also are shown in Fig. 5. It can be seen that this distribution gives a fair approximation to the experimental points, but using the χ^2 test we have been unable to find any binomial distributions that fit the experimental results with more than a negligible probability.

The theoretical emission probabilities calculated by Leachman[14] are in good agreement with the results given in Table II.

It should be pointed out that the discrepancy as reported earlier[13] between our preliminary value for $\bar{\nu}$ of Cf^{252} based on Pu^{240} and that of Crane, Higgins, and Thompson[8] has been partially resolved. The Pu^{240} source used in our preliminary work was rather thick, and about one-quarter of the fissions were lost by foil absorption. The error in the early value might be explained by assuming that the lower-energy fissions, which would be more easily lost by foil absorption, have a higher internal excitation and give off more neutrons than the high-energy fissions. Preliminary work indicates that this is true. We are comparing the average number of neutrons and neutron multiplicities with the energy mode of fission using a "back-to-back" fission chamber. The remaining discrepancy may be due to the different methods of absolute calibration.

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Relativistic Radiative Transitions*

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We have calculated the oscillator strength for electric dipole transitions with retardation between the $1s$ state and higher discrete states for a single Dirac electron in a Coulomb field. Numerical values for atomic number 82 show that the retarded relativistic oscillator strength for each shell is about 0.8 of the nonrelativistic value. We give numerical values for the relative intensity of the different K x-ray lines for lead, neglecting screening: the $K\alpha_1/K\alpha_2$ doublet intensity ratio is 1.93 as compared to the NR value of 2.0. We introduce an average "oscillator density" for transitions for discrete states, and extrapolate to the series limit to find the photoelectric cross section at threshold. Our numerical value for lead of 740 barns is 23% larger than the value given by Hulme *et al.* We also calculate the summed oscillator strength, using a cutoff at an arbitrary high energy to give convergence. The numerical results for lead are insensitive to the numerical value of the cutoff used: using 10 Mev, we find a summed oscillator strength of 0.85, with an increase of 0.01 for each factor of two increase in the cutoff energy. Our value of the summed oscillator strength is less than that calculated nonrelativistically by Thomas, Reiche, and Kuhn; or relativistically by Gell-Mann, Goldberger, and Thirring.

I. INTRODUCTION

THE oscillator strengths for electric dipole ($E1$) radiative transitions between discrete nonrelativistic (NR) states in a Coulomb field are given in closed form and tabulated numerically by Bethe.[1] The same approximation gives the Stobbe formula[1,2] for the atomic photoeffect. The much more difficult relativistic calculations of the photoeffect have been performed numerically by Hulme *et al.*[3] and approximate analytic forms given by Hall,[4] and Sauter.[5]

Jacobsohn's calculations[6] for discrete-discrete transi-

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[6] B. Jacobsohn, Ph.D. dissertation [University of Chicago, 1947 (unpublished)].

tions extend the nonrelativistic calculations in two ways: he gives relativistic oscillator strengths for dipole transitions from the L shell, and he gives NR calculations of $E2$ oscillator strengths. Also, Massey and Burhop[7] have calculated the relativistic nonretarded transition rate of $K\alpha$ x-rays of $_{79}$Au.

In the next section we shall present relativistic calculations of the oscillator strengths for $E1$ transitions with retardation from the $1s$ state in a Coulomb field to the $2p_{1/2}$, $2p_{3/2}$, $3p_{1/2}$, $3p_{3/2}$, $4p_{1/2}$, $4p_{3/2}$, $7p_{1/2}$, and $7p_{3/2}$ states. Numerical results are given for lead. We use Dirac wave functions for a Coulomb field. This means that for an actual atom we are neglecting the screening effects of the other atomic electrons. Even for the highly ionized $_{81}Pb^{+}$, our use of the one-particle Dirac equation neglects effects such as vacuum polarization. We shall also discuss possible comparisons with relative line intensities and absolute line widths that might be obtained in x-ray measurements of heavy elements. In Sec. III, we introduce an average "oscillator density" for discrete-discrete transitions. We then extrapolate this oscillator density to the series limit, to compare with the oscillator density at the series limit found for lead by Hulme *et al.* We find a value 23% larger than theirs. In the last section we calculate the summed oscillator strength for a single electron moving in the Coulomb field of a lead nucleus. Since Hall's asymptotic formula gives an oscillator density proportional to the reciprocal of the photon energy, we cut off the integration at a finite upper limit to obtain a numerical result. This numerical result is then compared to the dispersion-theoretic result of Gell-Mann, Goldberger, and Thirring.[8] Our present numerical results appear to contradict the GGT assumption that at very high energies a bound electron has the same forward scattering amplitude as a free electron.

TABLE I. Oscillator strengths for $_{82}$Pb.

$1s_{1/2}$ to	Nonrelativistic[a] f_{NR}	Relativistic[b] f_{rel}	Retarded[c] f_{ret}	For each shell f_{ret}/f_{NR}
$2p_{1/2}$	0.138	0.112	0.112	
$2p_{3/2}$	0.277	0.230	0.195	
L shell	0.416	0.342	0.317	0.76
$3p_{1/2}$	0.0263	0.0174	0.0190	
$3p_{3/2}$	0.0527	0.0476	0.0409	
$3d$	...	...	0.0031	
M shell	0.0790	0.0650	0.0630	0.80
$4p_{1/2}$	0.0097	0.0060	0.0064	
$4p_{3/2}$	0.0193	0.0175	0.0151	
$4d$	...	...	0.0016	
N shell	0.0290	0.0235	0.0231	0.80
$7p_{1/2}$	0.00160	0.00093	0.00101	
$7p_{3/2}$	0.00321	0.00297	0.00258	
$7d$	...	...	0.00032	
Q shell	0.00481	0.00390	0.00391	0.81

[a] The nonrelativistic dipole oscillator strengths are taken from Bethe, reference 1, and are independent of atomic number.
[b] The relativistic oscillator strengths for Pb are taken from WBP.
[c] The retarded relativistic dipole oscillator strengths for Pb are found by evaluating the formulas in the Appendix. The oscillator strengths for $E2$ transitions are taken from Jacobsohn, reference 6.

[7] H. S. W. Massey and E. H. S. Burhop, Proc. Roy. Soc. (London) **A153**, 661 (1936); Proc. Camb. Phil. Soc. **32**, 461 (1936); W. Laskar, J. Phys. et radium **16**, 644 (1955).
[8] Gell-Mann, Goldberger, and Thirring, Phys. Rev. **95**, 1612 (1954), referred to GGT.

II. CALCULATION OF OSCILLATOR STRENGTHS

The oscillator strength is proportional to the square of the matrix element of $\boldsymbol{\alpha}\cdot\mathbf{e}_\lambda \exp(-i\mathbf{k}\cdot\mathbf{r})$, where $\boldsymbol{\alpha}$ is the set of 3 Dirac matrices, and the photon has polarization direction $\mathbf{e}_\lambda$ and wave number $|\mathbf{k}|$. These squared matrix elements must be summed over the m values of the final state, and averaged over the photon polarization. Neglecting retardation we put $\exp(i\mathbf{k}\cdot\mathbf{r})$ equal to unity, and for absorption by a single electron in a $1s$ state we obtain the expressions given by WBP [Eqs. (44) and (45)] and Jacobsohn[6] [Eqs. (4.3a) and (4.3b)]. For $1s\rightarrow np_{1/2}$ transitions the oscillator strength $(f_{ab})^{rel}$ is

$$(f_{ab})_{rel}=(8/9)\epsilon_{ab}^{-1}\left[\int_0^\infty (\tfrac{1}{2}f_{1s}g_{p(1/2)} +\tfrac{3}{2}g_{1s}f_{p(1/2)})r^2dr\right]^2, \quad (1)$$

while for $1s\rightarrow np_{3/2}$ transitions we have

$$(f_{ab})_{rel}=(16/9)\epsilon_{ab}^{-1}\left[\int_0^\infty f_{1s}g_{p(3/2)}r^2dr\right]^2. \quad (2)$$

In these equations, ϵ_{ab} is the energy difference in units of mc^2, while f_{1s} and g_{1s} are the small and large components, respectively, for the Dirac radial wave function for the $1s$ state, with analogous notation for $f_{p(1/2)}$, $f_{p(3/2)}$, $g_{p(1/2)}$, and $g_{p(3/2)}$. We follow Bethe's notation for Dirac Coulomb wave functions, with normalizations $\int[|f_{1s}|^2+|g_{1s}|^2]r^2dr=$ unity. Explicit formulas for the large and small component radial wave functions are given by WBP for states from the $1s$ to the $4f$, along with numerical evaluation of the relevant paramenters for 6 different atoms from 29 to 100.

Equations (1) and (2) can be evaluated in a straightforward manner. Formulas and numerical values for the matrix elements for transitions from the $1s$ to the $2p_{1/2}$, $2p_{3/2}$, $3p_{1/2}$, $3p_{3/2}$, $4p_{1/2}$, and $4p_{3/2}$ are given by WBP. However, since retardation effects are appreciable, we shall not give the details of this relativistic $E1$ oscillator strength, neglecting retardation, which we denoted by f_{rel}.

The effects of retardation are included for $E1$ transitions by taking the spherically symmetrical part of $\exp(-i\mathbf{k}\cdot\mathbf{r})$, namely the spherical Bessel function $j_0(kr)=\sin kr/kr$. This extra expression is then inserted inside the integral, giving for the retarded-relativistic oscillator strength f_{ret} for transitions from the $1s$ to $np_{1/2}$ states:

$$(f_{ab})_{ret}=(8/9)\epsilon_{ab}^{-1}\left[\int_0^\infty (\tfrac{1}{2}f_{1s}g_{p(1/2)} +\tfrac{3}{2}g_{1s}f_{p(1/2)})\sin(kr)rdr/k\right]^2, \quad (3)$$

and for transitions to $np_{3/2}$ states:

$$(f_{ab})_{\rm ret}=(16/9)\epsilon_{ab}^{-1}\left[\int_0^\infty f_{1s}g_{p(3/2)}\sin(kr)rdr/k\right]^2. \quad (4)$$

We shall present the details of the calculations of $f_{\rm ret}$, $f_{\rm rel}$, and the nonrelativistic $f_{\rm NR}$ for the $1s$ to $2p_{3/2}$ transition. We shall present in the Appendix the formulas for $f_{\rm ret}$ for transitions to p states with principal quantum numbers 2, 3, 4, and 7; and we shall present in Table I numerical results for lead for these transitions for $f_{\rm ret}$, $f_{\rm rel}$, and also $f_{\rm NR}$.

For the $1s$ to $2p_{3/2}$ calculation, we evaluate Eq. (4) as

$$\begin{aligned}(f_{1s,2p(3/2)})_{\rm ret}&=(16/9)(\epsilon_3-\epsilon_1)^{-1}\Big\{{\rm Im}\Big[\int_0^\infty-(2Z/N_1)^{1/2+\gamma_1}\\&\times((2\gamma_1)!)^{-1/2}(1-\epsilon_1)^{1/2}(4N_1(N_1+1))^{-1/2}e^{-Zr/N_1}\\&\times r^{\gamma_1-1}(N_1+1)(2Z/N_3)^{1/2+\gamma_2}((2\gamma_2)!)^{-1/2}\\&\times(1+\epsilon_3)^{1/2}(4N_3(N_3+2))^{-1/2}e^{-Zr/N_3}\\&\times r^{\gamma_2-1}(N_3+2)e^{ik_3r}rdr/k_3\Big]\Big\}^2\\&=(16/9)(\epsilon_3-\epsilon_1)^{-1}(2Z/k_3)^2 64A_3'[(\gamma_1+\gamma_2)!]^2\\&\times[(2\gamma_1)!]^{-1}[(2\gamma_2)!]^{-1}N_3^{-(2\gamma_2+1)}R_3^{-2(\gamma_1+\gamma_2)}\\&\times[\sin((\gamma_1+\gamma_2)\varphi_3)/(\gamma_1+\gamma_2)]^2, \quad (5)\end{aligned}$$

where

$$\begin{aligned}&R_3^2=[(N_3+1)/2N_3]^2+(k_3/2Z)^2,\\&\phi_3=\tan^{-1}[N_3k_3/Z(N_3+1)],\\&k_3=137.0(\epsilon_3-\epsilon_1)\text{ in units of }a_0^{-1},\\&A_3'=(1-\epsilon_1)(1+\epsilon_3)/32N_3(N_3+2),\\&\gamma_1=(1-\alpha^2Z^2)^{1/2},\quad \gamma_2=(4-\alpha^2Z^2)^{1/2},\\&N_1=1,\qquad N_3=2.\end{aligned} \quad (6)$$

In these equations, ϵ_1 and ϵ_3 are the energies of the $1s$ and $2p_{3/2}$ states; N_1 and N_3 are the apparent principal quantum numbers of the $1s$ and $2p_{3/2}$ states (in this case equal to the integral nonrelativistic principal quantum numbers); γ_1 is used for all $j=1/2$ states, and γ_2 for all $j=3/2$ states; and Im means take the imaginary part. The photon wave number k is given in units of a_0^{-1}, where a_0 is the Bohr radius.

We neglect retardation effects in Eqs. (5) and (6) by taking $(k_3/2Z)$ much less than $(N_3+1)/2N_3$, so that R_3 becomes $(N_3+1)/2N_3$, and the angle φ_3 becomes $N_3k_3/Z(N_3+1)$. We then obtain the relativistic (nonretarded) oscillator strength,

$$\begin{aligned}(f_{1s,2p(3/2)})_{\rm rel}&=(16/9)(\epsilon_3-\epsilon_1)^{-1}2^{2(\gamma_1+\gamma_2+2)}16A_3'\\&\times[(\gamma_1+\gamma_2)!]^2[(2\gamma_1)!]^{-1}[(2\gamma_2)!]^{-1}\\&\times N_3^{2\gamma_1+1}(N_3+1)^{-2(\gamma_1+\gamma_2+1)}. \quad (7)\end{aligned}$$

(Our procedure here is to check $f_{\rm ret}$; of course, $f_{\rm rel}$ can be obtained more easily by evaluating the integral in Eq. (2), as was done by WBP.)

We can then check this result by going to the NR limit, for αZ much less than unity. In this limit γ_1 becomes 1, and γ_2 becomes 2. We have

$$(\epsilon_3-\epsilon_1)^{-1}A_3'=(3\alpha^2Z^2/8)^{-1}(\tfrac{1}{2}\alpha^2Z^2)(2)/32N_3(N_3+2), \quad (8)$$

and

$$(f_{1s,2p(3/2)})_{\rm NR}=\tfrac{2}{3}(2^{13}/3^9)=\tfrac{2}{3}(0.4162). \quad (9)$$

We use the statistical weight factor 2/3 since the $2p_{3/2}$ transitions absorb 2/3 of the NR oscillator strength for transitions to the $2p$ state. Our result $2^{13}/3^9=0.4162$ is in agreement with the Bethe[1] equation (41.4) and Table XVI.

In a similar manner, the retarded $E1$ oscillator strengths were calculated for transitions to the $2p_{1/2}$, $3p_{1/2}$, $3p_{3/2}$, $4p_{1/2}$, $4p_{3/2}$, $7p_{1/2}$, and $7p_{3/2}$ states. The numerical values of $f_{\rm ret}$, $f_{\rm rel}$, and $f_{\rm NR}$ are given in Table I, for $_{82}$Pb; while the lengthy formulas for $f_{\rm ret}$ are given in the Appendix. We also include in Table I as part of $f_{\rm ret}$ Jacobsohn's values[6] of the $E2$ oscillator strength, calculated nonrelativistically without retardation. ($f_{\rm ret}$ for each shell should include the retarded relativistic oscillator strength for each transition to that shell. Our use of Jacobsohn's $E2$ values represents a first step towards that end; but we see that the $E2$ effects are rather small, so it is a reasonable approximation to use an NR calculation of the $E2$ oscillator strength, and to neglect higher multipoles for transitions to discrete states.)

Table I shows that the relativistic oscillator strengths $f_{\rm rel}$ are consistently smaller than the nonrelativistic $f_{\rm NR}$ for the same transition. Retardation effects give a significant decrease in the $np_{3/2}$ oscillator strengths; but actually increase the $np_{1/2}$ oscillator strengths above the relativistic values. (This increase occurs because the retardation factor decreases the cancellation in the overlap integral.) After we include the estimated $E2$ oscillator strength, the retarded relativistic oscillator strength for each shell is about 80% of the NR value.

We shall not make a quantitative comparison of our results with those of Massey and Burhop[7] for $_{79}$Au, since they use Slater's screening constants as an approximate correction for the use of electron wave functions in an atomic rather than a Coulomb field.

In suggesting a comparison of our results with x-ray measurements, we are trying to learn how significant relativistic and retardation corrections are, as compared to the corrections due to screening effects. X-ray measurements of various lines are usually given in terms of line intensities I_{ab} in erg/sec.

$$I_{ab}=(2e^2\hbar/mc^3)\omega_{ab}^3f_{ab}, \quad (10)$$

where ω_{ab} is the angular frequency and the f_{ab} is the oscillator strength. In Table II we give NR intensity

TABLE II. X-ray intensity ratios for lead.[a]

From	Nonrelativistic ω_{cb}/ω_{ab}	Nonrelativistic $(f_{cb}/f_{ab})_{NR}$	Nonrelativistic $(I_{cb}/I_{ba})_{NR}$	Relativistic and retarded ω_{cb}/ω_{ab}	Relativistic and retarded $(f_{cb}/f_{ab})_{rel}$	Relativistic and retarded $(I_{cb}/I_{ab})_{rel}$	Relativistic and retarded $(f_{cb}/f_{ab})_{ret}$	Relativistic and retarded $(I_{cb}/I_{ab})_{ret}$
$2p_{1/2}(K\alpha_2)$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$2p_{3/2}(K\alpha_1)$	1.00	2.00	2.00	1.035	2.05	2.27	1.74	1.93
$3p_{1/2}(K\beta_3)$	1.185	0.191	0.318	1.196	0.155	0.265	0.170	0.291
$3p_{3/2}(K\beta_1)$	1.185	0.382	0.636	1.206	0.425	0.745	0.365	0.640
$3d$	1.185	0.022	0.037	...	...	...	...	...
$4p_{1/2}(K\beta_2)$	1.25	0.070	0.136	1.263	0.054	0.109	0.057	0.115
$4p_{3/2}(K\beta_2')$	1.25	0.140	0.273	1.267	0.156	0.317	0.135	0.275

[a] The subscripts ab refer to the $K\alpha_2$ transition and the subscripts cb to the transitions from other L and M and N states to the K state; ω is the photon's angular frequency; f the oscillator strength from Table I; and $I \sim \omega^3 f$ the line intensity. Screening is neglected.

ratios; and relativistic and retarded intensity ratios for $_{82}$Pb. The intensity ratio $I_{ab}/I_{cb} = (\omega_{ab}/\omega_{cb})^3(f_{ab}/f_{cb})$.

We see that neglect of retardation can be serious, especially for comparison between doublets. Thus the $K\alpha_1/K\alpha_2$ intensity ratio is 2.00, 2.27, and 1.93 for NR, relativistic, and retarded, respectively. Similarly the $K\beta_1/K\beta_3$ intensity ratio is 2.00, 2.81, and 2.20 for the 3 cases treated. We also note that the weak $3d$ to $1s$ $E2$ transitions could probably be observed with high-resolution x-ray spectroscopy.[9]

The x-ray intensity ratios given in Table II will be significantly changed by screening; in general the intensities for the transitions from states of high n and j will be decreased most.

The oscillator strengths given in Table I can also be used to calculate the radiative line width of the K-state (one electron missing from the K shell) which provides most of the line width of x-ray lines emitted by heavy atoms.

X-ray measurements of relative line intensities or of line widths for heavy atoms seem very rare. For example, Gohkale's recent work[10] only goes up to atomic number 50. California Institute of Technology measurements[11] resolved the $K\alpha_1$, $K\alpha_2$ and $K\beta_1$, $K\beta_3$ doublets for $_{74}$W, and partially resolved the $K\beta_2'$, $K\beta_2$ doublet; but they do not report relative intensities, or absolute line widths.

III. SERIES LIMIT

We can introduce a fictitious "oscillator density" to smooth out the violent fluctuations of the delta function oscillator strengths for transitions from the $1s$ to other bound states. Since as we approach the series limit from the discrete side the oscillator density does smooth itself out naturally due to the increasing energy density of "discrete states," we can hope to join the averaged oscillator density for the various discrete states to the true oscillator density (which is proportional to the cross section) for the continuum. For the NR oscillator strengths we can perform this calculation of the series limit analytically both from the discrete and the continuous side; but here we shall merely present the numerical results. For the relativistic retarded oscillator strengths, we shall approach the series limit from the discrete side, and compare with the numerical value given by Hulme *et al.*[3] for an approach from the continuum side.

For each discrete state we introduce the average oscillator density as the ratio of the oscillator strength to the energy difference between adjacent half-integral quantum number states. For example, for the NR case for $n=2$, the energy denominator is $Z^2R[(3/2)^{-2} - (5/2)^{-2}]$, where R is the Rydberg constant. This choice of denominator is rather arbitrary, but makes little difference for the cases of high quantum numbers, which are most important in determining the series limit. Since we are interested in Sec. IV in comparing the integrated values of NR and retarded oscillator densities, it is convenient to measure the energy denominator in units of the ionization energy I, which is Z^2R in the NR case, and 1.11 times this for the $1s$ electron bound in the Coulomb field of a lead nucleus. The energy denominator, in units of I, will be denoted by $\Delta\eta$; and the energy of the transition by η.

For the retarded case, we include Jacobsohn's NR calculation of the $E2$ oscillator strength. We also take η and $\Delta\eta$ as the average energy or energy difference for the shell, weighting 2:1 in favor of the $p_{3/2}$ state.

Numerical values of $f/\Delta\eta$ are given in Table III. In the NR case, Fig. 1 shows that the "oscillator density" for the L, M, N, and Q shells extrapolates smoothly to the continuum value at the series limit, given by the Stobbe formula.[1] One can show this in detail from the closed form for the NR oscillator strength to discrete states, or as follows: Bethe[1] gives the asymptotic value of the oscillator strength for high principal quantum number n as $1.6/n^3$. The energy spread $\Delta\eta$ for each state is asymptotically $-(d/dn)(1/n^2) = 2/n^3$. Hence $f/\Delta\eta$ is 0.8, in agreement with Stobbe's formula. (The precise numerical value is actually $2^7/3e^4 = 0.781$.)

Our values of $f_{ret}/\Delta\eta$ for the relativistic-retarded case for lead fall on a smooth curve for the L, M, N, and Q shells, which provides a useful check on our numerical work. However, the series limit value of 0.69

[9] *Note added in proof.*—E. Inglestam [Nova Acta Reg. Soc. Sci. Upsaliensis, Ser. IV, **10**, No. 5 (1936)] has observed the $3d$ to $1s$ transition.

[10] B. G. Gokhale, Ann. phys. **7**, 852 (1952); M. Mladjenovic, J. phys. radium **16**, 545 (1955).

[11] Watson, West, Lind, and DuMond, Phys. Rev. **75**, 505 (1949).

TABLE III. "Oscillator densities" for discrete transitions.

Shell	Nonrelativistic, any Z f_{NR}[a]	η[b]	$\Delta\eta$	$f_{NR}/\Delta\eta$	Retarded for lead f_{ret}[c]	η	$\Delta\eta$	$f_{ret}/\Delta\eta$
$n=2$ (L)	0.4162	0.750	0.2844	1.463	0.317	0.761	0.267	1.187
$n=3$ (M)	0.0791	0.889	0.0784	1.009	0.0631	0.894	0.0755	0.834
$n=4$ (N)	0.0290	0.938	0.0322	0.901	0.0231	0.941	0.0309	0.748
$n=7$ (Q)	0.00481	0.980	0.00589	0.817	0.00391	0.981	0.00554	0.706
Series limit: extrapolated from discrete[d]		1.0		0.78		1.0		0.69
From continuum[e]		1.0		0.781		1.0		0.56

[a] f_{NR} is from Bethe, reference 1.
[b] η is the energy of the transition from the 1s state, in units of the 1s ionization energy. $\Delta\eta$ is the energy difference between adjacent half-integral "states," in these units. The oscillator density is $f/\Delta\eta$.
[c] f_{ret} is the relativistic retarded oscillator strength from Table I.
[d] See Fig. 1 for extrapolation.
[e] NR from Stobbe's formula; retarded from Hulme's numerical value, reference 3.

obtained by extrapolation from the discrete side is in disagreement with the value 0.56 obtained from Hulme's numerical value of the atomic cross section. We do not understand the reason for this discrepancy.

Hulme *et al.*[3] give the atomic cross section for the photoeffect for lead at the series limit as 1510 barns. To find the cross section for a single K electron, we multiply by 4/5 to convert from the atomic cross section to the cross section for the K shell, and we also multiply by 1/2, giving 604 barns as the cross section for a single K electron at the series limit. Hulme's cross section $\sigma=604$ barns is converted to the oscillator density $df/d\eta=0.56$ by the equation:

$$\sigma=(2\pi^2e^2\hbar/mc)I^{-1}(df/d\eta)=1080(df/d\eta) \text{ barns.} \quad (11)$$

The numerical value uses the ionization energy $I=101.6$ kev for lead.

Our cross section for lead at the series limit is 740 barns, or 23% larger than Hulme's value.

IV. SUMMED OSCILLATOR STRENGTH

Gell-Mann, Goldberger, and Thirring[8] state that the summed oscillator strength for a bound electron should be unity. This statement is based on: (1) their proof of the validity of the dispersion relation between the forward scattering amplitude at one energy and the oscillator strength at all energies; (2) their assumption that at very high energies the forward scattering amplitude by a bound electron is e^2/mc^2, or identical with that for a free electron.

The GGT sum-rule is better than previous sum-rules in that the GGT sum-rule uses the oscillator strength for all multipoles, including retardation; while the Thomas-Reiche-Kuhn sum-rule uses only the non-retarded $E1$ oscillator strength. However the effects of higher multipoles and retardation on the oscillator strength are the same order of magnitude as relativistic effects, so we would like to check the GGT sum-rule for a relativistic system.

As discussed by GGT, it is not clear whether the summed oscillator strength for a bound Dirac electron even converges, since according to Sauter's[5] or Hall's equation[4] at very high energies the oscillator density is approximately proportional to $1/\eta$. In this paper we shall obtain a numerical value for the summed oscillator strength for lead by stopping the integral at an arbitrary upper limit η_m. We shall find that the summed oscillator strength is insensitive to the value chosen for the cutoff η_m. Our present arbitrary procedure is clearly less satisfactory than achieving an understanding for the lack of convergence of the integral of Hall's equation. (But note that the finite size of the lead nucleus does provide a cutoff.)

The separation of photoeffect from pair production integrated cross sections (or Rayleigh from Delbrück scattering amplitudes) has some arbitrary features. Brown[12] has proposed a separation that seems preferable to that of GGT in a calculation using the one-electron Dirac equation. (Note that in the many-electron Dirac equation we should include vacuum polarization effects in finding the electronic wave functions in a Coulomb field.) Brown argues that we should consider absorption cross sections for electron plus vacuum minus the absorption cross section for the vacuum: i.e., we should replace the GGT photoeffect cross section $\sigma_{P.E.}$ by $(\sigma_{P.E.}-\sigma_{P.P.})$, where $\sigma_{P.P.}$ is the change in the pair production cross section due to the presence of the K electron. (In the one-electron theory, $\sigma_{P.P.}$ is the cross section for pair production in which the produced electron would occupy the same state as the already-present K electron.) Brown's procedure would achieve convergence in the integrated cross section $\int(\sigma_{P.E.}-\sigma_{P.P.})$ and in the corresponding forward scattering amplitude. Brown's procedure would also give a smaller value for the integrated cross section than found in this paper, where we follow the GGT procedure, introducing an arbitrary high-energy cutoff to achieve convergence.[13]

As discussed by GGT, the major weakness in their sum-rule is the need for an assumption concerning the

[12] G. E. Brown (private communication).
[13] *Note added in proof.*—We, and also M. L. Goldberger, are now in agreement with Brown's procedure. Following Brown's procedure, we obtain a summed oscillator strength of 0.87 (J. S. Levinger and M. L. Rustgi, Houston meeting of the American Physical Society, February, 1956).

TABLE IV. Oscillator densities used for the lead photoeffect.

$\eta=h\nu/$ Ionization energy	Nonrelativistic[a] $df/d\eta$	Relativistic-retarded[b] $df/d\eta$
1.0	0.781	0.56
2	0.115	0.104
4	0.0153	0.0176
10	921×10^{-6}	2400×10^{-6}
20	102×10^{-6}	820×10^{-6}
40	10.6×10^{-6}	330×10^{-6}
100	0.504×10^{-6}	118×10^{-6}
400	0.0046×10^{-6}	28×10^{-6}

[a] The oscillator density per ionization energy is obtained from the cross section using Eq. (11). The NR cross section is Stobbe's equation.
[b] The relativistic-retarded cross section is based on Hulme's values at $\eta=1$, 3.5, and 11, joined by Miss White to Hall's high-energy cross section.

high-energy scattering amplitude. Thus in checking their sum-rule we are checking the validity of this assumption. (We are also checking the validity of our cut-off procedure, and the accuracy of the numerical values of the oscillator strengths for discrete states and the oscillator density to the continuum.) We would like to remark here that the paper on forward scattering by one of us[14] contains dubious mathematical approximations, as well as the dangerous use of the Born approximation for the intermediate state in photon scattering by a bound electron.[15]

The summed oscillator strength f_{ret} for discrete states is, from Table I, 0.407 for transitions to the L, M, N, and Q shells. For transitions to all other discrete states we used Bethe's NR value[1] of 0.035, multiplied by the factor 0.80 for $f_{\text{ret}}/f_{\text{NR}}$ which holds for the M, N, and Q shells. Thus the summed retarded oscillator strength for all discrete states for a single electron bound in the Coulomb field of a lead nucleus is 0.435. (Compare with the NR value of 0.564.)

For the integrated oscillator density for the continuum we use an oscillator density $df/d\eta$ based on Hulme's values at the series limit, at 350 kev ($\eta=3.5$), and at 1.12 Mev ($\eta=11.0$). For higher energies we used Miss White's numerical values[16] for joining Hulme's values to Hall's high-energy equation. The numerical values used are given in Table IV. We show for comparison the nonrelativistic $df/d\eta$ based on Stobbe's equation. (Note that the NR expression for $df/d\eta$ *vs* η holds for any value of Z.) We see that for the continuum the relativistic-retarded oscillator strength falls below the NR value, up to $\eta=3$, or $h\nu=300$ kev. The relativistic $1/h\nu$ tail has quite a small ordinate, so that the contribution of the tail to the integrated oscillator density is rather small—unless it is infinite.

Using the data of Table IV, we find for the relativistic oscillator density,

$$\int_1^{100}(df/d\eta)d\eta=0.41. \tag{12}$$

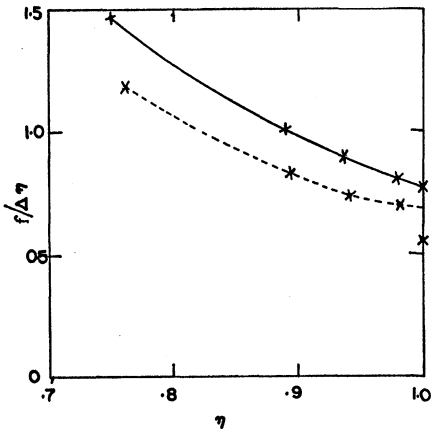

FIG. 1. Average "oscillator density" for transitions to discrete states *vs* the photon energy for a single electron in the Coulomb field of a lead nucleus. The numerical values are given in Table III. The solid line shows the nonrelativistic oscillator densities, while the dotted line shows those calculated relativistically, with retardation. The points are our calculations for the L, M, N, and Q shells; and at the series limit the NR photoeffect results of Stobbe, and the relativistic photoeffect result of Hulme *et al.*

(For comparison, the same integral for the NR oscillator density gives a slightly larger value of 0.43.) If we cut off the integral at $\eta_m=20$, we would reduce this value by 0.02; if we cut off the integral at $\eta_m=500$, we would increase the integral by 0.02. Thus the integrated oscillator density increases by about 0.01 for each factor of two increase in the cutoff used.

While we found in the previous section that our value for the oscillator density at the series limit, obtained by extrapolation from the discrete side, was 23% larger than Hulme's value, we do not change the integrated oscillator density greatly by changing to our value, since we still join on to Hulme's point at $\eta=3.5$. If we used our threshold oscillator density, the integral would be increased by 0.03.

Combining the integrated oscillator density (to $\eta_m=100$, or $h\nu_m$ about 10 Mev) with the sum over discrete states, we find

$$\sum_n(f_{0n})_{\text{ret}}=0.85. \tag{13}$$

Here the summation sign means that we sum over discrete states and integrate over the continuum, stopping at 10 Mev.

Our summed oscillator strength appears distinctly different from the value of unity given by GGT. We believe that the difference from unity is outside the numerical errors of the calculation: for instance, the use of our value instead of Hulme's for the threshold oscillator density gives an increase of only 0.03; and Hulme's quoted error of 6% for the calculational accuracy for the points at $\eta=3.5$ and 11 gives only another error of 0.02. Our use of an arbitrary cutoff η_m is suspect, but the final numerical result is insensitive to the cut-off value. We conclude that the forward scattering amplitude at very high energies by an

[14] J. S. Levinger, Phys. Rev. **87**, 656 (1952).
[15] Brown, Peierls, and Woodward, Proc. Roy. Soc. (London) **A227**, 51 (1954); Brenner, Brown, and Woodward, Proc. Roy. Soc. (London) **A227**, 59 (1954).
[16] Gladys White, National Bureau of Standards (private communication).

electron bound in the Coulomb field of a lead nucleus may be somewhat less than that for a free electron.

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APPENDIX

The retarded relativistic oscillator strengths for various $E1$ transitions from the $1s_{1/2}$ state are given in the equations below. We generally follow Bethe's notation[1] for the relativistic Coulomb wave functions, with γ_1 and γ_2 given by our Eq. (6). We use the additional notation

$$R^2=[(N+1)/2N]^2+(k/2Z)^2,$$
$$\phi=\tan^{-1}[Nk/Z(N+1)], \tag{A1}$$

where R, ϕ, and the apparent principal quantum number N are different for each final state. The energy $\epsilon=(1-\alpha^2Z^2/N^2)^{1/2}$ (in mc^2 units); and the photon wave number k (in a_0^{-1} units) is 137.0 times the energy difference of the quantum states.

For $np_{1/2}$ states, we use

$$\begin{aligned}&A=(3/2)[(1+\epsilon_1)(1-\epsilon)/32N(N-1)]^{1/2},\\&B=[1/2(1-\epsilon_1)(1+\epsilon)/32N(N-1)]^{1/2},\\&C=(A+B)(N-1),\\&D=(A-B)(n-1),\\&N=[n^2-2(n-1)(1-\gamma_1)]^{1/2}.\end{aligned} \tag{A2}$$

The general expression for the oscillator strength is found by expressing the $np_{1/2}$ wave function in terms of confluent hypergeometric functions,[1] and expressing the Laplace transform of a confluent hypergeometric function as a hypergeometric ${}_2F_1$ function. We have

$$\begin{aligned}f_{1s,\,np(1/2)}&=(32/9)(\epsilon-\epsilon_1)^{-1}(2Z/k)^2[(n-1)!]^{-1}\\&\times N^{-(2\gamma_1+1)}R^{-4\gamma_1}(2\gamma_1+n-1)![(2\gamma_1)!]^{-1}(2\gamma_1)^{-2}\\&\times\{D\,\mathrm{Im}[e^{2i\gamma_1\varphi}\,{}_2F_1(-n+2,2\gamma_1;2\gamma_1+1;e^{i\varphi}/RN)]\\&+C\,\mathrm{Im}[e^{2i\gamma_1\varphi}\,{}_2F_1(-n+1,2\gamma_1;\\&\qquad 2\gamma_1+1;e^{i\varphi}/RN)]\}^2.\end{aligned} \tag{A3}$$

Because the second and third terms in the hypergeometric function differ by unity, we can rewrite this expression using the binomial coefficient notation. Thus

$$\begin{aligned}&D\,\mathrm{Im}[e^{2i\gamma_1\varphi}\,{}_2F_1(-n+2,2\gamma_1;2\gamma_1+1;e^{i\varphi}/RN)]\\&\quad=2\gamma_1D\sum_{m=1}^{n-1}\binom{n-2}{m}(-1)^m(RN)^{-m}\\&\qquad\times\sin[(2\gamma_1+m)\varphi]/(2\gamma_1+m).\end{aligned} \tag{A4}$$

We shall give the detailed expression for the retarded oscillator strength for the transition to the $2p_{1/2}$ state.

$$\begin{aligned}f_{1s,\,2p(1/2)}&=(32/9)(\epsilon_2-\epsilon_1)^{-1}(2Z/k_2)^2N_2^{-(2\gamma_1+1)}\\&\times R_2^{-4\gamma_1}(2\gamma_1+1)\{(D_2+C_2)\sin(2\gamma_1\varphi_2)/2\gamma_1\\&\quad-(C_2/N_2R_2)\sin[(2\gamma_1+1)\varphi_2]/(2\gamma_1+1)\}^2,\end{aligned} \tag{A5}$$

where $n=2$, and $N_2=[4-2(1-\gamma_1)]^{1/2}$. The apparent principal quantum number N_2 is used to determine the energy ϵ_2, and from this the photon wave number k_2, then R_2 and φ_2, and also A_2, B_2, C_2, and D_2. [Note that in Eq. (A3), ϵ_1 refers to the energy of the $1s_{1/2}$ state, while the parameters without subscripts have different values depending on the value of n of the $np_{1/2}$ state.]

For the $3p_{1/2}$ state, we have

$$\begin{aligned}f_{1s,\,3p(1/2)}&=(16/9)(\epsilon_4-\epsilon_1)^{-1}(2Z/k_4)^2N_4^{-(2\gamma_1+1)}R_4^{-4\gamma_1}\\&\times(2\gamma_1+2)(2\gamma_1+1)\{(D_4+C_4)\sin(2\gamma_1\varphi_4)/2\gamma_1\\&-(D_4+2C_4)(N_4R_4)^{-1}\sin[(2\gamma_1+1)\varphi_4]/(2\gamma_1+1)\\&\quad+C_4(N_4R_4)^{-2}\sin[(2\gamma_1+2)\varphi_4]/(2\gamma_1+2)\}^2,\end{aligned} \tag{A6}$$

where $n=3$, and $N_4=[9-4(1-\gamma_1)]^{1/2}$.

For the $np_{3/2}$ state, we use

$$\begin{aligned}&A'=(1-\epsilon_1)(1+\epsilon)/32N(N+2),\\&N=[n^2-2(n-2)(2-\gamma_2)]^{1/2}.\end{aligned} \tag{A7'}$$

We find

$$\begin{aligned}f_{1s,\,np(3/2)}&=(64/9)(\epsilon-\epsilon_1)^{-1}(2Z/k)^2(2\gamma_2+n-2)!\\&\times[(2\gamma_2)!]^{-1}[(n-2)!]^{-1}A'N^{-(2\gamma_2+1)}R^{-2(\gamma_1+\gamma_2)}\\&\times[(\gamma_1+\gamma_2)!]^2[(2\gamma_1)!]^{-1}[(2\gamma_2)!]^{-1}(\gamma_1+\gamma_2)^{-2}\\&\times\{\mathrm{Im}[-(n-2)e^{i(\gamma_1+\gamma_2)\varphi}\,{}_2F_1(-n+3,\gamma_1+\gamma_2;\\&2\gamma_2+1;e^{i\varphi}/RN)+(N+2)e^{i(\gamma_1+\gamma_2)\varphi}\\&\quad\times{}_2F_1(-n+2,\gamma_1+\gamma_2;2\gamma_2+1;e^{i\varphi}/RN)]\}^2.\end{aligned} \tag{A7}$$

Here the hypergeometric functions cannot be written in terms of binomial coefficients; but they are polynomials with at most $n-1$ terms. The case $n=2$ is written as Eq. (5) of the text. The case $n=3$ gives

$$\begin{aligned}f_{1s,\,3p(3/2)}&=(64/9)(\epsilon_5-\epsilon_1)^{-1}(2Z/k_5)^2(2\gamma_2+1)\\&\times N_5^{-(2\gamma_2+1)}R_5^{-2(\gamma_1+\gamma_2)}[(\gamma_1+\gamma_2)!]^2\\&\times[(2\gamma_1)!]^{-1}[(2\gamma_2)!]^{-1}A_5'\{(N_5+1)\\&\times\sin[(\gamma_1+\gamma_2)\varphi_5]/(\gamma_1+\gamma_2)-(N_5+2)\\&\times(R_5N_5)^{-1}\sin[(\gamma_1+\gamma_2+1)\varphi_5]/\\&\qquad(2\gamma_2+1)\}^2,\\N_5&=[9-2(2-\gamma_2)]^{1/2}.\end{aligned} \tag{A8}$$