0.2$ and $\eta=0.95$ do not differ significantly in the case of the previous experiment.[17] The angular distribution for $\eta=0.95$ follows very closely that given by $I(\vartheta,\lambda)$, as can be easily understood by observing the energy dependence of Eq. (5). The normalized angular distribution for $\eta\doteq 0.2$ is somewhat lower but the differences are small, of the order of five

[21] K. Mitchell, Phil. Mag. **40**, I, 351 (1949).

[22] In the numerical computation of $I^{(0)}(x,\lambda)$, apart from the exponential integral term, we have used the terms $n=1$ and 2 in Eq. (12). To the value of $I^{(0)}(x,\lambda)$ thus obtained, the contribution $B^{-1}I^{(1)}(x,\lambda)$ was added, as given by Eq. (15a). This same experimental distribution has been compared with different theories (reference 7).

percent or less. This similarity is due to two reasons. In the first place, the angular distribution predicted by Schiff's intrinsic theory does not depend very sensitively on the energy. On the other hand, when λ is small, the behavior of the function $J(\vartheta,\lambda)$ is very similar to that of $I(\vartheta,\lambda)$.

In general, the theoretical predictions on the angular distribution are in very good agreement with the experiment of Fig. 1. In the large-angle region, the theoretical curve is somewhat below the ion chamber points, but it fits very closely the film data.

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APPENDIX A

Consider the well-known Bessel transform of the Gaussian function:

$$\int_0^\infty \vartheta d\vartheta J_0(y\vartheta)\exp(-\vartheta^2\alpha/\lambda)=\frac{\lambda}{2\alpha}\exp(-y^2\lambda/4\alpha). \tag{A, 1}$$

By multiplying Eq. (A, 1) by $\alpha e^{-\alpha}$ and integrating with respect to α from 0 to ∞, Eq. (6) is obtained immediately.

APPENDIX B

Consider Eq. (10). By integrating over τ we get

$$I^{(0)}(x)=2\int_0^\infty e^{-\alpha}d\alpha\left[Ei\left(-\frac{x\alpha}{\lambda}\right)-Ei\left(-\frac{x\alpha}{\lambda+\alpha}\right)\right], \tag{B, 1}$$

where $x=\vartheta^2$. The first integral is easily evaluated by means of a partial integration:

$$\int_0^\infty e^{-\alpha}Ei\left(-\frac{x\alpha}{\lambda}\right)d\alpha=-\ln\left(1+\frac{\lambda}{x}\right). \tag{B, 2}$$

Remembering the series development of $Ei(-u)$ (see reference 20), the following expression is obtained:

$$-\int_0^\infty e^{-\alpha}Ei\left(-\frac{x\alpha}{\lambda+\alpha}\right)d\alpha=\ln\left(\frac{\lambda}{x}\right)-e^\lambda Ei(-\lambda)-\sum_{n=1}^\infty\frac{(-x)^n}{n!n}V_n(\lambda), \tag{B, 3}$$

where

$$V_n(\lambda)=\int_0^\infty e^{-\alpha}\left(\frac{\alpha}{\alpha+\lambda}\right)^n d\alpha. \tag{B, 4}$$

Combining Eqs. (B, 2) and (B, 3), we get

$$\frac{I^{(0)}(x)}{2}=-\ln\left(1+\frac{x}{\lambda}\right)-e^\lambda Ei(-\lambda)-\sum_{n=1}^\infty\frac{(-x)^n}{n!n}V_n(\lambda). \tag{B, 5}$$

In order to evaluate the $V_n(\lambda)$, we introduce $u=\alpha+\lambda$ in Eq. (B, 4) and obtain

$$V_p=e^\lambda\int_\lambda^\infty e^{-u}\left(1-\frac{\lambda}{u}\right)^p du =e^\lambda\sum_{\nu=0}^\infty\binom{p}{\nu}(-\lambda)^\nu\int_\lambda^\infty\frac{e^{-u}}{u^\nu}du. \tag{B, 6}$$

By partial integrations, Eq. (B, 6) leads to

$$V_p=1+e^\lambda Ei(-\lambda)\sum_{\nu=1}^\infty\binom{p}{\nu}\frac{\lambda^\nu}{(\nu-1)!} -\sum_{\mu=1}^\infty(-\lambda)^\mu\sum_{\nu=\mu+1}^\infty\binom{p}{\nu}(-1)^\nu\frac{(\nu-\mu-1)!}{(\nu-1)!}. \tag{B, 7}$$

Of course, in the case of interest in Eq. (B, 5), $p=n$ is an integer and the series of Eq. (B, 7) reduce to simple polynomials in λ, which may be readily evaluated (in that case, replace the upper limits of the first, second and third summations by m, $n-1$ and n, respectively). From the definition (B, 4) we find also the following relation for integer n:

$$V_n(\lambda)=\frac{1}{(n-1)!\lambda^n}\left(\lambda^2\frac{d}{d\lambda}\right)^n e^\lambda Ei(-\lambda). \tag{B, 8}$$

Using Eqs. (5c) and (B, 8), the expansion for $J^{(0)}(x,\lambda)$ is obtained:

$$\frac{J^{(0)}(x,\lambda)}{2}=\frac{I^{(0)}}{2}+\frac{x\lambda}{(x+\lambda)^2}+\lambda+\lambda e^\lambda Ei(-\lambda)(1+\lambda) +\sum_{n=1}^\infty\frac{(-x)^n}{(n-1)!}\left[V_n-\left(2+\frac{1}{n}\right)V_{n+1} +\left(1+\frac{1}{n}\right)V_{n+2}\right]. \tag{B, 9}$$

Equations (B, 5), (B, 7), and (B, 9) are especially useful for the case of small angles.

APPENDIX C

In this section, a simple method for reducing the functions $R_n(x)$ to closed expressions is outlined. Consider first

$$R_1(x)=\sum_{\nu=1}^\infty\frac{x^\nu}{(\nu+1)!\nu}=\frac{1}{x}\int_0^x[\bar{E}i(x)-\ln(\gamma x)]dx. \tag{C, 1}$$

This integral may be performed by partial integrations and Eq. (12d) is readily obtained.

Now consider $R_n(x)$. This function, according to its definition (12b), consists of n series. Let $R_{n,j}$ be the jth series of $R_n(x)$. Then, obviously, we have

$$R_{n,1}(x)=\frac{1}{x^n}\int_0^x R_{n-1,1}(x)x^{n-1}dx, \qquad (C, 2)$$

$$R_{n,j}(x)=\int_0^x R_{n,j-1}(x)dx. \qquad (C, 3)$$

These integrals may be reduced again to closed expressions by means of partial integrations. In this way, for $n=2$, Eq. (12e) is obtained. For the sake of completeness we give the result for $R_3(x)$:

$$6R_3(x)=[\bar{E}i(x)-\ln(\gamma x)][1-2x-x^2/2] -\frac{e^x}{x^3}\left(2-\frac{3}{2}x^3+\frac{x^4}{2}\right)+\frac{2}{x^3}(1+x+x^2/2) -\frac{7}{6}-\frac{5}{3}x(1-x). \qquad (C, 4)$$

These closed expressions usually involve differences of large numbers, but, with the use of accurate tables, they greatly simplify the problem of evaluating the leading $R_n(x)$.

APPENDIX D

In order to evaluate Eq. (15), we first develop in series the Bessel function and perform the integration over y. This leads to

$$I^{(1)}(x)=2\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}(n+1)\int_0^1 \tau d\tau\int_0^{\infty}\frac{e^{-\alpha}d\alpha}{(\tau+\lambda/\alpha)^{n+2}} \times[\psi(n+1)-\ln(\tau+\lambda/\alpha)]. \qquad (D, 1)$$

Performing the remaining integrations in the order τ then α, the following result is obtained after a lengthy calculation:

$$I^{(1)}(x)/2=[e^{\lambda}Ei(-\lambda)(1+\lambda)+1]\ln\gamma +\tfrac{1}{2}[(\ln\gamma\lambda)^2+\pi^2/6]-V_1'-\tfrac{1}{2}V_0''+L\left(\frac{x}{x+\lambda}\right) +\ln(1+x/\lambda)[\ln(\gamma^2\lambda)+(1/2)\ln(1+x/\lambda)] -\sum_{n=1}^{\infty}\frac{(-x)^n}{n!}\left[\psi(n)\left(V_{n+1}+\frac{V_n}{n}\right) +V_{n+1}'+\frac{V_n'}{n}-\frac{V_n}{n^2}\right]. \qquad (D, 2)$$

Here V_n' is an abbreviation for $(\partial V_p/\partial p)_{p=n}$ and $V_0''=(\partial^2 V_p/\partial p^2)_{p=0}$. The functions V_n as well as their derivatives may be easily calculated from Eq. (B, 7). The function $L(u)$ is defined in Eq. (15c); the following are useful expressions:

$$L(u)=-\sum_{n=1}^{\infty}\frac{u^n}{n^2} \quad \text{for } |u|\leq 1,$$

$$L(1-u)=\ln(1-u)\ln u-L(u)=\pi^2/6 \quad \text{for } 0\leq u\leq 1,$$

$$L\left(\frac{u}{1+u}\right)=-L(-u)+\tfrac{1}{2}[\ln(1+u)]^3 \quad \text{for } u\geq 0, \qquad (D, 3)$$

$$L(u)=\tfrac{1}{2}(\ln u)^2-L(1/u)+\ln(-1)\ln u-\pi^2/3 \quad \text{for } u\geq 1.$$

The general result given in Eq. (D, 2) is very complicated. A great simplification is attained if terms of order λ or higher are neglected in the expression for V_n and its derivatives. This is equivalent to the simplifications $V_n=1$, $V_n'=V_0''=0$. Using these approximations and performing a similar simplification in the first term of Eq. (D, 2), then Eq. (15a) is readily obtained. In Eq. (15b) the following relation has been used:

$$-\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\psi(x)=e^{-x}[\bar{E}i(x)-\ln x]. \qquad (D, 4)$$

It is not difficult to prove directly that the series in Eq. (15b) behaves asymptotically as

$$-\sum_{n=1}^{\infty}\frac{(-x)^n}{n!n}\psi(n-1)\to -2\ln\gamma\ln x-\tfrac{1}{2}(\ln x)^2 +\frac{\pi^2}{12}-\frac{3}{2}(\ln\gamma)^2-\sum_{n=0}^{\infty}\frac{n!}{(n+1)x^{n+1}}. \qquad (D, 5)$$

By using the well-known asymptotic development of $e^{-x}\bar{E}i(x)$, we readily get

$$h(x)\to\sum_{n=1}^{\infty}\frac{n!n}{(n+1)x^{n+1}}-2\ln\gamma\ln x -\frac{1}{2}(\ln x)^2+\frac{\pi^2}{12}-\frac{3}{2}(\ln\gamma)^2. \qquad (D, 6)$$

Inserting Eq. (D, 6) into Eq. (15a), we reobtain Eq. (15d).