Nucleon Structure and the $n-p$ Mass Difference*

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The possibility of understanding the $n-p$ mass difference by means of electromagnetic interactions among the particles of a structured nucleon is pointed out. The success of this approach depends on the same general features that seem to be necessary in understanding the nucleon moments, namely, the enhancement of the annihilation of a π^+, π^- pair. Thus, further confirmation is provided for the hypothesis of a pion-pion attraction. Information on the spatial extent of the pion cloud is obtained, namely that $\rho \approx 2.4$ nucleon Compton wavelengths, where ρ^{-3} is related to the average of the inverse cube of the distance between the pions and the core.

I. INTRODUCTION

THE possibility of understanding the neutron-proton mass difference by treating the nucleon as a structured system has already been presented.[1] The argument proceeds on the assumption that the mass difference results from the electromagnetic interactions among the particles (charged pions and charged core) that compose the nucleon structure. To calculate the mass difference, electromagnetic interactions are investigated that arise in the neutron but not in the proton, and vice versa. On the basis of the charge-independence hypothesis, the mass difference between the charged and neutral pion and the electromagnetic interactions among charged pions affect the neutron and the proton in the same way and, therefore, need not be calculated. In order to isolate the effects that might contribute to the mass difference, we assume that the charged and uncharged core particles have the same mass.[2]

Two main effects, which are discussed in detail together with minor effects in Sec. II, contribute to the difference in the electromagnetic energy between the two systems. These are (1) the electrostatic energy in the neutron between the negatively charged pion cloud and the positively charged core, and (2) the interaction in the proton between the magnetic moment of the charged core and the magnetic field produced by the pion currents.

The electrostatic interaction in the neutron would lead to a neutron mass smaller than the proton mass, the amount being given in Mev by

$$E_e = \tfrac{2}{3} P(T=1)\rho_1^{-1},$$

where $P(T=1)$ is the total probability that any number of pions occur in the field with total isotopic spin $T=1$, and ρ_1 is related to their average distance (in pion Compton wavelengths) from the core. The factor $\frac{2}{3}$ enters because the pion cloud is negative in the $T=1$ state with probability $\frac{2}{3}$. Since the pion probabilities are small, E_e is relatively small. For example, if ρ_1 is taken to be as small as 2 nucleon Compton wavelengths, $E_e \approx 250$ kev for $P(T=1)=10\%$.

The magnetic interaction in the proton between the pions and the core contains a term that depends on the annihilation current.[3] The corresponding term is absent in the neutron, since the core is neutral. This contribution is proportional to the *amplitude* of the two-pion state, and is describable in terms of the annihilation of a $\pi^+-\pi^-$ pair, producing virtual radiation which is absorbed by the charged core. An equivalent statement is that the current loop formed by annihilation of the pair produces a magnetic field with which the core magnetic moment interacts. The direction of the current loop must be such that the resulting magnetic moment adds to the magnetic moment of the core. This implies that the magnetic field at the core due to the loop is parallel to the core moment (in the no-pion state). The energy is then negative and the mass of the proton is thus reduced.

An approximate expression for this energy is derived in Sec. II, and is found to be[4]

$$E_m = -6\mathfrak{M}(c)\Delta\mathfrak{M}_p\rho^{-3}, \tag{1}$$

where $\mathfrak{M}(c)$ is the magnetic moment of the charged core, $\Delta\mathfrak{M}_p$ is the contribution to the proton moment due to the annihilation term, and ρ is a distance depending on the spatial extent of the pion cloud.

There is, of course, a magnetic interaction in the neutron, and, in fact, to order g^2 in a straightforward weak-coupling theory, such an interaction lowers the neutron and proton mass by the same amount. Therefore, on this basis the electrostatic energy in the neutron would make it lighter. But the increased effect of the annihilation current due to pion-pion attraction, which was postulated in the preceding paper in order to understand the magnetic moments, here leads to an increased magnetic energy in the proton. In fact, since most of the anomalous moment came from the annihilation term, we expect the proton magnetic energy to greatly dominate that of the neutron. Furthermore, the amplitude

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[1] W. G. Holladay and R. G. Sachs, Phys. Rev. **96**, 810 (1954).
[2] A difference in core mass can be compensated by a small change in the dimensions of the pion cloud.
[3] Annihilation current was defined in the preceding paper [W. G. Holladay, Phys. Rev. **101**, 1198 (1956)] to be those terms in the pion current operator which create or annihilate π^+, π^- pairs.
[4] Note the similarity of this expression with the expression for hyperfine splitting of atomic levels.

effect and the small size of the nucleon system together with the ρ^{-3} dependence allow the proton magnetic energy to be larger than the electrostatic energy for the neutron. Therefore, pion-pion attraction with the concomitant enhancement of the annihilation current, which seems necessary to understand the nucleons' anomalous moments, also leads in a natural way to a proton mass smaller than that of the neutron.

Equation (1) can be used to get an approximate value of ρ^{-3}. If E_m is set equal to the total $n-p$ mass difference and $\Delta\mathfrak{M}_p$ to the total anomalous moment, and if $\mathfrak{M}(c)$ is taken to be one nuclear magneton, we find

$$\rho = 2.4\hbar/M_n c,$$

where M_n is the nucleon mass. The electrostatic energy and the other contributions to the moment anomaly will slightly reduce this value of ρ.

So far the discussion has proceeded in rather general terms, i.e., without an explicit form of the two-pion wave function. In Sec. III the $n-p$ mass difference is calculated with the specific form of the correlated two-pion function[5] assumed in the magnetic moment calculation. It will be shown that reasonable values of the parameters in the function can be chosen to fit both the $n-p$ mass difference and the moments.

II. ELECTROMAGNETIC INTERACTIONS IN THE STRUCTURED NUCLEON

In this section we discuss the $n-p$ mass difference without an explicit form of the pion wave function. Of the electromagnetic interactions to be considered, namely, electrostatic and magnetostatic, we now examine the former. The operator for the electrostatic energy is

$$E_e = e\frac{1+\tau_3}{2}\int\frac{\rho(\mathbf{r})}{r}d^3r, \tag{2}$$

where $\rho(\mathbf{r})$, the pion charge density operator, is

$$\rho(\mathbf{r}) = \frac{-ie}{\hbar}[\pi(\mathbf{r})\psi(\mathbf{r}) - \pi^*(\mathbf{r})\psi^*(\mathbf{r})],$$

in which $\psi(\mathbf{r})$ is a component of the charged pion field and $\pi(\mathbf{r})$ is its conjugate. For a spherical wave expansion of the pion field, E_e becomes

$$E_e = \frac{1+\tau_3}{2}\frac{e^2}{2}\sum_{s,s'}\delta_{l,l'}\delta_{m,m'} \times\left\{\left[\left(\frac{\omega}{\omega'}\right)^{\frac{1}{2}}+\left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}}\right](a_s^*a_{s'}-b_{-s'}^*b_{-s}) +\left[\left(\frac{\omega}{\omega'}\right)^{\frac{1}{2}}-\left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}}\right](-1)^m(a_s^*b_{-s'}^*-a_{s'}b_{-s})\right\} \times\int\frac{j_l(kr)j_{l'}(k'r)r^2dr}{r},$$

the notation being that of Sachs,[6] Appendix I. In the one- and two-pion states, the following matrix elements contribute to the electrostatic energy between the pions and the core, the energy values being expressed in Mev:

Neutron:

$$\langle N=1,(-)|E_e|N=1,(-)\rangle = -\tfrac{2}{3}P_1(1)\rho_1^{-1},$$

$$\langle N=2,L=1,T=1,(-,0)|E_e|N=2,L=1,T=1,(-,0)\rangle = -\tfrac{2}{3}P_1(2,1)({}_{1,1}\rho_{1,1}^{-1}),$$

$$\langle 2,0,1(-,0)|E_e|2,0,1,(-,0)\rangle = -\tfrac{2}{3}P_0(2,1)({}_{0,1}\rho_{0,1}^{-1}).$$

Proton:

$$\langle 2,0,0,(+,-)|E_e|2,0,1,(+,-)\rangle = \pm(4/3)\sqrt{2}[P_0(2,0)P_0(2,1)]^{\frac{1}{2}}({}_{0,0}\rho_{0,1}^{-1}),$$

$$\langle 2,1,0,(+,-)|E_e|2,1,1,(+,-)\rangle = \pm(4/3)\sqrt{2}[P_1(2,0)P_1(2,1)]^{\frac{1}{2}}({}_{1,0}\rho_{1,1}^{-1}),$$

$$\langle N=0|E_e|2,0,1,(+,-)\rangle = \pm(2/\sqrt{3})[P(0)P_0(2,1)]^{\frac{1}{2}}(\rho_{0,1}^{-1}),$$

where $+$, $-$, or 0 in the parentheses refer to charge on the pions. $P_1(1)$ is the one-pion probability with $L=1$ and $P_{L=1}(N=2,\,T=1)$ is the two-pion probability with total angular momentum $L=1$ and total isotopic spin $T=1$. The quantities denoted by ρ with various indices are distances in units of the pion Compton wavelength (μ^{-1}) which depend on the spatial extent of the pion cloud. More explicitly,

$$\rho_1^{-1} = \int\frac{drr^2}{\mu r}\left\{\frac{1}{2}\sum_{k,k'}\left[\left(\frac{\omega}{\omega'}\right)^{\frac{1}{2}}+\left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}}\right] \times g_1(k)g_1(k')j_1(kr)j_1(k'r)\right\}, \tag{3}$$

$${}_{L,T}\rho_{L',T'}^{-1} = \int\int\frac{drdr'r^2r'^2}{\mu r}\sum_{l,l'}\left\{\frac{1}{2}\sum_{\substack{k,k'\\k_1,k_1'}}\left[\left(\frac{\omega}{\omega'}\right)^{\frac{1}{2}}+\left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}}\right] \times g_{l,L,T}(k,k_1)g_{l',L',T'}(k',k_1') \times j_l(kr)j_l(k_1r')j_{l'}(k'r)j_{l'}(k_1'r')\right\} \tag{4}$$

$$\rho_{0,1}^{-1} = \int\frac{r^2dr}{\mu r}\sum_l\left\{\frac{1}{2}\sum_{k,k'}\left[\left(\frac{\omega}{\omega'}\right)^{\frac{1}{2}}-\left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}}\right] \times g_{l,0,1}(k,k')j_l(k'r)j_l(kr)\right\}. \tag{5}$$

The functions g are the Fourier-Bessel transforms of the pion radial functions. Hence if we set $(\omega'/\omega)^{\frac{1}{2}}=1$,

[5] See Eq. (3), preceding paper.

[6] R. G. Sachs, Phys. Rev. 87, 1100 (1952). In addition, we decompose the pion field into the various states as given in Sec. II in Sachs' paper.

the function in the curly bracket in Eq. (3) is the square of the one-pion radial function; in Eq. (4) is the product of a pair of two-pion radial functions, the first with quantum numbers l, L, T, the second with quantum numbers l', L', T', and in Eq. (5) the function in curly brackets vanishes. The quantities l and l' are one-pion angular momentum quantum numbers.

Note that in the proton an antisymmetric radial state[7] is involved in each matrix element. In particular the radial function in the no-pion, two-pion cross term must be antisymmetric, since the pion charge density operator has antisymmetric dependence on the radial variables in the pair annihilation and creation terms. This cross term will be quite small, because to annihilate one another, the two pions have to be close together, but as we saw in the last paragraph, the antisymmetric radial function vanishes when the pions coincide. The other matrix elements in the proton should also be small, for they not only contain antisymmetric radial functions but also depend on the *probability* of the two-pion state.

The electrostatic energy for the neutron lowers its mass as compared to that of the proton, by the amount estimated in the Introduction to be about 250 kev.

We now investigate the energy of the magnetic-magnetic interactions between the core and the pions, which in the no-recoil approximation can be written[8]

$$E_m = -\frac{1+\tau_3}{2}\mathfrak{M}(c)\cdot\mathbf{H}(c),$$

where $\mathfrak{M}(c)$ is the core moment operator and $\mathbf{H}(c)$ the magnetic field produced at the core by the pions. An estimate of E_m may be obtained by use of the Biot-Savart law,

$$\mathbf{H}(c) = \frac{1}{c}\int\frac{\mathbf{r}\times\mathbf{j}(\mathbf{r})}{r^3}d^3r,$$

if $\mathbf{r}$ is the vector away from the core to d^3r and $\mathbf{j}(\mathbf{r})$ is the current density of the pions. Hence

$$E_m = -\frac{1+\tau_3}{2c}\int\frac{\mathfrak{M}(c)\cdot(\mathbf{r}\times\mathbf{j})}{r^3}d^3r. \tag{6}$$

We are interested in the expectation value of this operator in the ground state of the nucleon, which has total angular momentum $\frac{1}{2}$. For a spin-$\frac{1}{2}$ core, the three terms of the dot product in Eq. (6) give equal contributions, so that the expectation value, $\langle E_m\rangle$ of E_m in the ground state of the nucleon is the same as that of the quantity

$$E_m' = -6\frac{1+\tau_3}{2}\mathfrak{M}_z(c)\frac{1}{2c}\int\frac{(\mathbf{r}\times\mathbf{j})_z}{r^3}d^3r.$$

The pion current appears in both this expression and in the magnetic moment operator in about the same way. Furthermore, the nucleon mass is independent of the direction of the spin, hence we are allowed to take it oriented along the axis of quantization; therefore it is possible to use many of the results of the magnetic moment calculation in the calculation of $\langle E_m'\rangle$. In particular for those states with a charged core

$$\langle E_m'\rangle = -6\mathfrak{M}_z(c)\langle\mathfrak{M}_z(\pi)\rangle\rho^{-3},$$

where $\langle\mathfrak{M}_z(\pi)\rangle$ is the magnetic moment contribution of the pions in those states and ρ^{-3} is related to the average of r^{-3}.

If consideration is limited to the one- and two-pion states, for the neutron the two contributing states, $|N=1,\ (-)\rangle$ and $|N=2,\ (-,0)\rangle$, provide relatively small magnetic moments[9] and therefore a relatively small contribution to the lowering of the neutron mass.

For the proton numerous states contribute, but since the annihilation term yields the largest moment, it also gives the largest magnetic-magnetic interaction energy. To calculate this, we make a spherical wave expansion of E_m', so that

$$E_m' = -6\frac{(1+\tau_3)}{2}\mathfrak{M}_z(c)\sum_{s,s'}\frac{e}{2(k_0k_0')^{\frac{1}{2}}}\times\{a_s^*a_{s'} - b_s^*b_{s'} + (-1)^{m'}a_s^*b_{-s'}^* + (-1)^m b_{-s}a_{s'}\}\times\langle l,m|L_z|l',m'\rangle\int\frac{j_l(kr)j_{l'}(k'r)r^2dr}{r^3}.$$

The annihilation term, then, is

$$\langle N=0|E_m'(p)|N=2\rangle = -6\mathfrak{M}(c)\left\{\frac{\pm e}{3^{\frac{1}{2}}}[P(0)P_1(2,1)]^{\frac{1}{2}}\times\sum_{l,k,k'}[l(l+1)(2l+1)]^{\frac{1}{2}}\frac{g_{l,\ L=1,\ T=1}(k,k')}{(k_0k_0')^{\frac{1}{2}}}\times\int\frac{j_l(kr)j_l(k'r)r^2dr}{r^3}\right\} = -6\mathfrak{M}(c)\Delta\mathfrak{M}_p\rho^{-3},$$

where the functions $g_l(k,k')$ are the Fourier-Bessel transforms of the two pion radial function, $\Delta\mathfrak{M}_p$ is the magnetic moment obtained from the no-pion, two-pion cross term and

$$\rho^{-3} = \frac{\{\text{quantity in curly brackets}\}}{\{\text{quantity in curly bracket with } r^{-3}=1\}}.$$

The sign of this interaction energy is given by the sign of the magnetic moments and is such as to lower the proton mass compared to that of the neutron, since, in the

[7] By the Bose principle an antisymmetric radial state must be associated with the $L=0$, $T=1$ and $L=1$, $T=0$ two-pion states.

[8] It is worthwhile to point out that the $\mathbf{A}\cdot\mathbf{P}$ interaction between the vector potential set up by the pions and the momentum $\mathbf{P}$ of the core is quite small since $\mathbf{A}$ vanishes along the axis of a circular current loop.

[9] We are assuming here a relative enhancement of the two-pion state over the one-pion state.

neutron, the annihilation term does not contribute to the magnetic energy. For $\mathfrak{M}(c)=e/2M$ and $\Delta\mathfrak{M}_p=-1.91(e/2M)$, the choice of $\rho=2.4/M$ yields -1.3 Mev for the magnetic energy in the proton. This value of ρ will be slightly modified by the minor effects we have neglected. Hence, the annihilation term which was so important in the magnetic moment calculation, also appears to be capable of accounting for the $n-p$ mass difference.

III. EFFECT OF CORRELATIONS

The preceding paper emphasized the effect of a pion-pion correlation on the calculated nucleon anomalous moments. A specific two-pion wave function with a pion-momentum cut-off and a correlation parameter was used there and various values of these parameters giving the correct moments were listed in Table I. In this section the $n-p$ mass difference is calculated on the basis of the same wave function.[10] We shall show that reasonable values of these parameters can be chosen to fit both the $n-p$ mass difference and the moments.

To calculate the proton magnetic energy with a specific two-pion wave function, it is convenient to write E_m in the plane wave representation:

$$E_m=\frac{1+\tau_3}{2}\frac{ie4\pi}{V}\times\sum_{\mathbf{k},\mathbf{k}'}\frac{(a_{\mathbf{k}'}{}^*a_{\mathbf{k}}-b_{\mathbf{k}'}{}^*b_{\mathbf{k}}+a_{\mathbf{k}'}{}^*b_{-\mathbf{k}}{}^*+a_{\mathbf{k}}b_{-\mathbf{k}'})\mathfrak{M}(c)\cdot\mathbf{k}'\times\mathbf{k}}{(k_0k_0')^{\frac{1}{2}}(\mathbf{k}-\mathbf{k}')^2}.$$

The no-pion, two-pion matrix element of E_m for the proton is

$$\langle 0|E_m(p)|2\rangle=\left[\frac{P(0)P_1(2,1)}{3}\right]^{\frac{1}{2}}\frac{ie}{\pi^2}\times\int\int\frac{\mathfrak{M}(c)\cdot(\mathbf{k}\times\mathbf{k}')f(\mathbf{k},\mathbf{k}')d^3kd^3k'}{(\mathbf{k}+\mathbf{k}')^2(k_0k_0')^{\frac{1}{2}}}. \quad (7)$$

If Eq. (4) in the preceding paper is used for $f(\mathbf{k},\mathbf{k}')$ ($+i$ is appropriate for the proton), this expression becomes

$$\langle 0|E_m(p)|2\rangle=-\frac{eK^2}{\pi^3}C'(\gamma,K)\left[\frac{P(0)P_1(2,1)}{3}\right]^{\frac{1}{2}}\mathfrak{M}(c)\times\int\int\frac{[\boldsymbol{\sigma}\cdot(\mathbf{k}\times\mathbf{k}')]^2}{(\mathbf{k}+\mathbf{k}')^2k_0{}^2k_0'^2}\times\left(\frac{K^2}{K^2+k^2}\right)^{5/4}\left(\frac{K^2}{K^2+k'^2}\right)^{5/4}\frac{\gamma}{\gamma+[(\mathbf{k}+\mathbf{k}')^2/kk']}d^3kd^3k',$$

which by integration over angles becomes

$$=\frac{-8eK^2C'}{\pi}\left[\frac{P(0)P_1(2,1)}{3}\right]^{\frac{1}{2}}\mathfrak{M}(c)\times\int\int\frac{dkdk'k^4k'^4}{k_0{}^2k_0'^2}\left(\frac{K^2}{K^2+k^2}\right)^{5/4}\left(\frac{K^2}{K^2+k^2}\right)^{5/4}\times\frac{1}{(2kk')^3}\left\{-4\gamma k^2k'^2+[4k^2k'^2-(k^2+k'^2)]\log\frac{(k+k')^2}{(k-k')^2}+[(\gamma kk'+k^2+k'^2)^2-4k^2k'^2]\log\frac{\gamma kk'+(k+k')^2}{\gamma kk'+(k-k')^2}\right\}.$$

Some approximations believed to be accurate to within about 5% have been used to obtain a value for this integral. The values of γ, K/μ, and P_1 which fit both the neutron anomalous moment and the $n-p$ mass difference are

P_1	γ	K/μ	
9%	0.5	6.3	(8)
20%	1.1	5.5	

In the calculation of the $n-p$ mass difference a neutron electrostatic energy of 300 kev when $P_1=9\%$ and 700 kev for $P_1=20\%$ has been taken into account.

Because the r^{-3} in the operator E_m has the effect of introducing a correlation between the pions in the calculation [the $(\mathbf{k}+\mathbf{k}')^2$ in the denominator of Eq. (7)], the explicit introduction of correlation does not produce as profound an effect in the $n-p$ mass difference as in the magnetic moments.[10] For example, even if $\gamma=\infty$ the correct mass difference can be calculated for $K/\mu=7$, and $P_1=9\%$.

IV. CONCLUSIONS

The electromagnetic interaction in the proton between the positively charged core and the magnetic field set up by the annihilation current has the correct sign to lower the proton mass. On the assumption of a strong pion-pion attraction, it has been shown how this term can dominate all other electromagnetic interactions in the nucleon which affect the mass difference. Further, it is possible to deduce that $\rho=2.4\hbar/M_nc$, where ρ^{-3} is related to the average of the inverse cube of the distance of the pion cloud from the core. Reasonable parameters (pion-momentum cut-off and correlation parameter) can be chosen in a two-pion wave function from which the moments and the mass difference can be calculated. These parameters cannot be uniquely fixed because of unprecise information on the pion probabilities, but their approximate values are given by expression 8.

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[10] We do not mean to impart any unusually great significance to this particular wave function. It merely forms a basis for orientation. It turns out that for this particular wave function the correlation *per se*, which had such a profound influence on the magnetic moments, does not greatly affect the $n-p$ mass difference. Other ways of introducing the correlation into the wave function can lead to an equally great influence on both the moments and the mass difference.