Self-Energy of Dirac Particles*

KERSON HUANG†
Department of Physics and Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts
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Weisskopf's analysis of the electron self-energy in 1939 showed that the part of the self-energy that is due to the spin of the particle is negative. An extension of this analysis with the addition of a Pauli term to the Hamiltonian shows that the additional self-energy that arises is also negative and only adds to the original spin self-energy. The physical significance of this negative spin energy is discussed in detail.

With the choice of an appropriate cutoff, the total self-energy of a Dirac particle with a Pauli anomalous magnetic moment can be made negative. A possible explanation of the neutron-proton mass difference in terms of their difference in electromagnetic self-energy, as first pointed by Feynman and Speisman, is considered in the light of such an analysis of the self-energy.

I. INTRODUCTION

IN the present framework of quantum electrodynamics, the electromagnetic self-energy of the electron is a divergent quantity. In many problems, however, it is possible to obtain physically meaningful results with the process of mass renormalization, whereby the infinite self-energy is covariantly isolated out as a mass "correction" and incorporated into the observed mass of the particle. The possibility of the renormalization procedure has not made the theory complete and self-consistent. For example, it has not thrown any light on the question whether the present quantum electrodynamics can be given a mathematical reformulation without any change in its physical content, in such a way that only finite quantities appear throughout the theory, or that some entirely new and heretofore unknown fields come into play when one considers the interaction between matter and high frequency quanta, so as to make all self-energies finite. In the present stage of development, these two approaches are actually equivalent. They merely represent different ways of saying that the interaction between matter and radiation according to the existing theory must be modified at high frequencies. Exactly how it is going to be modified is something completely unknown. To crudely simulate such a modification in the theory, one may introduce a certain cut-off frequency, above which one assumes that the interaction between matter and radiation becomes negligible. If one takes such a cut-off procedure seriously, then the electromagnetic self-energy of the electron becomes finite (being a function of the cut-off frequency), and assumes a physical meaning.

It was in the spirit of such an approach that Weisskopf[1] calculated and analyzed in detail the self-energy of the electron in 1939. It was shown there that the self-energy of the electron according to the electron-positron field theory of Dirac can be decomposed into several parts. There is the static Coulomb self-energy, which is the direct analog of the classical electromagnetic self-energy of a point charge, and which consists of terms diverging logarithmically with the cut-off frequency. There are two other terms which are specifically nonclassical in origin: first, there is the "fluctuation energy," which diverges quadratically, and which represents the kinetic energy of the electron in its "Brownian motion" in the fluctuating electromagnetic vacuum. And then there is the "spin energy," which consists of terms diverging logarithmically and quadratically, and which represents the additional energy arising from the electric and magnetic fields associated with the spin of the electron. This "spin energy" is negative in contradistinction to all other terms, and, as shown in W, is equal to twice the *static* attractive energy of a system of magnetic moment distributions. On adding up all these terms, there is a calcellation of the quadratically divergent terms, so that we have a total electron self-energy that is positive and diverges only logarithmically.

It is interesting, in the light of the analysis mentioned above, to consider the self-energy of a Dirac particle which possesses an anomalous magnetic moment in the form of a "Pauli term." The addition of the Pauli term leaves the charge and spin of the particle unchanged, but increases or decreases its total magnetic moment, depending on the sign of the anomalous magnetic moment. It is therefore not surprising to find, with subsequent analysis, that, by introducing an anomalous moment, the spin energy alone is affected in the self-energy of the particle. Since the spin energy is negative, it is made *more negative* by the addition of a positive anomalous moment, and *less negative* by a negative anomalous moment. As is pointed out by Feyman and Speisman,[2] this *could be* the reason why the proton has a slightly smaller mass than the neutron. Assume that before one "turned on" the electromagnetic coupling the proton and neutron both had the same "mechanical

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† Now at The Institute for Advanced Study, Princeton, New Jersey.

[1] V. F. Weisskopf, Phys. Rev. 56, 72 (1939), hereafter referred to as W.

[2] R. P. Feynman and G. Speisman, Phys. Rev. 94, 500 (1954); G. Speisman, Ph.D. Thesis, California Institute of Technology, 1955 (unpublished). See also A. Petermann, Helv. Phys. Acta 27, 441 (1954), for a slightly different approach.

mass." Assume further, that when the electromagnetic field is "turned on" the proton and neutron are Dirac particles with a "Pauli term," and differing from each other only in their electric charge ($+e$ for proton and 0 for neutron) and the anomalous magnetic moment ($+1.79$ and -1.91 nuclear magnetons, respectively). Then it will be seen that the electromagnetic self-energy tends to decrease the mass of the proton and to increase the mass of the neutron. By choosing an appropriate cut-off frequency, it is thus possible to arrive at the experimentally observed neutron-proton mass difference of 2.52 electron masses. The value of the cut-off frequency thus obtained can be related to a "radius" of the nucleon, for distances smaller than which the present electrodynamics breaks down. Feynman and Speisman found a cut-off radius of the order of the nucleon Compton wavelength—a result that is not unreasonable. However, the validity of adding a pure Pauli term to represent the electromagnetic interaction of a nucleon with such high-frequency quanta as is involved in the self-energy must be seriously questioned. This and other questions related to the logical basis for such an interpretation of the neutron-proton mass difference will be discussed later.

The main part of this paper will be devoted to an analysis and physical interpretation of the various terms that arise in the electromagnetic self-energy of a Dirac particle with a "Pauli term" anomalous magnetic moment. The spirit of the analysis will be the same as in W, but calculations will be considerably simplified by making use of the covariant quantum electrodynamics. An understanding of the effect of the Pauli term on the self-energy helps us in understanding even better the self-energy without the Pauli term. While it is unlikely that the neutron-proton mass difference may be explained without bringing in the details of meson theory, one hopes that the physical reason for this difference can be understood on a simple basis.

II. CALCULATION OF THE SELF-ENERGY

(a) General Considerations

Our purpose is to calculate the electromagnetic self-energy of a Dirac particle with an anomalous moment represented by a "Pauli term." The Hamiltonian density for the system under consideration is (with $\hbar=c=1$)

$$H=H_0+H_{\text{int}},$$

where H_0 is the field-free Hamiltonian of the Dirac and the electromagnetic field, and

$$H_{\text{int}}(x)=-j_\nu(x)A_\nu(x)-\tfrac{1}{2}M_{\alpha\beta}(x)F_{\alpha\beta}(x), \tag{1}$$

$$j_\nu(x)=ie\bar{\psi}(x)\gamma_\nu\psi(x), \tag{2}$$

$$M_{\alpha\beta}(x)=\mu(e/2m)\bar{\psi}(x)\sigma_{\alpha\beta}\psi(x). \tag{3}$$

The notations are the usual ones, with $\bar{\psi}=\psi^*\beta$, ψ being the field operator for the Dirac field, $F_{\alpha\beta}=(\partial/\partial x_\alpha)A_\beta-(\partial/\partial x_\beta)A_\alpha$, A_α being the 4-vector potential, is the quantized field tensor for the electromagnetic field. The value of the anomalous magnetic moment in units of the nuclear magneton $e/2m$ (m=nucleon mass, $mc^2=931$ Mev) is denoted by μ. The summation convention is employed whereby all repeated Greek indices are summed from 1 to 4. Charge symmetrization of the current operators is not necessary, as usual, provided one always sees to it that the vacuum expectation value of the current is zero.

The equation of motion for the field operators are

$$(\gamma_\nu\partial_\nu+m^0)\psi(x)=[ie\gamma_\nu A_\nu(x)+\tfrac{1}{2}\mu(e/2m)\sigma_{\alpha\beta}F_{\alpha\beta}(x)]\psi(x), \tag{4}$$

$$\Box A_\nu(x)=-j_\nu(x)+\partial_\alpha M_{\alpha\nu}(x), \tag{5}$$

where $\partial_\nu\equiv\partial/\partial x_\nu$, and $\Box\equiv\partial^2/\partial x_\nu\partial x_\nu=\nabla^2-\partial^2/\partial t^2$ is the d'Alembertian operator. (Note that Heaviside electromagnetic units are being used.) It is seen that the effect of the Pauli term is to add to the current operator j_ν of the Dirac field a term $-\partial_\alpha M_{\alpha\nu}$, so that the total source for the electromagnetic field is

$$J_\nu(x)=j_\nu(x)-\partial_\alpha M_{\alpha\nu}(x), \tag{6}$$

that is,

$$\Box A_\nu(x)=-J_\nu(x). \tag{7}$$

The constant m^0 in (4) denotes the "nonelectromagnetic mass" of the nucleon. It is the mass of the particle before one "turns on" the electromagnetic interaction.

The self-energy we shall calculate is the expectation value of $\mathcal{H}=\int Hd^3x$ for a state in which there is one Dirac particle at rest and no photons present, and we shall calculate it only to order e^2 in perturbation theory, treating H_{int} as the perturbing operator. The calculation will be carried out in the Heisenberg representation, and the problem reduces to the following: Given the Hamiltonian $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_{\text{int}}$, where $\mathcal{H}_0$ is already diagonal, diagonalize $\mathcal{H}$ to second order in e. This is achieved by performing a canonical transformation on $\mathcal{H}$:

$$\mathcal{H}^*=e^{iS}\mathcal{H}e^{-iS}=\mathcal{H}_0+\mathcal{H}_{\text{int}}+i[S,\mathcal{H}_0]+i[S,\mathcal{H}_{\text{int}}]-\tfrac{1}{2}[S,[S,\mathcal{H}_0]]+\cdots.$$

Requiring

$$[S,\mathcal{H}_0]=i\mathcal{H}_{\text{int}},$$

we obtain

$$\mathcal{H}^*-\mathcal{H}_0=\tfrac{1}{2}e^{iS}\mathcal{H}_{\text{int}}e^{-iS}, \tag{8}$$

where it is understood that the right side should be expanded to order e^2 only. Hence an equivalent form of (8) is

$$\mathcal{H}^*-\mathcal{H}_0=\tfrac{1}{2}\mathcal{H}_{\text{int}}{}^*, \tag{9}$$

where $\mathcal{H}_{\text{int}}{}^*$ is obtained from $\mathcal{H}_{\text{int}}$ by expanding all operators occurring in $\mathcal{H}_{\text{int}}$ in terms of the free-field operators, and the result taken to order e^2.

The self-energy of the nucleon in its rest frame, δm, is defined by the following equation:

$$\mathcal{H}^* - \left(\mathcal{H}_0 + \delta m \int \bar{\psi}\psi d^3x\right) = \tfrac{1}{2}\mathcal{H}_{\text{int}}^* - \delta m \int \bar{\psi}\psi d^3x \equiv 0.$$

Integrating over the time coordinate and taking the expectation value for the state in which there is one free nucleon at rest and no photons, we obtain

$$\tfrac{1}{2}\int d^4x \langle \mathcal{H}_{\text{int}}^* \rangle - \delta m \int d^4x \langle \bar{\psi}\psi \rangle = 0,$$

so that

$$\delta m = \tfrac{1}{2}\left\langle \int d^4x \mathcal{H}_{\text{int}}^* \right\rangle_1 \tag{10}$$

is obtained as an invariant quantity. The bracket $\langle\ \rangle_1$, denotes the "one-particle part" of an operator, defined by

$$\left\langle \int d^4x \mathcal{H}_{\text{int}}^* \right\rangle_1 = \left\langle \int d^4x \mathcal{H}_{\text{int}}^* \right\rangle \Big/ \left\langle \int d^4x \bar{\psi}\psi \right\rangle, \tag{11}$$

where on the right side the expectation values are taken for a state with one nucleon at rest and no photons present. The quantity $\langle \int d^4x \bar{\psi}\psi \rangle$ is singular, equal to the limit of $(2\pi)^4\delta(p-p')$ as p_ν, p_ν' both approach the 4-momentum of a free nucleon. Hence one can also write

$$\left\langle \int d^4x \mathcal{H}_{\text{int}}^* \right\rangle_1 = [(2\pi)^4\delta(p-p')]^{-1} \left\langle \int d^4x \mathcal{H}_{\text{int}}^* \right\rangle, \tag{12}$$

evaluated in the limit as p_ν, p_ν' both approach the free-nucleon 4-momentum. The "division" by the δ function has an obvious symbolic meaning.

The operators occurring in $\mathcal{H}_{\text{int}}^*$ satisfy the field equations (4) and (5), which can be transformed into integral equations satisfying the desired boundary conditions as follows[3]:

$$\psi(x) = \psi^0(x) - \int d^4x' S^{\text{ret}}(x-x')[ie\gamma_\nu A_\nu(x') + \tfrac{1}{2}\mu(e/2m)\sigma_{\alpha\beta}F_{\alpha\beta}(x')]\psi(x'), \tag{13a}$$

$$\bar{\psi}(x) = \bar{\psi}^0(x) - \int d^4x' \bar{\psi}(x')[ie\gamma_\nu A_\nu(x') + \tfrac{1}{2}\mu(e/2m)\sigma_{\alpha\beta}F_{\alpha\beta}(x')]S^{\text{adv}}(x'-x), \tag{13b}$$

$$A_\nu(x) = A_\nu^0(x) + \int d^4x' D^{\text{ret}}(x-x') J_\nu(x'), \tag{13c}$$

where ψ^0, $\bar{\psi}^0$ and A_ν^0 are the free-field operators given by solutions to the corresponding homogeneous equations of (4) and (5), with boundary conditions which are not important for the present discussions. The various Green's functions in (11) are defined by

$$S^{\text{ret}}(x) = (\gamma_\nu\partial_\nu - m)\Delta^{\text{ret}}(x), \quad \Delta^{\text{ret}}(x) = \bar{\Delta}(x) - \tfrac{1}{2}\Delta(x), \tag{14a}$$

$$S^{\text{adv}}(x) = (\gamma_\nu\partial_\nu - m)\Delta^{\text{adv}}(x), \quad \Delta^{\text{adv}}(x) = \bar{\Delta}(x) + \tfrac{1}{2}\Delta(x), \tag{14b}$$

$$D^{\text{ret}}(x) = \Delta^{\text{ret}}(x)|_{m=0} = \frac{1}{4\pi}\frac{\delta(|\mathbf{r}|-t)}{|\mathbf{r}|}, \tag{14c}$$

and the functions $\bar{\Delta}(x)$ and $\Delta(x)$ are defined by[4]

$$\bar{\Delta}(x) = \mathrm{P}\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot x}}{k^2+m^2}, \tag{15}$$

$$\Delta(x) = -2\pi i \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}\delta(k^2+m^2)\epsilon(k), \tag{16}$$

where P denotes the Cauchy principal value of the integral, and $\epsilon(k) = +1$ if $k_0 > 0$, and -1 if $k_0 < 0$, $(k_0 = ik_4)$. The Fourier transforms of these functions will be denoted by the same symbol, e.g.,

$$\bar{\Delta}(k) = \mathrm{P}\frac{1}{k^2+m^2}.$$

The free-field operators obey the usual commutation rules[4]:

$$\{\psi^0(x), \bar{\psi}^0(y)\} = -iS(x-y), \quad [A_\nu^0(x), A_\lambda^0(y)] = i\delta_{\nu\lambda}D(x-y), \tag{17}$$

where $S(x) = (\gamma_\nu\partial_\nu - m)\Delta(x)$, $D(x) = \Delta(x)|_{m=0}$.

To solve H_{int} to second order, it is sufficient to solve the field operators to first order. This is achieved by iterating the integral equations (11) once. We write

$$\psi(x) = \psi^0(x) + \psi'(x), \quad \bar{\psi}(x) = \bar{\psi}^0(x) + \bar{\psi}'(x), \quad A_\nu(x) = A_\nu^0(x) + A_\nu'(x), \tag{18}$$

where

$$\psi'(x) = -\int d^4x' S^{\text{ret}}(x-x')[ie\gamma_\nu A_\nu(x') + \tfrac{1}{2}\mu(e/2m)\sigma_{\alpha\beta}\partial_\alpha' A_\beta(x')]\psi^0(x'),$$

$$\bar{\psi}'(x) = -\int d^4x' \bar{\psi}^0(x')[ie\gamma_\nu A_\nu(x') + \tfrac{1}{2}\mu(e/2m)\sigma_{\alpha\beta}\partial_\alpha' A_\beta(x')]S^{\text{adv}}(x'-x), \tag{19}$$

$$A_\nu'(x) = \int d^4x' D^{\text{ret}}(x-x') J_\nu^0(x'),$$

where J_ν^0 is the operator defined by (6) but with $\bar{\psi}^0$, ψ^0

[3] C. N. Yang and D. Feldman, Phys. Rev. **79**, 972 (1950).

[4] For details see J. Schwinger, Phys. Rev. **75**, 651 (1949).

replacing $\bar{\psi}$, ψ, respectively. We shall also write

$$J_\nu(x)=J_\nu^0(x)+J_\nu'(x), \tag{20}$$

where

$$J_\nu'(x)=\bar{\psi}^0[ie\gamma_\nu-\mu(e/2m)\partial_\alpha\sigma_{\alpha\nu}]\psi' +\bar{\psi}'[ie\gamma_\nu-\mu(e/2m)\partial_\alpha\sigma_{\alpha\nu}]\psi^0. \tag{21}$$

The interaction Hamiltonian density can alternatively be taken to be $-J_\nu(x)A_\nu(x)$, and to second order it is given by

$$H_{\text{int}}^*=-J_\nu^0A_\nu'-J_\nu'A_\nu^0, \tag{22}$$

where we have omitted the term $-J_\nu^0A_\nu^0$ which obviously has vanishing one-particle part.

We see that H_{int}^* is split covariantly into two parts. The first term on the right side of (22) represents the self-interaction of the nucleon. It is the interaction of A_ν', which is the electromagnetic field generated by the charge and current in the nucleon, with the (unperturbed) charge-current distribution J_ν^0 of the nucleon itself. This term alone represents the entire self-energy of the particle in classical theory. It contains the Coulomb self-energy and the attractive energy of the "spin current" of the Dirac particle, denoted respectively by "Coulomb energy" and "spin energy." We shall denote the whole term "interaction energy."

The second term of (22) represents the interaction of the vacuum electromagnetic field A_ν^0 with the charge and current J_ν' of the nucleon that is produced under its forced motion caused by zero-point fluctuations of the electromagnetic field. It is hence a sort of kinetic energy of the nucleon due to its "Brownian motion" in the fluctuating vacuum. It is of purely quantum-mechanical origin and will be denoted "fluctuation energy" in accordance with W. Hence,

$$\delta m=\delta m_{\text{int}}+\delta m_{\text{fluc}}, \tag{23}$$

with

$$\delta m_{\text{int}}=-\tfrac{1}{2}\left\langle\int d^4xJ_\nu^0A_\nu'\right\rangle_1,$$
$$\delta m_{\text{fluc}}=-\tfrac{1}{2}\left\langle\int d^4xJ_\nu'A_\nu^0\right\rangle_1. \tag{24}$$

(b) The Fluctuation Energy

Let us first examine the fluctuation energy. The only contribution to this energy comes from the term j_ν' in J_ν', and thus the Pauli anomalous moment contributes no "Brownian motion" to the particle in the electromagnetic vacuum. This can be seen by the following argument:

Writing $J_\nu'=j_\nu'-\partial_\alpha M_{\alpha\nu}'$ according to (6) and (21), we see that

$$\delta m_{\text{fluc}} =-\tfrac{1}{2}\left\langle\int d^4xj_\nu'A_\nu^0\right\rangle_1-\tfrac{1}{2}\left\langle\int d^4x\tfrac{1}{2}M_{\alpha\beta}'F_{\alpha\beta}^0\right\rangle_1, \tag{25}$$

where the second term on the right is the contribution due to the Pauli term. Now $M_{\alpha\beta}'$ is the magnetic moment density produced under the forced motion of the nucleon in the vacuum electromagnetic field $F_{\alpha\beta}^0$. It must be of the form $c_1F_{\alpha\beta}^0+c_2\tilde{F}_{\alpha\beta}^0$, where c_1, c_2 are constants, and $\tilde{F}_{\alpha\beta}^0$ is the tensor dual to $F_{\alpha\beta}^0$. Hence

$$\left\langle\int d^4xM_{\alpha\beta}'F_{\alpha\beta}^0\right\rangle_1 =c_1\left\langle\int d^4xF_{\alpha\beta}^0F_{\alpha\beta}^0\right\rangle_0+c_2\left\langle\int d^4x\tilde{F}_{\alpha\beta}^0F_{\alpha\beta}^0\right\rangle_0 =c_1\left\langle\int d^4x(\mathbf{E}_0^2-\mathbf{H}_0^2)\right\rangle_0+c_2\left\langle\int d^4x\mathbf{E}_0\cdot\mathbf{H}_0\right\rangle_0=0.$$

This result is also verified by a direct calculation. Whence

$$\delta m_{\text{fluc}}=-\tfrac{1}{2}\left\langle\int d^4xj_\nu'A_\nu^0\right\rangle_1, \tag{26}$$

where

$$j_\nu'=ie(\bar{\psi}^0\gamma_\nu\psi'+\bar{\psi}'\gamma_\nu\psi^0).$$

Substituting the expression for $\bar{\psi}'$ and ψ' from (16), one obtains

$$\delta m_{\text{fluc}}=\frac{e^2}{2}\left\langle\int d^4xd^4x'[\bar{\psi}^0(x)\gamma_\nu S^{\text{ret}}(x-x')\gamma_\lambda\psi^0(x') +\bar{\psi}^0(x')\gamma_\lambda S^{\text{adv}}(x'-x)\gamma_\nu\psi^0(x)]A_\nu(x')A_\nu(x)\right\rangle_1. \tag{27}$$

To evaluate the one-particle parts above, we shall find the following relationships useful[5]:

$$\langle\{A_\nu^0(x),A_\lambda^0(x')\}\rangle_0=\delta_{\nu\lambda}D^{(1)}(x-x'),$$
$$\langle[\psi^0(x),\bar{\psi}^0(x')]\rangle_0=-S^{(1)}(x-x'), \tag{28}$$

where $\langle\ \rangle_0$ denotes vacuum expectation value, and

$$S^{(1)}(x)=(\gamma_\nu\partial_\nu-m)\Delta^{(1)}(x),\quad D^{(1)}(x)=\Delta^{(1)}(x)|_{m=0}, \tag{29}$$

$$\Delta^{(1)}(x)=2\pi\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot x}\delta(k^2+m^2). \tag{30}$$

In (27), only the symmetric parts of $S^{\text{ret}}(x-x')$ and $S^{\text{adv}}(x'-x)$ survive, so that they can both be replaced by $\bar{S}(x-x')=(\gamma_\nu\partial_\nu-m)\bar{\Delta}(x-x')$ [see (15)]. The final expression obtained is, when transformed into momentum space,[6]

$$\delta m_{\text{fluc}}=-\tfrac{1}{2}e^2\bar{u}(p)\int\frac{d^4k}{(2\pi)^4}D^{(1)}(p-k)\gamma_\nu\bar{S}(k)\gamma_\nu u(p), \tag{31}$$

[5] See reference 4, particularly, page 672.

[6] It is understood that p_ν shall be put equal to the 4-momentum of a free Dirac particle at rest at the end of the calculation, i.e., $p_\nu=(0,0,0,ip_0)$, where $\beta p_0=m$.

where $u(p)$ is a normalized Dirac spinor of 4-momentum p:

$$\bar{u}(p)(\gamma_\nu p_\nu - im) = (\gamma_\nu p_\nu - im)u(p) = 0, \quad \bar{u}(p)u(p) = 1. \tag{32}$$

(c) Interaction Energy

The interaction self-energy is defined by (28). The free-field current $J_\nu{}^0$ may be expressed differently by transforming $j_\nu{}^0$ into Gordon form[7]:

$$j_\nu{}^0 = ie\bar{\psi}^0\gamma_\nu\psi^0 = (e/2m)[-is_\nu + \partial_\alpha(\bar{\psi}^0\sigma_{\alpha\nu}\psi^0)], \tag{33}$$

where

$$s_\nu = \bar{\psi}^0(\partial_\nu\psi^0) - (\partial_\nu\bar{\psi}^0)\psi^0. \tag{34}$$

The space and time components of $j_\nu{}^0$ are

$$\begin{aligned}\mathbf{j}^0 &= (e/2m)[-i\mathbf{s} + \operatorname{curl}(\bar{\psi}^0\boldsymbol{\sigma}\psi^0) - i(\partial/\partial t)(\bar{\psi}^0\boldsymbol{\alpha}\psi^0)],\\ j_4{}^0 = i\rho^0 &= (e/2m)[-is_4 + i\operatorname{div}(\bar{\psi}^0\boldsymbol{\alpha}\psi^0)].\end{aligned} \tag{35}$$

The operators $(\partial/\partial t)(\bar{\psi}^0\boldsymbol{\alpha}\psi)$ and $\operatorname{div}(\bar{\psi}^0\boldsymbol{\alpha}\psi^0)$ have vanishing diagonal matrix elements, but the non-diagonal elements do not vanish.

The term $\mathbf{s}$ in (35) is just the Schrödinger convection current for a spinless particle. The second term, $\operatorname{curl}(\psi^0\boldsymbol{\sigma}\psi^0)$ represents the current associated with the spin (the current from the Zitterbewegung),[8] and the last term $(\partial/\partial t)(\psi^0\boldsymbol{\alpha}\psi^0)$ is the time rate of change of the electric moment density according to relativity. The last two terms add up to give the covariant spin current $\partial_\alpha(\bar{\psi}^0\sigma_{\alpha\nu}\psi^0)$. The fourth component of this, $i\,\mathbf{div}(\psi^0\boldsymbol{\alpha}\psi^0)$, has zero diagonal matrix elements. This insures that the total charge of the particle is related only to the convection current $\mathbf{s}$. Note that the effect of the Pauli term is just to add to the strength of the second term in (33), the spin current. Hence

$$J_\nu{}^0 = (e/2m)[-is_\nu + (1+\mu)\partial_\alpha(\bar{\psi}^0\sigma_{\alpha\nu}\psi^0)]. \tag{36}$$

From (27) and (19) we have

$$\begin{aligned}\delta m_{\text{int}} &= -\tfrac{1}{2}\left\langle \int d^4x d^4x' D^{\text{ret}}(x-x') J_\nu{}^0(x) J_\nu{}^0(x')\right\rangle_1\\ &= -\tfrac{1}{2}\left\langle \int d^4x d^4x' \bar{D}(x-x') J_\nu{}^0(x) J_\nu{}^0(x')\right\rangle_1.\end{aligned} \tag{37}$$

It is only $\bar{D}(x-x')$, the symmetric part of $D^{\text{ret}}(x-x')$, that survives, where

$$\bar{D}(x-x') = \bar{\Delta}(x-x')\,|_{m=0}.$$

Using the explicit form (36) for $J_\nu{}^0$, one obtains, after some straightforward reduction:

$$\delta m_{\text{int}} = \delta m_{ss} + \delta m_{s\sigma} + \delta m_{\sigma\sigma}, \tag{38}$$

where[6]

$$\begin{aligned}\delta m_{ss} &= \tfrac{1}{2}(e/2m)^2\left\langle \int d^4x d^4x' \bar{D}(x-x') s_\nu(x) s_\nu(x')\right\rangle_1\\ &= \tfrac{1}{2}(e/2m)^2(2\pi)^{-4}\bar{u}(p)\\ &\quad\times\int d^4k \bar{D}(p-k)(k+p)^2 S^{(1)}(k)u(p),\end{aligned} \tag{39}$$

$$\begin{aligned}\delta m_{s\sigma} &= -\tfrac{1}{2}i(1+\mu)(e/2m)^2\left\langle \int d^4x d^4x' \bar{D}(x-x')\right.\\ &\quad\times[s_\nu(x)\partial_\lambda{}'(\bar{\psi}^0(x')\sigma_{\lambda\nu}\psi^0(x'))\\ &\quad\left.+\partial_\nu(\bar{\psi}^0(x)\sigma_{\nu\lambda}\psi^0(x))s_\lambda(x')]\right\rangle_1\\ &= \tfrac{1}{2}i(1+\mu)(e/2m)^2(2\pi)^{-4}\bar{u}(p)\\ &\quad\times\int d^4k \bar{D}(p-k)(p_\nu - k_\nu)[\sigma_{\nu\lambda}S^{(1)}(k)\\ &\quad - S^{(1)}(k)\sigma_{\nu\lambda}](p_\lambda + k_\lambda)u(p).\end{aligned} \tag{40}$$

$$\begin{aligned}\delta m_{\sigma\sigma} &= -\tfrac{1}{2}(1+\mu)^2(e/2m)^2\left\langle \int d^4x d^4x' \bar{D}(x-x')\right.\\ &\quad\left.\times\partial_\alpha(\bar{\psi}^0(x)\sigma_{\alpha\nu}\psi^0(x))\partial_\beta{}'(\bar{\psi}(x')\sigma_{\beta\nu}\psi(x'))\right\rangle_1\\ &= \tfrac{1}{2}(1+\mu)^2(e/2m)^2(2\pi)^{-4}\bar{u}(p)\int d^4k(p-k)_\nu\\ &\quad\times(p-k)_\lambda\bar{D}(p-k)\sigma_{\nu\alpha}S^{(1)}(k)\sigma_{\lambda\alpha}u(p).\end{aligned} \tag{41}$$

(d) Connection with Conventional Methods

Before we proceed to evaluate these expressions explicitly, we may remark that our purpose in writing δm_{int} in this form is to show its dependence on the anomalous moment μ clearly. If μ were zero, there would be no particular advantage to this decomposition, and it would have been better to proceed with the origin form $ie\bar{\psi}^0\gamma_\nu\psi^0$ for $j_\nu{}^0$. For the part of δm_{int} that is independent of μ, one thus gets

$$\begin{aligned}\delta m_{\text{int}}(\mu=0) &= -\tfrac{1}{2}e^2(2\pi)^{-4}\bar{u}(p)\\ &\quad\times\int d^4k \bar{D}(p-k)\gamma_\nu S^{(1)}(k)\gamma_\nu u(p).\end{aligned} \tag{42}$$

Together with the fluctuation energy (31), (which has no contribution from the Pauli anomalous moment), one gets

$$\begin{aligned}\delta m(\mu=0) &= -\tfrac{1}{2}e^2(2\pi)^{-4}\bar{u}(p)\int d^4k\gamma_\nu[D^{(1)}(p-k)\bar{S}(k)\\ &\quad+\bar{D}(p-k)S^{(1)}(k)]\gamma_\nu u(p)\\ &= -\tfrac{1}{4}ie^2(2\pi)^{-4}\bar{u}(p)\\ &\quad\times\int d^4k\gamma_\nu D_F(p-k)S_F(k)\gamma_\nu u(p),\end{aligned} \tag{44}$$

[7] See, for example, W. Pauli, *Handbuch der Physik* (Verlag Julius Springer, Berlin, 1933), second edition, Vol. 24, Part 1, p. 238.

[8] K. Huang, Am. J. Phys. **20**, 479 (1952).

where

$$S_F(k)=2/(\gamma_\nu k_\nu - im),\quad D_F(k)=-(2i/k^2),\tag{45}$$

are the familiar Feynman propagation functions. Equation (44) is what one would write immediately for the self-energy of an electron, using the standard techniques of Feynman and Dyson. The transformation from (43) to (44) is based on the fact that the combination

$$D^{(1)}(p-k)\bar{S}(k)+\bar{D}(p-k)S^{(1)}(k)$$

differs from $D_F(p-k)S_F(k)$ by terms that vanish upon carrying out the required integrations. Splitting D_FS_F in the manner we do covariantly separates out the "fluctuation" and "interaction" self-energies.

(e) Results of the Calculation

To calculate δm_{fluc} and δm_{int} from (31) and (38)–(41) we shall introduce a cutoff in the rest frame of the nucleon by understanding the momentum space integration to be carried out as follows:

$$\int d^4k=\int_0^{K_0}K^2dK\int d\Omega\int_{-\infty}^{+\infty}dk_0,\tag{46}$$

where $K=|\mathbf{k}|$, and K_0 denotes the cut-off momentum. $d\Omega$ is an element of solid angle in $\mathbf{k}$-space, and $k_0=-ik_4$. All integrations are then elementary, and one obtains as final results:

$$\begin{aligned}\delta m&=\delta m_{\text{fluc}}+\delta m_{\text{int}},\\ \delta m_{\text{int}}&=\delta m_{ss}+\delta m_{s\sigma}+\delta m_{\sigma\sigma},\\ \delta m_{\text{fluc}}&=(\alpha/2\pi)m\nu^2,\end{aligned}\tag{47}$$

$$\begin{aligned}\delta m_{ss}=2(\alpha/2\pi)m\log[\nu+(1+\nu^2)^{\frac{1}{2}}]&\\ +\tfrac{3}{4}(\alpha/2\pi)m\{\nu(1+\nu^2)^{\frac{1}{2}}&\\ -\log[\nu+(1+\nu^2)^{\frac{1}{2}}]\},&\end{aligned}\tag{48a}$$

$$\begin{aligned}\delta m_{s\sigma}=-\tfrac{1}{2}(1+\mu)(\alpha/2\pi)m\{\nu(1+\nu^2)^{\frac{1}{2}}&\\ -\log[\nu+(1+\nu^2)^{\frac{1}{2}}]\},&\end{aligned}\tag{48b}$$

$$\begin{aligned}\delta m_{\sigma\sigma}=-(5/4)(1+\mu)^2(\alpha/2\pi)m\{\nu(1+\nu^2)^{\frac{1}{2}}&\\ -\log[\nu+(1+\nu^2)^{\frac{1}{2}}]\},&\end{aligned}\tag{48c}$$

where $\nu=K_0/m$ is the cut-off parameter, and

$$\alpha=e^2/4\pi\hbar c=1/137.$$

The first term in (48a) represents the static Coulomb self-energy of the particle, as can be verified by a direct calculation:

$$\begin{aligned}\delta m_{\text{Coul}}&=\tfrac{1}{2}\left\langle\int\frac{\rho^0(\mathbf{r}t)\rho^0(\mathbf{r}'t)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}d\mathbf{r}'dt\right\rangle\\ &=2m(\alpha/2\pi)\log[\nu+(1+\nu^2)^{\frac{1}{2}}].\end{aligned}\tag{49}$$

All other terms in δm_{int} then represent the effect of the spin current, and will be denoted by δm_{spin}, the spin energy, in the notation of W. Thus

$$\delta m_{\text{int}}=\delta m_{\text{Coul}}+\delta m_{\text{spin}},$$

with δm_{Coul} given by (49) and

$$\begin{aligned}\delta m_{\text{spin}}=-[1+3\mu+(5/4)\mu^2]\{\nu(1+\nu^2)^{\frac{1}{2}}&\\ -\log[\nu+(1+\nu^2)^{\frac{1}{2}}]\}.&\end{aligned}\tag{50}$$

The total self-energy is then

$$\begin{aligned}\delta m=3(\alpha/2\pi)m\log[\nu+(1+\nu^2)^{\frac{1}{2}}]&\\ +(\alpha/2\pi)m[\nu^2-\nu(1+\nu^2)^{\frac{1}{2}}]&\\ -[3\mu+(5/4)\mu^2](\alpha/2\pi)m\{\nu(1+\nu^2)^{\frac{1}{2}}&\\ -\log[\nu+(1+\nu^2)^{\frac{1}{2}}]\},&\end{aligned}\tag{51}$$

where the first term, which is independent of μ, is the well-known self-energy of a Dirac particle without anomalous moment, as given by W. The second term is negligible for large ν, and the remaining quadratically divergent terms represent the additional self-energy to the anomalous magnetic moment.

These results can be obtained more readily if one uses the Feynman-Dyson technique of calculation. The self-energy (51) is then the sum of contributions from four Feynman diagrams shown in Fig. 1. Diagram (a) gives rise to the terms independent of μ, the two diagrams (b) give rise to the terms linear in μ, and (c) gives the terms proportional to μ^2. The method we have employed, however, is more adapted to a physical interpretation of the self-energy.

III. PHYSICAL INTERPRETATION[9]

We have seen in the foregoing developments that the self-energy can be covariantly separated into two parts, δm_{fluc} and δm_{int}. The former is independent of the anomalous moment, and its physical significance has been discussed in previous sections and in W. The present discussion will deal mainly with the interaction energy δm_{int}. This can again be split into two terms: the static Coulomb self-energy δm_{Coul} and the spin energy δm_{spin}. The introduction of the Pauli anomalous moment affects δm_{spin} alone. It serves merely to put a multiplicative constant on δm_{spin}, while leaving unchanged its functional dependence on the cut-off parameter. For a physical understanding of the effect of the Pauli term, it is thus sufficient to examine δm_{spin} in more detail.

First let us consider the pure Dirac case, with $\mu=0$:

$$\begin{aligned}\delta m_{\text{spin}}=&-\tfrac{1}{2}\left\langle\int d^4xd^4x'D^{\text{ret}}(x-x')j_\nu(x)j_\nu(x')\right\rangle_1\\ &-\tfrac{1}{2}\left\langle\int d^4xd^4x'D^c(x-x')\rho(x)\rho(x')\right\rangle_1,\end{aligned}\tag{52}$$

where

$$\rho(x)=e\bar{\psi}(x)\beta\psi(x),\quad D^c(x-x')=\frac{\delta(t-t')}{|\mathbf{r}-\mathbf{r}'|}.\tag{53}$$

[9] From now on we shall drop the superscript on the free-field operators ψ^0, A_ν^0, etc. as no confusion will arise.

Since the separation between δm_{spin} and δm_{Coul} is a noncovariant one, our use of the Lorentz gauge so far, convenient for formal calculations, becomes clumsy for physical interpretation. It is hence desirable to go over to the Coulomb gauge for our purpose here. We shall use a superscript c on the electromagnetic potentials to denote Coulomb gauge. The equations defining the vector and scalar potentials are

$$\Box \mathbf{A}^c(x) = -\mathbf{j}(x) + \text{grad}\frac{\partial \varphi^c}{\partial t}, \quad \nabla^2 \varphi^c(x) = -\rho(x),$$

from which one obtains

$$\Box \mathbf{A}^c(x) = -\mathbf{I}(x), \quad \varphi^c(x) = \int d^4x' D^c(x-x')\rho(x'), \tag{54}$$

where

$$\mathbf{I}(x) = \text{curl}\mathbf{M}(x), \quad \text{div}\mathbf{I} \equiv 0, \tag{55}$$

$$\mathbf{M}(x) = \frac{e}{2m}\bar{\psi}(x)\boldsymbol{\sigma}\psi(x) + \frac{e}{2mi}\,\text{curl}\int \frac{d^4x}{4\pi} D^c(x-x') \times \left\{ \mathbf{s}(x') + \frac{\partial}{\partial t'}[\bar{\psi}(x')\boldsymbol{\alpha}\psi(x')] \right\}, \tag{56}$$

with $\mathbf{s}(x)$ as defined in (34). Thus we have $\mathbf{div}\mathbf{A}^c \equiv 0$ as an operator identity. If the matter field under consideration were a classical field, we would have $\mathbf{I}(x)$ equal to the transverse part of the current vector $\mathbf{j}(x)$, which would be just $(e/2m)\,\mathbf{curl}(\bar{\psi}\boldsymbol{\sigma}\psi)$. However, the transverse part of the operator $\mathbf{j}(x)$ contains extra terms which are not identically zero (only their diagonal matrix elements are zero). This very fact is responsible for the additional term in (56), which involves the convection current $\mathbf{s}$ and the electric moment $\bar{\psi}\boldsymbol{\alpha}\psi$. It has, of course, zero diagonal matrix elements. $\mathbf{I}(x)$ then is the effective transverse current density responsible for the magnetic field and the transverse electric field of the particle.

The magnetic and electric fields set up by the particle are given by

$$\begin{aligned} \mathbf{H} &= \text{curl}\mathbf{A}^c, & \mathbf{E}_T &= -\partial \mathbf{A}^c/\partial t, \\ \mathbf{E} &= \mathbf{E}_T + \mathbf{E}_L, & \mathbf{E}_L &= -\text{grad}\,\varphi^c, \end{aligned} \tag{57}$$

where $\mathbf{E}_T$, $\mathbf{E}_L$ are the transverse and longitudinal electric fields, respectively. It is easily verified that $\int \mathbf{E}_T \cdot \mathbf{E}_L d^3x = 0$. The quantity $\int d^4x(\mathbf{H}^2 - \mathbf{E}^2)$ is both Lorentz invariant and gauge invariant:

$$\int d^4x(\mathbf{H}^2 - \mathbf{E}^2) = \int d^4x\, j_\nu A_\nu = \int d^4x(\mathbf{I}\cdot\mathbf{A}^c - \rho\phi^c), \tag{58}$$

from which it can also be deduced that

$$\begin{aligned} \int d^4x \mathbf{I}\cdot\mathbf{A}^c &= \int d^4x(\mathbf{H}^2 - \mathbf{E}_T^2), \\ \int d^4x \rho\phi^c &= \int d^4x \mathbf{E}_L^2. \end{aligned} \tag{59}$$

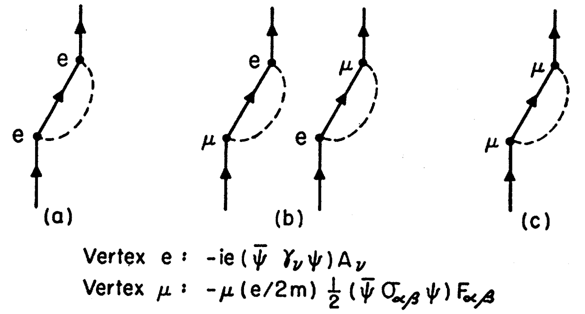

FIG. 1. Feynman diagrams for the self-energy of a Dirac particle with a "Pauli term" anomalous magnetic moment.

Hence

$$\begin{aligned} \delta m_{\text{Coul}} &= \tfrac{1}{2}\left\langle \int d^4x \mathbf{E}_L^2 \right\rangle_1, \\ \delta m_{\text{spin}} &= -\tfrac{1}{2}\left\langle \int d^4x(\mathbf{H}^2 - \mathbf{E}_L^2) \right\rangle_1 \\ &= -\tfrac{1}{2}\left\langle \int d^4x \mathbf{I}\cdot\mathbf{A}^c \right\rangle_1 \\ &= -\tfrac{1}{2}\left\langle \int d^4x d^4x' D^{\text{ret}}(x-x')\mathbf{I}(x)\cdot\mathbf{I}(x') \right\rangle_1. \end{aligned} \tag{60}$$

The last equation justifies the term "spin energy," for it clearly shows that δm_{spin} is the attractive energy of the system of transverse currents $\mathbf{I}(x)$ which owes its existence to the spin of the particle. The transversality of $\mathbf{I}(x)$ enables one to picture it as a collection of current loops, the mass motion associated with which gives rise to the intrinsic spin.[8]

For the case $\mu = 0$, it has been shown in W that (60) can even be reduced further. There is a close cancellation of terms to give

$$\delta m_{\text{spin}}(\mu=0) = -\tfrac{1}{2}(e/2m)^2 \left\langle \int d^4x d^4x' D^c(x-x') \times \text{curl}[\bar{\psi}\boldsymbol{\sigma}\psi(x)]\cdot\text{curl}[\bar{\psi}\boldsymbol{\sigma}\psi(x')] \right\rangle_1. \tag{61}$$

Thus for this case δm_{spin} is twice the static attraction of the spin currents. The simplicity of this result is also reflected in the fact that, as shown in W, the total energy in the transverse electromagnetic field is zero:

$$\frac{1}{2}\left\langle \int d^4x(\mathbf{H}^2 + \mathbf{E}_T^2) \right\rangle_1 = 0, \quad (\text{for } \mu = 0), \tag{62}$$

from which one can immediately conclude that

$$\frac{1}{2}\left\langle \int d^4x(\mathbf{H}^2 - \mathbf{E}_T^2) \right\rangle_1 = \left\langle \int d^4x \mathbf{H}^2 \right\rangle_1, \quad (\text{for } \mu = 0), \tag{63}$$

whence,

$$\delta m_{\text{spin}}(\mu=0) = -\left\langle \int d^4x \mathbf{H}^2 \right\rangle_1. \tag{64}$$

The simplicity of this result, unfortunately, is not carried over to the case when μ is not zero. It will be seen from (54) and (55) that by making μ different from zero, $\mathbf{A}^c$ and $\mathbf{I}$ are not simply multiplied by $(1+\mu)$ as one might expect if the system were classical. The electric and magnetic fields, then, are not simply multiplied by $(1+\mu)^2$ when we put in the Pauli term. The term which contains $\mathbf{s}$ in (56) is independent of μ, and is the cause of the complication. This term, as we have seen, is quantum mechanically necessary to insure the transversality of the magnetic field. With the addition of a Pauli term, (62) and (63) are no longer true. Instead, one obtains, after some straightforward calculations,

$$\frac{1}{2}\left\langle \int d^4x (\mathbf{H}^2 + \mathbf{E}_T^2) \right\rangle_1 = -2\mu(1+\mu)m(\alpha/2\pi)[f(\nu)+g(\nu)] \tag{65a}$$

and from (60), (50):

$$\frac{1}{2}\left\langle \int d^4x (\mathbf{H}^2 - \mathbf{E}_T^2) \right\rangle_1 = [1+3\mu+(5/4)\mu^2]m(\alpha/2\pi)[f(\nu)-g(\nu)], \tag{65b}$$

where

$$f(\nu)=\nu(1+\nu^2)^{\frac{1}{2}}, \quad g(\nu)=\log[\nu+(1+\nu^2)^{\frac{1}{2}}], \tag{66}$$

and ν is the cut-off parameter. The introduction of an anomalous moment changes the gyromagnetic ratio so that the delicate balance that produces result (62) is upset. Nevertheless, the interpretation of δm_{spin} as a spin energy still stands. We must now imagine that in addition to the static attraction between the spin "current loops," there is also an interaction with the convection current of the particle in intermediate states—a purely quantum-mechanical effect.

To talk in a more picturesque way, one may perhaps make the following semiclassical picture: Drawing again from the interpretation of the Zitterbewegung as a circular motion which give rise to the spin and magnetic moment of a Dirac particle,[8] the Pauli term merely adds to these "current loops" responsible for the magnetic moment. Thus it can be said that there are two kinds of current loops: the Dirac current loops and the Pauli current loops. Now we can write δm_{spin} in the form

$$\delta m_{\text{spin}} = -m(\alpha/2\pi)[f(\nu)-g(\nu)][(1+\mu)^2+\mu(1+\tfrac{1}{4}\mu)].$$

The first term expresses the effect of simply adding the Pauli current loops to the Dirac current loops. The presence of the second term demonstrates the fact that such a semiclassical picture is only qualitatively correct. However, such a viewpoint is helpful in the understanding of why δm_{spin} is negative when μ is not zero: the addition of μ to the Dirac intrinsic moment increases the absolute magnitude of the negative spin energy, just as an increase in the current of two loops increases their energy of attraction.

It may be well to discuss in this connection a fact which at first sight appears paradoxical. The results of our calculations show that $\delta m_{\text{spin}} = -\frac{1}{2}\langle \int d^4x (\mathbf{H}^2 - \mathbf{E}_T^2)\rangle_1$, which differs from the total energy residing in the transverse electromagnetic field $\frac{1}{2}\langle \int d^4x (\mathbf{H}^2 + \mathbf{E}_T^2)\rangle_1$. In fact, for $\mu=0$, the total transverse electromagnetic energy is zero, while δm_{spin} is nonzero and negative. How are we to understand this difference in terms of the usual intuitive picture that the self-energy is the work required to "create" the particle in question? Let us first consider the building up of a charge aggregate classically. If one assembles elementary charges, originally infinitely far from one another, and build up a single charge, then the work one must do is

$$\tfrac{1}{2}\int \mathbf{E}^2 d^3x,$$

equal to the energy in the field set up by the final system. Thus all the work goes into setting up the electric field, and this work is the self-energy of the final system. Now imagine that one tries to assemble a system of closed-loop currents. To make the picture even more concrete, imagine that originally one has a number of identical thin charged flywheels, which rotate about the same axis but are infinitely separated from one another. Now one wishes to bring them together along the axis to form a single flywheel. How much work is required? The answer depends on whether the angular velocity of these flywheels during the assembling is kept constant. If each flywheel maintains the same angular velocity throughout, then the work done is equal to the field energy set up by the system (ignoring the electric effect), i.e.,

$$\tfrac{1}{2}\int \mathbf{H}^2 d^3x.$$

In so doing, one also finds that the total angular momentum of the system (mechanical+field) is different before and after the assembling. On the other hand, if one stipulates that no external torque shall be in effect, so that the total angular momentum shall be kept constant, then one must take energy away from the flywheels, allowing them to slow down, with the result that one does a negative amount of work, equal to *minus twice* the field energy of the final system,[10]

$$-\int \mathbf{H}^2 d^3x,$$

[10] See, for instance, J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941).

which is the proper interpretation of $\delta m_{\rm spin}$. Now if one wants to form a semiclassical picture of the "creation" of the Dirac particle by assembling elementary charges and elementary "flywheels," one must keep in mind that the total angular momentum of the system is quantized, and must always be $\frac{1}{2}\hbar$.[11]

IV. THE NEUTRON-PROTON MASS DIFFERENCE

Feynman and Speisman[2] pointed out that the neutron-proton mass difference of 2.52 electron masses may arise from a difference in their electromagnetic self-energy. Experimentally, the proton and neutron are observed to possess anomalous magnetic moments[12] μ_P, μ_N respectively (over and above the Dirac intrinsic moment of +1 and 0 respectively), with

$$\mu_P = +1.79 \quad \text{Nuclear magnetons,} \qquad \mu_N = -1.91 \quad \text{Nuclear magnetons.} \tag{67}$$

One way to account for the moments is to introduce Pauli terms with the proper respective strengths (67) in the equation of motion for the proton and the neutron. Such a Pauli term represents correctly the interaction of the nucleon with radiation of long wavelength (long compared to the nucleon Compton wavelength).

Suppose now one takes this Pauli term seriously, and compares the self-energies δm_P of the proton with δm_N of the neutron. Since the neutron possesses no charge, δm_N will be solely proportional to μ^2, corresponding to the Feynman diagram (c) of Fig. 1. The proton, on the other hand, will have contributions from all Feynman diagrams of Fig. 1. From (51) one can immediately write down:

$$\begin{aligned}\delta m_P = (\alpha/2\pi)m\{3\log[\nu+(1+\nu^2)^{\frac{1}{2}}]+[\nu^2-\nu(1+\nu^2)^{\frac{1}{2}}] \\ -[3\mu_P+(5/4)\mu_P^2][\nu(1+\nu^2)^{\frac{1}{2}} \\ -\log(\nu+(1+\nu^2)^{\frac{1}{2}})]\}, \quad (68)\\ \delta m_N = -(\alpha/2\pi)m(5/4)\mu_N^2[\nu(1+\nu^2)^{\frac{1}{2}} \\ -\log(\nu+(1+\nu^2)^{\frac{1}{2}})].\end{aligned}$$

Since μ_P and μ_N are almost equal in magnitude, the main difference between δm_P and δm_N comes from the quadratically divergent term linear in μ_P. With a suitably chosen cut-off parameter ν, δm_P, δm_N are both finite and represent corrections to the nonelectromagnetic masses $m_P{}^0$, $m_N{}^0$, which appear in the equations of motion (4) (leaving out the term $ie\gamma_\nu A_\nu$ for the neutron). One assumes that

$$m_P{}^0 = m_N{}^0,$$

in accordance with the principle of charge independence. That is, the proton and the neutron are assumed to interact is exactly the same manner with field other

[11] We are indebted to M. Gell-Mann for an illuminating discussion concerning this point.

[12] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).

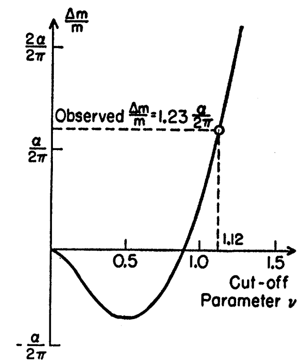

FIG. 2. Neutron-proton mass difference, taken as the difference in their electromagnetic self-energy, calculated as a function of the cut-off parameter $\nu = K_0/m$, where K_0 is the cut-off momentum.

the electromagnetic field—a fact which has been supported by numerous experiments in nuclear and meson physics.[12] The observed masses will then be

$$m_P = m_P{}^0 + \delta m_P, \quad m_N = m_N{}^0 + \delta m_N,$$

and the neutron-proton mass difference is just the difference in electromagnetic self-energy:

$$\Delta m = m_N - m_P = \delta m_N - \delta m_P.$$

From (60):

$$\begin{aligned}\Delta m/m = 3(\alpha/2\pi)\{[\mu_P-(5/12)(\mu_N^2-\mu_P^2)] \\ \times[\nu(1+\nu^2)^{\frac{1}{2}}-\log(\nu+(1+\nu^2)^{\frac{1}{2}})] \\ -\tfrac{1}{3}[\nu^2-\nu(1+\nu^2)^{\frac{1}{2}}]-\log[\nu+(1+\nu^2)^{\frac{1}{2}}]\} \\ = (\alpha/2\pi)\{6.15\nu(1+\nu^2)^{\frac{1}{2}}-\nu^2 \\ -8.16\log[\nu+(1+\nu^2)^{\frac{1}{2}}]\}. \quad (69)\end{aligned}$$

A plot of $\Delta m/m$ as a function of ν is shown in Fig. 2. The observed value[12] of

$$\Delta m = 2.52 \text{ electron masses,}$$

or

$$\Delta m/m = 1.23(\alpha/2\pi), \tag{70}$$

is seen to occur at $\nu = 1.12$. This yields a cut-off momentum $K_0 = 1.12mc$, or an "effective radius" of the nucleon:

$$r_0 = \hbar/K_0 = 0.9\hbar/mc = 2\times10^{-14} \text{ cm.} \tag{71}$$

Our philosophy here can be stated as follows: In ignorance of a consistent field theory which would give a finite self-energy, we attempt to simulate the actual state of affairs by using the existing quantum electrodynamics together with a cutoff. Our procedure consists of retaining only the electromagnetic interaction *outside* of the cut-off radius, and discarding the "inside" contributions (which diverges in the existing theory). The radius (71) hence defines a sphere surrounding the nucleon, in such a way that the electromagnetic self-energy in the region *outside* of this sphere, calculated with the existing electrodynamics, would account for the neutron-proton mass difference.

We have further assumed, in the present model, that in the "outside" region the electromagnetic properties of a nucleon are reasonably accounted for through the Pauli term. For our model to be valid, the cut-off radius should be of the order of, or not much smaller than, the extension of the virtual meson cloud about a nucleon. Otherwise one cannot employ the Pauli term, and must explicitly bring in meson theory.[13]

The smallness of the value (71) is a happy result from the point of view of the magnetic moments of nuclei. Experimental evidence seems to point to the deduction that the intrinsic magnetic moment of nucleons are not affected when they are in nuclear matter.[12] This means that the internucleon distance in normal nuclear matter (about 10^{-13} cm) must be large compared to the size of the meson cloud about a nucleon.

To make a better model, one probably should use two different cutoffs, one for the charge distribution and another for the current distribution as Feynman and Speisman did. However, since the main effect comes from the current distribution, the self-energy is not sensitive to such a modification, as borne out by the fact that Feynman and Speisman obtained about the same radius (71) for the nucleon.

Recently, there have been attempts to estimate the radius of the proton from experiments. Zemach,[14] and others, have tried to deduce the size of the proton magnetic moment distribution from the hyperfine structure of hydrogen. Zemach obtains an upper bound for the proton radius of about 5×10^{-14} cm, and a lower bound of zero. The neutron radius as deduced from experiments on the neutron-electron interaction[15] also points to a "small" meson cloud (less than 10^{-14} cm). These seem to be reassuring. However, experiments on the elastic scattering of high-energy ($\sim$200-Mev) electrons by hydrogen performed by Hofstadter and co-workers[16] yield an effective proton radius of the order of 10^{-13} cm, or about five times our cut-off radius (71). There is at present no quantitative way to relate our cut-off radius to the various effective radii determined from experiments. However, the largeness of Hofstadter's radius makes one feel unsafe in ignoring the detailed structure of the meson cloud about the nucleon.

It is generally believed that the nucleon acquires additional electromagnetic interaction via virtual emission and absorption of mesons in intermediate states. On the basis of Lorentz covariance, one can argue that the added interaction that is linear in the electromagnetic field must be of the general form[17]

$$\sum_{n=1}^{\infty}\{\epsilon_n(x)j_\nu(x)\Box^n A_\nu(x)+\tfrac{1}{2}\mu_n(x)M_{\alpha\beta}(x)\Box^n F_{\alpha\beta}(x)\}, \quad (72)$$

where $\epsilon_n(x)$, $\mu_n(x)$ are invariant functions whose explicit forms can only be obtained from meson theory. For interaction with photons of long wavelength, one can make a Taylor expansion of these functions and retain only the constant terms $\epsilon_1(0)$, $\epsilon_2(0)$, $\cdots$, $\mu_1(0)$, $\mu_2(0)$, $\cdots$. The constant $\mu_1(0)$ is identified with the anomalous magnetic moment, and $\epsilon_1(0)$ with the intrinsic electron-neutron interaction.[18] One may reasonably expect that using the form factor $\mu_1(x)$ instead of $\mu_1(0)$ for the anomalous moment will not yield results qualitatively different. It probably would still give a neutron-proton mass difference of the correct sign, with the same physical reasons. However, we have no idea as to how important the rest of the terms are in (72). We can only say, by virtue of the relation $\Box A_\nu=-j_\nu$, that their contributions to the self-energy can all be written in the form

$$\left\langle\int d^4x d^4x' G(x-x')j_\nu(x)j_\nu(x')\right\rangle_1, \quad (73)$$

indicating that they also represent current interactions (since they cannot contribute to the charge of the particle), similar to the anomalous moment term, but with some function $G(x-x')$ replacing $D^{\text{ret}}(x-x')$. Until the explicit form of $G(x-x')$ is calculated from some reliable meson theory, we have no sure way of telling whether these terms are negligible.

One may hope that in the very end it is only the sort of "current-loop attraction" that we have discussed that is important for the neutron-proton mass difference. But one can never quite guess the devious ways of nature. For the moment, the model discussed here serves to point out that in spite of the fact that the proton is charged while the neutron is not, it is possible, on account of their structure as Dirac particles, for the proton to have *less* electromagnetic self-energy than the neutron.

ACKNOWLEDGMENTS

The author is deeply indebted to Professor V. F. Weisskopf for suggesting this problem and for his illuminating discussions on all phases of this work.

[13] W. G. Holladay and R. G. Sachs [Phys. Rev. **96**, 810 (1954)] have considered the neutron-proton mass difference in a specific meson theory.

[14] A. C. Zemach (private communication). *Note added in proof.*—This work has now been published [Moellering, Zemach, Klein, and Low, Phys. Rev. **100**, 441 (1955)].

[15] Hughes, Harvey, Goldberg, and Stafue, Phys. Rev. **90**, 497 (1953).

[16] R. Hofstadter and R. W. McAllister, Phys. Rev. **98**, 217 (1955).

[17] G. Salzman, Phys. Rev. **99**, 973 (1955).

[18] L. L. Foldy, Phys. Rev. **87**, 675 (1952).