Nuclear Radius and Nuclear Forces*

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The difference between the radius of the nuclear matter distribution and the nuclear force radius, $R_N \simeq 1.4A^{\frac{1}{3}} \times 10^{-13}$ cm, for heavy nuclei ($A > 100$) is interpreted as a consequence of the finite range of nuclear forces. Assuming that the nuclear matter distribution coincides with the charge distribution as determined at Stanford ($R_C = 1.12A^{\frac{1}{3}} \times 10^{-13}$ cm is the distance at which the charge density falls to one half value) we sum up the nuclear interactions of an incident nucleon for various proposed internucleon potentials, $V(r)$. We also evaluate contributions from the spin, charge, and matter polarizations induced in the nuclear distributions by the incident nucleon as a test of the convergence of these calculations. The aim here is to infer some features of nuclear forces which satisfy saturation requirements and at the same time give rise to an appreciable nuclear attraction for an incident nucleon at R_N. Analyses of the scattering of neutrons and protons by heavy nuclei suggest a nuclear attraction $\gtrsim 14$ Mev at a distance R_N.

These considerations are primarily sensitive to the long range behavior of the direct, central part of $V(r)$. The key point which emerges from them is that the nuclear forces must contain long range ($\sim$ meson Compton wavelength) direct, central attractions which will be felt by an incident nucleon at R_N before the shorter range repulsions (hard cores, many-body forces, or exchange interactions), which are responsible for saturation, become effective. Such interactions can be constructed phenomenologically, but are not found in recent meson-theoretically deduced potentials.

I. INTRODUCTION

THE discussion presented in this paper is concerned with the difference in nuclear radii as observed and interpreted in various experiments which measure different properties of the nuclear structure. It is limited to nuclei of large mass number, $A > 100$. The two classes of experiments of immediate concern here are the one group which measures the "nuclear force radius" (R_N) as opposed to that which determines the "radius of the charge distribution" (R_C) in nuclei.

By "nuclear force radius" is meant the radius at which an impinging nuclear particle (neutron, proton, etc.) first feels the influence of the nuclear forces. Analyses of neutron cross sections at various energies indicate a nuclear radius of roughly $1.4A^{\frac{1}{3}}$. (Unless specifically stated otherwise, all lengths in this paper are in units of 10^{-13} cm.)[1] The approximate $A^{\frac{1}{3}}$ variation of the radius expresses the well-known phenomenon of saturation of nuclear densities. More detailed considerations of the total, reaction, and elastic scattering cross sections of neutrons with kinetic energy in the range from thermal energies up to the order of 10 Mev as presented in the cloudy crystal ball analysis of Feshbach, Porter, and Weisskopf,[2,3] give radii which are slightly larger for intermediate mass nuclei with $A\sim 50$ and smaller for the heavy nuclei with $A\sim 200$. The optical analysis of Taylor[4] for high-energy total neutron cross sections in the energy range 50 to 400 Mev agrees with this conclusion. We take as a representation of the nuclear force radius

$$R_N = (1.26A^{\frac{1}{3}} + 0.75). \tag{1}$$

Equation (1) thus predicts radii of $1.45A^{\frac{1}{3}}$ for $A\sim 60$ and of $1.39A^{\frac{1}{3}}$ for $A\sim 200$.

This expression for the nuclear radius is obtained from a simple model which pictures the nuclear potential well to have sharp edges. For a rounded-off nuclear well, corresponding to a surface of finite thickness, we must know the depth of the potential well at the radius, R_N, in Eq. (1) before we can interpret this number. Toward this end, we may appeal to proton scattering cross sections. The analysis of Woods and Saxon[5] has shown that it is possible to fit the observed differential elastic scattering cross sections for $\sim$20-Mev protons on medium and heavy nuclei with a nuclear potential which decreases smoothly to zero in a distance $\sim$2. A square-well model fails here because it predicts considerably too much large-angle relative to small-angle elastic scattering.

More important, however, for the discussion here is the presence of the Coulomb barrier. A proton with energy less than $\sim Z/A^{\frac{1}{3}}$ Mev incident on a nucleus of charge Z and mass number A must tunnel through the Coulomb barrier. Its probability of reaching the nuclear surface and of initiating a reaction is a very sensitive function of the height and width of the barrier. The

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[1] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), see Chap. 1, Sec. 4 for a qualitative discussion of the nuclear size as deduced from various experiments.

[2] Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954). In this paper, the radius is actually taken to be $1.45A^{\frac{1}{3}}$.

[3] Later work (personal communication with V. F. Weisskopf) as discussed in the Brookhaven National Laboratory Report BNL-331 C-21, on the conference "Statistical aspects of the nucleus" held January 24–26, 1955 (see especially p. 16) reproduces a better fit with data on neutron scattering angular distributions and on cross sections for compound nucleus formation as a function of mass number if the radius is altered to a value in close agreement with Eq. (1).

[4] T. B. Taylor, Ph.D. thesis, Cornell, 1954 (Phys. Rev. to be published); also Phys. Rev. **92**, 577 (1954).

[5] R. G. Woods and D. S. Saxon, Phys. Rev. **95**, 577 (1954). Experimental and earlier theoretical analyses are referred to in this work.

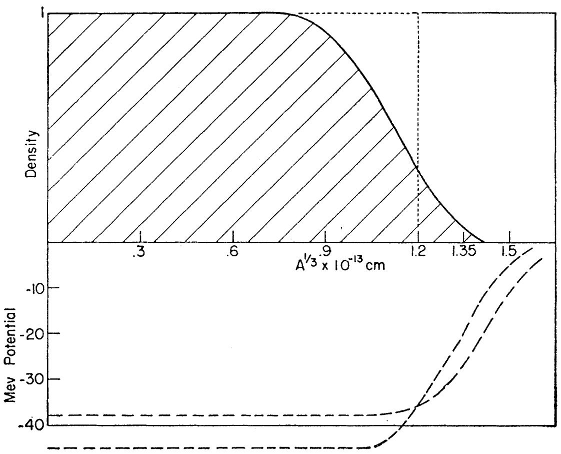

FIG. 1. The shaded region represents the charge distribution for Au (Ravenhall and Yennie, reference 13); the dotted line shows the uniform distribution with the same root mean square radius ($=1.2A^{\frac{1}{3}}$). The dashed lines show the nuclear wells (the depth gives the magnitude of the real part) which Saxon, Melkanoff, and Nodvik have used to match proton scattering data. The analytic form of these wells is $V_0/1+e^{(r-r_0)/a}$ with $a=0.49$, and $V_0=-38$ Mev $r_0=1.42A^{\frac{1}{3}}$ for the shallow curve (reference 5) and $V_0=-45$ Mev and $r_0=1.33A^{\frac{1}{3}}$ for the deep one (reference 8). The illustrated potentials and charge distributions are for Au.

observed cross sections thus give a relatively sensitive determination of the radius at which the attractive nuclear forces overcompensate the repulsive Coulomb barrier.[6] If the proton had to tunnel its way through a Coulomb barrier produced by the nuclear charge distribution with radius R_C, its cross section for formation of a compound nucleus would be much smaller than if the barrier stopped at the larger distance,[7] R_N. Saxon, Melkanoff, and Nodvik[5,8] have shown that it is possible to fit the data with a well of sloping sides and with the Coulomb barrier overcompensated by the nuclear forces at a distance slightly larger than R_N. The potential which they use is shown in Fig. 1. We assert then a basic premise for future discussions: an incident nuclear particle feels a nuclear force attraction which overcompensates the Coulomb barrier for protons at a distance from the center of the nucleus given by R_N of Eq. (1). For our purposes, it will suffice to approximate Eq. (1) to $R_N=1.4A^{\frac{1}{3}}$, and to require that the magnitude of the attractive nuclear forces at a distance R_N be at least 14 Mev.[8,9]

The above value of the nuclear force radius is to be contrasted with the radius of the proton, or electric charge, distribution, as determined by high-energy electron scattering at Stanford[10] and Michigan[11] and by the energy levels of μ-mesic atoms at Columbia.[12] The charge profile and size are both known from the recent work of Hofstadter, McIntyre, and collaborators[10] as analyzed by Ravenhall and Yennie.[13] The charge distribution is calculated to decrease to half-value at a radius of close to $R_C=1.12A^{\frac{1}{3}}$, with a surface thickness of 2.38 representing the distance between the 10 percent and 90 percent values of the density, for a mass number of $A=197$. [See Fig. 1.] It is not yet determined experimentally how the surface thickness scales with atomic number but this uncertainty will have no effect on the discussion here for heavy nuclei ($A>100$).

In this work, our aim is to infer some properties of nuclear forces on the basis of this difference between the charge and nuclear force radii. Although a complete nuclear force theory does not exist, it may still prove fruitful to study this difference in radii as a reflection of the effects of separated portions of the nuclear force.

It is generally agreed that the difference in radii is in part a measure of the finite range of nuclear forces. We observe that for medium-heavy nuclei, R_N-R_C is approximately equal to one meson Compton wavelength, or the characteristic nuclear force length. In this work, we adopt the interpretation that the difference between R_N and R_C is *entirely* a consequence of the finite nuclear force range, and we use the magnitude of this difference as a lever to learn some properties of the nuclear forces themselves.[14] That is, we *assume* that

[6] The presence of the Coulomb barrier is thus of value here in that it permits a "normalization" of the energy scale.

[7] For $Z=50$ and $E_p=5$ Mev, this reduction factor is of the order of three.

[8] Personal communication with Dr. D. S. Saxon and Dr. S. Moszkowski. More extensive calculations on elastic proton scattering from various nuclei, including Al, Cu, Ag, and Au, confirm the initial result of Woods and Saxon in reference 5 that the nuclear potential extends appreciably beyond the radius R_C. Some of these calculations are reported in the Brookhaven report (reference 3) in the session on "Elastic Scattering" by D. S. Saxon. In all cases, the best fit with experiment is achieved with a nuclear potential at least 14 Mev deep at $R_N=1.4A^{\frac{1}{3}}$

[9] The analysis of J. S. Blair, Phys. Rev. **95**, 1218 (1954) on elastic α-particle scattering provides further evidence in support of a large nuclear force radius. He fits the experimental angular distributions of α particles elastically scattered by heavy nuclei [G. W. Farwell and H. E. Wegner, Phys. Rev. **95**, 1212 (1954)] with a nuclear model which assumes that the nucleus is opaque out to a radius of $1.5A^{\frac{1}{3}}$. Further work along these lines by Wall, Rees, and Ford with similar results is reported by D. S. Saxon in reference 3, pp. 51–52. See also Wall, Rees, and Ford, Phys. Rev. **97**, 726 (1955).

[10] Hofstadter, Fechter, and McIntyre, Phys. Rev. **92**, 978 (1953); Hofstadter, Hahn, Knudsen, and McIntyre, Phys. Rev. **95**, 512 (1954); Yennie, Wilson, and Ravenhall, Phys. Rev. **92**, 1325 (1953); Yennie, Ravenhall, and Wilson, Phys. Rev. **95**, 500 (1954).

[11] Pidd, Hammer, and Raka, Phys. Rev. **92**, 436 (1953).

[12] V. L. Fitch and J. Rainwater, Phys. Rev. **92**, 789 (1953); L. N. Cooper and E. M. Henley, Phys. Rev. **92**, 801 (1953).

[13] D. G. Ravenhall and D. R. Yennie, Phys. Rev. **96**, 239 (1954).

[14] To fix this point more clearly we consider a slowly moving proton which is incident on a heavy nucleus. We ask what factors help a proton which comes up against the high Coulomb barrier corresponding to radius R_C tunnel its way through as if the barrier were lower and cut off at the larger radius R_N. A first suggestion might be to take into account the polarizability of the proton in the electric field of the nucleus. This corresponds schematically to the proton tossing out its charge on a π^+ meson, coasting freely as a neutron, while the lighter π^+ meson bucks the barrier with greater ease, and then catching the π^+ back again. However, the effect of this induced electric dipole moment was calculated to be negligible with meson parameters adjusted to give the correct anomalous magnetic moments. (The entire magnetic moment contribution is itself negligible.) Along this line of thought we may consider that the proton throws out its π mesons to be caught by a nucleon inside the nucleus. This "forward pass" effect (as contrasted with the above "fumble") then contributes the nuclear force attraction to help balance the Coulomb barrier.

the *charge* distribution determined at Stanford coincides with the entire nuclear *matter* distribution. We then determine if various meson-theoretically inspired and phenomenological nuclear force theories which have been proposed are suitable to account for the radius difference, R_N-R_C.

The nuclear forces of primary interest in this work are those which satisfy the saturation requirements in heavy nuclei. It is a simple matter to write down attractive two-body forces extending over a range of the order of 1×10^{-13} cm which will account for R_N-R_C. However, we shall see that the dual demands that the nuclear forces account for saturation and at the same time give rise to an initial attraction of magnitude $\geqslant 14$ Mev for an incident nucleon at a distance R_N, provide severe requirements for the forces to satisfy. The reason for this is that the saturation phenomenon is an expression of the weakness of the average attraction experienced by a nucleon in nuclear matter. Only if the nuclear forces contain long-range direct, central attractions which will be felt by an incident nucleon at a distance R_N before the repulsions, which are responsible for saturation, become effective, can they completely account for the difference, R_N-R_C. Thus, the agents which serve to establish the saturation of nuclear forces, whether they be hard cores, many-body repulsions, and/or exchange potentials, must be characterized by a short range relative to the attractive interactions. This is the key point which emerges from our considerations.

Alternatively, the difference between R_N and R_C can be trivially explained on the basis of an entirely different approach to this problem which assumes that the neutron distribution extends beyond the proton one. Such proposals have been made by Johnson and Teller[15] and by Swiatecki[16] and are based on the excess in neutron number over proton number, and on Coulomb effects. In the absence of definite experimental information on the relative sizes of the neutron and proton distributions[17] in nuclei, these proposals provide equally valid approaches to an explanation of R_N-R_C. Further experimental as well as theoretical work is required in order to establish to what extent the radius difference can be explained simply on the basis of a larger matter than charge radius. As indirect evidence in support of the assumption of equal neutron and proton radii, we note the observations on the lighter nuclei[12,18] such as copper and titanium, which have roughly equal numbers of neutrons and protons, but for which the radius difference $(R_N-R_C)/A^{\frac{1}{3}}$ is essentially the same as for lead with a neutron to proton ratio of 1.5. We also note Williams[19] optical-model analysis of the 1.4-Bev neutron cross sections which indicates small matter radii in close agreement with R_C.

We assert then as a second basic premise for these discussions: the nuclear *matter* distribution coincides with the charge distribution as determined at Stanford. The best fit to Au is given in Fig. 1.

On the basis of the two preceding basic premises, we seek to establish features of the nuclear forces which are necessary to account for the observed difference in radii. At first sight, it may seem that a study of the difference between R_N and R_C in heavy nuclei is a rather indirect approach to the nature of nuclear forces. A study of the deuteron and of nucleon-nucleon scattering is surely more direct. However, the two-body problem in nuclear physics has proved to be very complicated and is nowhere near solution. It is evident, especially from the work of Lévy,[20] Brueckner and Watson,[21] and Blatt and Kalos,[22] that a fit to two-body data at various low and intermediate energies is quite sensitively affected by the introduction of infinite repulsive cores with different singlet and triplet ranges and by the depth and slope of the potential at the edge of the core. This means that the calculations of the two-body interactions are sensitive to regions of small particle separations ($\sim$0.4) where the static potential picture is certainly deficient. It also means that it is difficult to make an unambiguous choice of the exchange properties of the potential as well as of its profile for larger separations ($\sim\hbar/\mu c=1.4$). This point is clearly illustrated by the fact that the Lévy potential and the Brueckner-Watson potential both give a satisfactory account of themselves on low energy data, but are very different from one another insofar as concerns their exchange nature, tensor contributions, and space variation.[23]

On the other hand, a study of the nuclear radius difference can be used to cast light on one distinct feature of the nuclear force. Its main advantage is that, to lowest order, for two-body interactions it measures predominantly the effects of the long range behavior of the direct part of the central potential for heavy

[15] M. H. Johnson and E. Teller, Phys. Rev. **93**, 357 (1954).
[16] W. J. Swiatecki, Phys. Rev. **98**, 203–204 (1955).
[17] See however W. N. Hess and B. J. Moyer, Phys. Rev. **96**, 859(A) (1954). W. N. Hess, University of California Radiation Laboratory Report UCRL-2670, 1954 (unpublished); L. N. Cooper and W. Tobocman, Phys. Rev. **97**, 243 (1955). The experiments of Hess and Moyer on the indirect pickup process provide qualitative indications of a 0.8 neutron surface layer in Pb which is not present in lighter elements with $N=Z=A/2$. However, the stripping analysis of Cooper and Tobocman suggests a neutron surface layer of 0.6 for mirror nuclei Mg^{25} and Al^{25}.
[18] Lyman, Hanson, and Scott, Phys. Rev. **84**, 626 (1951); A. E. Glassgold, Ph.D. thesis, Massachusetts Institute of Technology Physics Department, 1954 (unpublished).
[19] R. W. Williams, Phys. Rev. **98**, 1387, 1393 (1955) and personal communication, Coor, Hill, Hornyak, Smith, and Snow, Phys. Rev. **98**, 1369 (1955). Similar results are obtained from the scattering of 860-Mev protons. Chen, Leavitt, and Shapiro, Phys. Rev. **99**, 857 (1955).
[20] M. M. Lévy, Phys. Rev. **88**, 725 (1952).
[21] K. A. Brueckner and K. M. Watson, Phys. Rev. **92**, 1023 (1953).
[22] J. M. Blatt and M. A. Kalos, Phys. Rev. **92**, 1563 (1953).
[23] We note, for example, that the Lévy potential contains roughly four times as much direct central attraction as does the Brueckner-Watson potential and two-thirds as much long range tensor force. Henceforth, we shall mean by Brueckner-Watson potential the potential which they derive using only the gradient coupling version of pseudoscalar meson theory (no pair term).

nuclei, $A>100$; i.e., of $V_d(r)$ in an expression for the potential of the form

$$V(r)=V_d(r)+V_\tau(r)\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}+V_\sigma(r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}+V_{\sigma\tau}(r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}+V_t(r)S_{12}+V_{t\tau}(r)\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}S_{12}, \quad (2)$$

where S_{12} is the tensor force operator.

It measures only the effects of $V_d(r)$ because, to leading order in $A\gg 1$, there are equal numbers of spin-up and spin-down particles in the nucleus, so that the last four terms of Eq. (2) average approximately to zero. They vanish identically in the sum over nuclear particles for spin zero nuclei. In general, their contributions are reduced by a factor of $A^{\frac{2}{3}}$. This is because there are $\sim A^{\frac{2}{3}}$ particles near the facing surface of the nucleus which interact strongly with the incident nucleon, in virtue of the short range of the nuclear forces, and all but one of a few of these $A^{\frac{2}{3}}$ particles will be paired in spin-up and spin-down states. Also, the second term of Eq. (2) is proportional to the neutron excess and is responsible for only a small contribution to the calculations following in comparison with the $V_d(r)$ term. This approach thus has the advantage of being primarily sensitive to one portion of the nuclear potential: the long-range behavior of $V_d(r)$, which determines at what distance from the nucleus an impinging nucleon first experiences an appreciable nuclear interaction. By the same token it cannot tell whether $V(r)$ falls precipitously up to the edge of a repulsive core or simply increases gradually as $r\rightarrow 0$.

II. OUTLINE AND SUMMARY

In this section, we present a brief outline of the calculations performed and results obtained in the succeeding paragraphs.

Firstly, in Sec. III, we sum up the nuclear interactions of a nucleon outside the nucleus with the nucleons comprising the nucleus. In this calculation we neglect the influence of the external nucleon on the nuclear matter density, which we assume to correspond to the charge distribution measured at Stanford.[10] We also neglect the requirement of antisymmetrizing the wave function of the incident nucleon with those in the nucleus. To first order, it is only the direct ($\boldsymbol{\sigma}$- and $\boldsymbol{\tau}$-independent) parts of the nuclear potential which are of importance. We consider in this work several different models of nuclear forces, all of which have in common the property of satisfying the saturation requirements in nuclei. Two of these models are rooted in meson field theory; one is the Lévy[20] potential of pseudoscalar meson theory, supplemented by sufficient three-body repulsions to meet the saturation requirements; the second is the Brueckner-Watson[21] potential, deduced on the basis of the gradient-coupling version of pseudoscalar meson theory, which is also consistent with the saturation requirements. Two more phenomenological models are also discussed. The Lévy and Brueckner-Watson models have in common the feature that the main contributions to the central forces result from two-meson exchanges so that the interaction is of characteristic range of the order of one-half of the meson Compton wavelength. We find that the direct part of the Brueckner-Watson potential is too small to account for the difference between R_N and R_C. The Lévy potential, on the other hand, has a very strong Wigner attraction; however, when enough three-body repulsion of the form predicted by the same meson theory as used in deducing the two-body attraction is introduced to be consistent with nuclear stability,[24] we find again that the difference between R_N and R_C cannot be accounted for. In contrast, a simple phenomenological Yukawa or Hulthén potential with a range of one meson Compton wavelength and a depth appropriate to the binding energy of the deuteron has the desired property. Calculations are presented for two different potentials of this type. In the course of these calculations we verify that our conclusions are insensitive to details of the assumed shape of the tail of the nuclear charge distribution.

In Sec. IV, we calculate the induced spin, charge, and matter polarization of the nuclear density due to the interactions with the approaching nucleon. These effects are found to be small for the incident nucleon at a distance $\gtrsim 1.35A^{\frac{1}{3}}$ from the center of the nucleus and indicate the convergence of our calculations.

Section V is devoted to a consideration of the effect of taking into account the Pauli principle by antisymmetrizing the wave function of the incident nucleon with the nucleons in the nucleus. This effect serves to weaken the nuclear attraction due to direct terms, since the Pauli principle inhibits the close approach of two fermions, and to introduce exchange energies. Its contributions are shown to be small. These exchange contributions do not influence our conclusions which are presented in Sec. VI and may be summarized briefly as follows.

The observed difference between the nuclear force radius, R_N, and the charge radius, R_C, may be explained in either of two ways. On the one hand we may assume that the nuclear charge and matter distributions coincide, in which case the potential between nucleons must contain an appreciable amount of long range, direct, central attractive interaction. If we require that this same internucleon potential account for nuclear saturation, the repulsive cores, many-body repulsions, and/or exchange interactions which stabilize the nucleus against collapse must be of shorter range than the attractive interactions. Recent meson-theoretically deduced nuclear potentials are not consistent with these requirements. On the other hand, we may assume that the radius of the neutron distribution extends beyond the radius, R_C, of the proton distribution. Our calculations show that the matter distribution must extend

[24] S. D. Drell and K. Huang, Phys. Rev. 91, 1527 (1953).

at least 10% beyond the charge radius if we accept the indications of meson field theory as manifest in the potentials of Lévy[20] or of Brueckner and Watson.[21]

We emphasize at this point that the nuclear radius, R_N, as used in this discussion, represents the distance at which an incident nucleon feels an appreciable nuclear attraction, 14 Mev. Our calculations are valid only for the incident nucleon "outside" of the nucleus, and tell nothing about the strength or shape of the nuclear interaction for a particle "inside" of the nucleus.

III. FIRST-ORDER INTERACTION ENERGY

In this section, we compute directly the interaction energy between a nucleon outside of the nucleus and the nucleons in the nucleus as a function of distance, making the following simplifying approximations:

(1) We neglect the influence of the external nucleon on those in the nucleus, which we take to be distributed according to the Stanford charge distribution.

(2) We treat the external nucleon as distinguishable; i.e., we ignore the requirement of the Pauli principle that its wave function be antisymmetrized with those in the nucleus.

(3) We assume that the interaction between a nucleon outside with one or a pair of nucleons inside nuclear matter is unchanged from what it would be if these (two or three) interacting nucleons were isolated from the rest of the nuclear matter.

Approximations (1) and (2) are analyzed and discussed in the following sections. Their effects are calculated in order to determine the regions of validity which they limit for our discussions here. Approximation (3) is an assumption as to the linearity of the nuclear force theory. We remark that, because of the short range character of nuclear forces, the important contributions to the interaction energy which we calculate below come from nucleons near the surface of the nucleus facing the incident particle. It is the nucleons in the region between $1.3A^{\frac{1}{3}}$ and $1.0A^{\frac{1}{3}}$ of the density distribution shown in Fig. 1, where the nuclear density is gradually increasing, which contribute most of the energy. Nonlinearities of the type discussed by Schiff,[25] which operate in regions of high density, may thus be relatively unimportant in these calculations. However, if present, such nonlinearities would have an important influence on the saturation conditions. Our point of view is to assume the linearity of the theory in calculating the interaction energy and in considering the saturation requirements on the two- and three-body potentials studied in this work.

We consider in this calculation only heavy nuclei ($A\sim 176$) so that we can replace the sum over nuclear particles by an integration over the nuclear density. As discussed in the last paragraph of Sec. I, it is consistent with our approximations to neglect the last four terms of Eq. (2). They average approximately to zero in the sum over nuclear coordinates. Upon summing over the charges of the nuclear particles, we obtain for Eq. (2)

$$V(r)=V_d(r)+[(2Z-A)/A]\langle\tau_{z1}\rangle V_\tau(r), \qquad (3)$$

with $\langle\tau_{z1}\rangle=\pm 1$ for incident proton or neutron, respectively.[26] The second term is seen to be proportional to the neutron excess. Since $(1-2Z/A)<1/5$ for most nuclei, its effect is reduced relative to the first term. A big contribution from the $V_\tau(r)$ term would reflect itself in a difference between neutron and proton radii of heavy nuclei. This difference suggests itself as a possible means of investigating the importance of Heisenberg (or charge exchange) forces in nuclei. For our purposes here we shall confine our attentions primarily to the first term, $V_d(r)$. We shall, however, briefly consider the second term in a discussion of the potential model of Brueckner and Watson which has a fair amount of Heisenberg force.

If we consider first just two-body interactions, the basic integral to be calculated is

$$E_n=\sum_{i=1}^{A} V_d(|\mathbf{R}_0-\mathbf{r}_i|)=\frac{A}{v}\int \rho(r)d\mathbf{r}V_d(|\mathbf{R}_0-\mathbf{r}|), \qquad (4)$$

where $\rho(r)$ is the nuclear matter density, $v=\int\rho(r)d\mathbf{r}$ is the nuclear volume, and R_0 is the coordinate of the external nucleon.

The integral in Eq. (4) can be reduced to a canonical form for these calculations on the basis of the assumed spherical symmetry of the nuclear distribution. A simple change in variable in the polar angle integration gives directly

$$\int d\Omega V_d(|\mathbf{R}_0-\mathbf{r}|)=(2\pi/rR_0\mu^2)\int_{|R_0-r|\mu}^{(R_0+r)\mu} dyyV_d(y), \qquad (5)$$

where $1/\mu=1.4$ is the meson Compton wavelength in units $\hbar=c=1$. Introducing Eq. (5) into (4) we have the interaction energy as a function of R_0 for a given potential, nuclear density, and A. We introduce the following notation to put the expression in dimensionless form:

$$\eta=\mu R_0/A^{\frac{1}{3}};\quad x=\mu r/A^{\frac{1}{3}}$$

and use the experimental result for the charge density that it occupies a volume equivalent to a uniform distribution of radius $1.16A^{\frac{1}{3}}=0.83A^{\frac{1}{3}}/\mu$ cm. This gives finally

$$E_n=2.6(A^{\frac{1}{3}}/\eta)\int_0^\infty \rho(x)xdx\int_{A^{\frac{1}{3}}|\eta-x|}^{\infty} yV_d(y)dy, \qquad (6)$$

where we have replaced to a very good approximation

[25] Such nonlinearities are contained in theories such as those given by Schiff [L. I. Schiff, Phys. Rev. **83**, 1 (1951)].

[26] In Eq. (3), the charge dependent term is altered if we give up the basic premise No. II (coincidence of neutron and proton distributions). This situation is discussed later.

the upper limit of the integral in Eq. (5) by ∞, as permitted by the short range of the forces.

The evaluation of Eq. (6) is effected by a straightforward procedure of elementary and/or graphical integrations.

We consider as a first model of the potential, a saturating combination of short range direct and exchange forces with no repulsive cores. A simple prototype of this class of interactions replaces $V_i(r)$ in Eq. (2) (for all subscripts i) by a step function

$$V_i(r)=\lambda_i\epsilon_i(r);\quad \epsilon_i(r)=\begin{cases}1 & r<\rho_i\\ 0 & r>\rho_i\end{cases}. \tag{7}$$

The saturation conditions for such a potential include the requirement that $\lambda_d\geqslant 0$. This is because it is only the V_d term of Eq. (2) that operates for a nucleus in the collapsed state, and its coefficient must be positive to prevent the nuclear collapse that would accompany a negative A^2 contribution to the energy.[27] However, for $\lambda_d>0$, the energy of interaction calculated in Eq. (6) is positive, which means a net repulsion of the external nucleon by the nuclear force effects when it first gets within the force range. If we assume, for example, a range $\rho_d\sim 1$, a nucleon incident on a heavy nucleus would first feel the nuclear force effects when it approached within $(1.2A^{\frac{1}{3}}+1)\simeq R_N$, and at this distance the nuclear interaction would be repulsive rather than attractive. We learn from this example that if we have simple potentials wells and achieve saturation by means of suitable exchange mixtures, we cannot explain the difference between R_N and R_C, once we have assumed that the charge and matter radii coincide.

It is apparent, however, that if saturation is achieved through repulsive cores in addition to exchange mixtures, a net attraction can be obtained in Eq. (6). As an example of this, we consider the Brueckner-Watson[21] potential which is deduced from gradient-coupling pseudoscalar meson theory carried through fourth order in the coupling parameter and thereby allowing for one and two meson exchanges. In the deduction of this force the theory is supplemented in two ways: repulsive cores are introduced at short distances and multiple scattering arguments are appealed to in order to allow a selection of a particular set of the fourth order interactions. The fourth order interactions which are retained in the analysis of Brueckner and Watson are those for which two virtual mesons are simultaneously present in the intermediate states in a perturbation expansion. All fourth order contributions which arise as nonadiabatic, velocity-dependent corrections to the second order, one-meson exchange interaction are dropped.

We accept this potential as indicative in general features of what results from the gradient-coupling

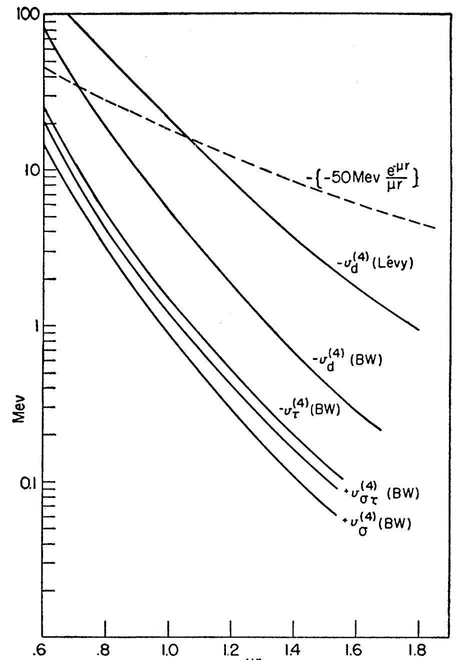

FIG. 2. Logarithmic plot of the magnitudes of the various contributions to the Lévy and Brueckner-Watson potentials as labeled *vs* distance. $-V_d{}^4$ (Lévy) represents Eq. (9) for the Lévy potential and $-V_d{}^4$ (Brueckner-Watson) represents Eq. (8) for the Brueckner-Watson potential. The two-meson exchange contributions to the Brueckner-Watson potential which are proportional to $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}$, $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}$ and $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}$ [see Eq. (2)] are represented by $+V_\sigma{}^{(4)}$ (Brueckner-Watson), $-V_\tau{}^{(4)}$ (Brueckner-Watson), and $V_{\sigma\tau}{}^{(4)}$ (Brueckner-Watson). The one-meson exchange terms, Eq. (27), and the additional two-meson exchange contributions to the tensor force are not indicated. The dashed line represents a Yukawa potential of sufficient depth to bind the deuteron.

meson theory; i.e., from the pseudoscalar theory when the pair term is completely suppressed. It has the virtues of giving a reasonably satisfactory fit to low and intermediate energy two-body data and of predicting reasonable saturation behavior for heavy nuclei.[28] The main feature of concern to us here is the short range and moderate strength of the direct term, V_d, as illustrated in Fig. 2. In units $z=\mu r$ it reads[29]

$$\begin{aligned}V_d=&-\gamma z^{-4}[(4+4z+z^2)e^{-z}K_1(z)\\&\qquad+z(2+2z+z^2)e^{-z}K_0(z)], &&\text{for}\quad z>0.32;\\=&+\infty, &&\text{for}\quad z\leqslant 0.32;\end{aligned}\tag{8}$$

with

$$\gamma=6(\mu/\pi)(g^2/4\pi)^2(\mu/2M)^4=1.73\text{ Mev}.$$

The short range of the direct part is due to the fact that it arises from two-meson exchange and is thereby

[27] For a discussion of the various saturation conditions see reference 1, Chapter III.

[28] Brueckner, Levinson, and Mahmoud, Phys. Rev. **95**, 217 (1954); K. A. Brueckner, Phys. Rev. **96**, 908 (1954) and **97**, 1351 (1955).

[29] A repulsive core radius of 0.32 $(1/\mu)$ cm represents the appropriately weighted average of the singlet and triplet core radii in reference 21 (units corrected from reference 21 as indicated in reference 28).

approximately characterized by the range $1/2\mu$. The longer range contribution resulting from one meson exchange in the gradient coupling gives rise only to the long-studied tensor and $V_{\sigma\tau}$ terms and does not contribute in this first-order analysis.

Introducing Eq. (8) into Eq. (6), we have calculated the interaction energy as a function of the particle distance η. We have carried through these calculations both for the Stanford charge distribution and for the uniform distribution with the same root mean square radius ($\equiv 1.2A^{\frac{1}{3}}$) as drawn in Fig. 1 in order to test the sensitivity of our results to the tail of the charge distribution. The results are presented graphically in Fig. 3 and tabulated in Table I for an arbitrary nucleus of mass number $A=176$. The important point here is that, at a distance of $1.35A^{\frac{1}{3}}$, the nuclear attraction is 4.3 Mev, or less than one third of what is required to make the nuclear force radius appear to be this big. We conclude that the nuclear forces have contributed insufficiently in this case to make the nuclear force radius appear to be $R_N=1.4A^{\frac{1}{3}}$. It is seen that this conclusion applies to both calculations, with the Stanford and with the uniform charge distributions.

We observe that the energies obtained with the Stanford and with the uniform distributions correspond to each other much more closely for $R_0=1.35A^{\frac{1}{3}}$ than for $R_0=1.45A^{\frac{1}{3}}$. This may be understood in terms of the repulsive core and short range singular character of the internucleon potential. For large separations, the short range nature of the interactions weights heavily the energy contribution due to the portion of the matter distribution which extends beyond $1.2A^{\frac{1}{3}}$. However, for smaller separations the repulsive core is effective in preventing the matter distributions and incident nucleon from approaching to within $0.32/\mu$ cm of each other. The repulsive core in Eq. (8) has no influence in calculation for $R_0=1.45A^{\frac{1}{3}}$, since there is less than one percent of the matter distribution (Fig. 1) within a distance $0.32/\mu$ cm of an incident nucleon. It does, however, enter into the evaluation of the interaction energy for the incident nucleon at $1.40A^{\frac{1}{3}}$ and at $1.35A^{\frac{1}{3}}$, since the tail of the Stanford charge distribution extends out to these distances. For these cases, we have performed the calculations so as to obtain the maximum possible attraction by sharply cutting off the matter distribution at a separation of $0.32/\mu$ cm between the incident nucleon and the nucleus, and increasing its value uniformly in the region within 0.5

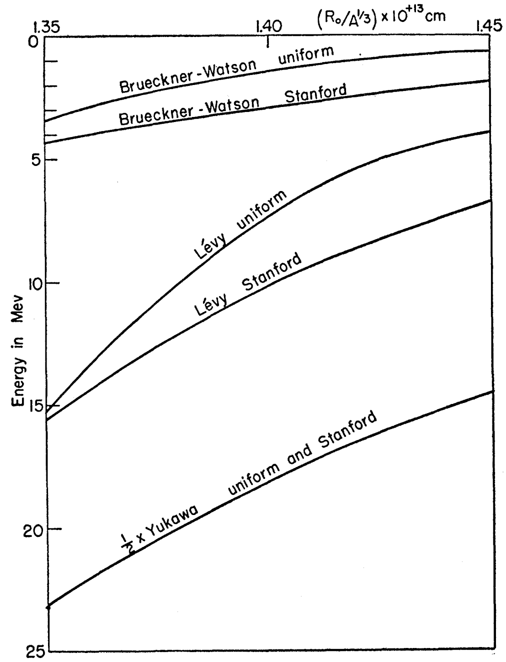

FIG. 3. Graph of results listed in Table I. The energies calculated with the Yukawa potential are reduced by a factor two in this graph.

of the cutoff to preserve its normalization to A particles. By this procedure we avoid all repulsive contributions to the interaction arising due to the hard cores and at the same time take full advantage of the deepest portion of the internucleon attraction at the edge of the hard core. Various modifications of this procedure altered the calculated energies only by several percent. Also a decrease in the core radius to its minimum value (which applies for triplet states) of $0.30/\mu$ cm increased the attraction by less than 5%. On the basis of this discussion we feel that our conclusion above is safely insensitive to the shape of the tail of the charge distribution. We do not carry this analysis to smaller distances than $R_0=1.35A^{\frac{1}{3}}$ since the approximations represented in Eq. (6) do not warrant it. (This point is discussed more fully in the following sections.) In the final section, we discuss the A dependence of this result.

From Fig. 2, it is clear that the V_τ interaction term in the Brueckner-Watson potential contributes in Eq. (3) an energy which is less than 5 percent of that due

TABLE I. First-order interaction energies, Eq. (6), computed for the Brueckner-Watson potential, Eq. (8), the Lévy potential, Eq. (9), and the Yukawa potential, Eq. (16). The calculations were carried out for atomic number $A=176$ and for both the Stanford and equivalent uniform charge distributions shown in Fig. 1. The results of this table are graphed in Fig. 3.

	Brueckner-Watson potential [Eq. (8)]		Lévy pot. [Eq. (9)]		Yukawa pot. [Eq. (16)]	
R_0	Stanford	Uniform	Stanford	Uniform	Stanford	Uniform
$1.35A^{\frac{1}{3}}$	−4.3 Mev	−3.5 Mev	−15.6 Mev	−15.2 Mev	−45 Mev	−46 Mev
$1.40A^{\frac{1}{3}}$	−2.9 Mev	−1.4 Mev	−10.0 Mev	− 7.2 Mev	−35 Mev	−36 Mev
$1.45A^{\frac{1}{3}}$	−1.8 Mev	−0.64 Mev	− 6.9 Mev	− 3.9 Mev	−28 Mev	−29 Mev

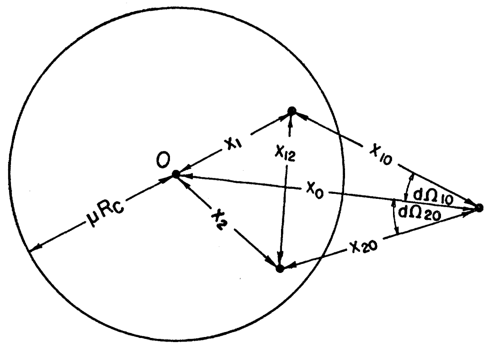

FIG. 4. Coordinates for carrying out the integrations, Eq. (12) for the three-body interaction energy.

to the direct term, V_d. We can thus neglect it insofar as concerns the first-order energies.

Another more dated version of the nuclear forces as deduced from pseudoscalar meson theory is that due to Lévy.[30] In this version, the "pair term" of the pseudoscalar meson theory, which is omitted in the gradient coupling version used by Brueckner and Watson, is responsible for a very strong Wigner attraction which reads to leading order

$$\begin{aligned} V_d &= -\gamma z^{-2} K_1(2z), \quad z>0.38; \\ &+\infty, \qquad\qquad\quad z\leqslant 0.38; \end{aligned} \tag{9}$$

with

$$\gamma = 6(\mu/\pi)(g^2/4\pi)^2(\mu/2M)^2 = 151 \text{ Mev}.$$

As in the gradient coupling version, the main central force contribution results from two-meson exchanges and therefore is of range $\simeq 1/2\mu$. In both versions of the pseudoscalar theory, i.e., with or without the pair term, the longer-ranged single meson exchange terms give rise only to a tensor and $V_{\sigma\tau}$ part in Eq. (2).

In contrast with the Brueckner-Watson interaction, this nuclear force is sufficiently strong to give rise to an appreciable nuclear attraction at distances $R_N \sim 1.4A^{\frac{1}{3}}$. This is seen in Fig. 3 and Table I. The Lévy central potential is seen to be ~ 4 times stronger than the Brueckner-Watson at a distance ~ 1 and to give rise to ~ 4 times as strong a nuclear attraction. However, the Lévy potential is not a saturating interaction. One way of constructing a saturating nuclear force theory and still keeping Eq. (9) is to add three-body repulsive forces to provide the nuclear stability. The effect of such three-body repulsions will also be felt by the nucleon approaching the nuclear surface.

We will now outline the calculation of such three-body contributions to the nuclear force experienced by the incident nucleon. In this calculation, we will take the form of the three-body repulsion from the pseudoscalar meson theory with pair term, as previously studied:[24]

$$V_3 = \beta \frac{K_1(\{r_{12}+r_{23}+r_{31}\}\mu)}{\mu^3 r_{12} r_{23} r_{31}}; \quad r_{ij} \equiv |\mathbf{r}_i - \mathbf{r}_j|; \tag{10}$$

i.e., we take the three-body force from the same theory that gives the two-body attraction, Eq. (9). We shall find that the minimum amount of three-body repulsion consistent with saturation essentially cancels the above calculated attraction due to the two-body forces.

Because of the complexity of this calculation we only outline the procedure here in the approximation of a uniform nuclear matter density. The essential difficulty here which is not present in the nuclear saturation problem is the following: in the saturation problem the usual procedure is to calculate the interaction energies between particles in an infinite nuclear medium of fixed density, whereas we must now take careful account of the nuclear surface since it is the distance from the surface that is the critical parameter.

We write first Eq. (10) in dimensionless form, denoting as in Fig. 4 by $x_0 = \mu r_0$ the coordinate of the incident nucleon relative to the center of the nucleus:[31]

$$\begin{aligned} V_3 &= \beta \frac{K_1(x_{10}+x_{20}+x_{12})}{x_{10}x_{20}x_{12}} \\ &= \beta \int_1^\infty \frac{t\,dt}{(t^2-1)^{\frac{1}{2}}} \frac{e^{-x_{10}t}}{x_{10}} \frac{e^{-x_{20}t}}{x_{20}} \frac{e^{-x_{12}t}}{x_{12}}. \end{aligned} \tag{11}$$

The interaction energy is then given by

$$\begin{aligned} E_{n3} = \left[\tfrac{1}{2}A^2/(v\mu^3)^2\right] \int\int d\mathbf{x}_1 d\mathbf{x}_2 \beta \\ \times \int_1^\infty \frac{t\,dt}{(t^2-1)^{\frac{1}{2}}} \frac{e^{-x_{10}t}}{x_{10}} \frac{e^{-x_{20}t}}{x_{20}} \frac{e^{-x_{12}t}}{x_{12}}. \end{aligned} \tag{12}$$

The procedure we found most direct for the evaluation of Eq. (12) develops as follows:

(1) Transform the volume integrals to integrals over the inter-particle separations and use the x_0 axis as the polar axis for the angular integrations; *viz.*,

$$\begin{aligned} &\int\int d\mathbf{x}_1 d\mathbf{x}_2 \\ &= \int_{x_0-x_1}^\infty x_{10}^2 dx_{10} \int_{x_0-x_2}^\infty x_{20}^2 dx_{20} \int d\Omega_{10} \int d\Omega_{20}, \end{aligned} \tag{13}$$

with the limits on the polar angle integral expressed in terms of the nuclear radius R_C by

$$\begin{aligned} (x_0-x_{10})^2 \leqslant x_{10}^2 + x_0^2 - 2x_{10}x_0\cos\theta_{10} \leqslant (\mu R_C)^2, \\ (x_0-x_{20})^2 \leqslant x_{20}^2 + x_0^2 - 2x_{20}x_0\cos\theta_{20} \leqslant (\mu R_C)^2. \end{aligned} \tag{14}$$

[30] A. Klein, Phys. Rev. 94, 1061 (1954). He has presented a strong argument for the suppression of the pair term on the basis of the absence of strong S-wave pion nucleon scattering and the relation between this and the nuclear force problem. See also Phys. Rev. 89, 1158 (1953) and reference 24.

[31] W. Magnus and F. Obberhettinger, *Special Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1949).

(2) Expand the exponential factor involving the separation x_{12} in a series of spherical harmonics with x_0 as the polar axis; *viz.*,[31]

$$\frac{e^{-x_{12}t}}{x_{12}}=4\pi\sum_{l,m}(x_{10}x_{20})^{-\frac{1}{2}}K_{l+\frac{1}{2}}(x_>t)I_{l+\frac{1}{2}}(x_<t)\times Y_l^m(\theta_{10})Y_l^{m*}(\theta_{20}),\quad(15)$$

with $x_>$ and $x_<$ the larger and smaller, respectively, of x_{10} and x_{20}.

(3) Observe that by cylindrical symmetry only the $m=0$ terms contribute, and for these

$$\int d\Omega_{10}Y_l^0=[(2l+1)\pi]^{\frac{1}{2}}\int_{\cos\Theta_{10}}^1 d\mu P_l(\mu);$$

here $\cos\Theta_{10}$ is the solution of Eq. (14) with the equality on the right.

(4) Evaluate directly the first few l-terms.

The series in l converges rapidly in practice for the parameters of interest to us. For our hypothetical nucleus with $A^{\frac{1}{3}}=5.6$, the equivalent uniform nuclear radius is $\mu R_C=(6/7)A^{\frac{1}{3}}=4.8$. For the incident nucleon at a distance of $1.45A^{\frac{1}{3}}=R_0$, $x_0=5.8$ and the important values of x_{10} and x_{20} will be of the order of unity. As the integral representation Eq. (11) is peaked at $t\sim1$, the important arguments of the Bessel functions in Eq. (15) are also of the order of unity. The other two exponential factors in Eq. (12) weigh heavily against contributions from larger arguments. However, the Bessel function of imaginary argument in Eq. (15), $I_{l+\frac{1}{2}}(x_<t)$, is peaked at $x_<t\sim l$. This means that for large l, the peak in $l_{l+\frac{1}{2}}$ is beat down by the exponentials in the integrand in Eq. (12), and so the contribution of large l values to the energy in Eq. (12) is considerably reduced. It is this rapid convergence in l that supports the change in variable indicated in step 1.

Carrying out these calculations with the above indicated approximations and procedures we obtain for $A=176$ and $R_0=1.45A^{\frac{1}{3}}$ contributions of 1.9 Mev, 3.6 Mev, 1.2 Mev, and <0.5 Mev from the first $l=0$, 1, 2, and 3 values, so that the three-body energy contributions adds to 7 Mev, for $\beta=450$ Mev, as given by the full pair term of the pseudoscalar meson theory.[24] The corresponding two-body attraction for a uniform nucleus is seen in Table I to be only -3.9 Mev. It is possible to reduce the constant, β, of the three-body repulsion by no more than a factor of two consistent with the saturation requirement.[32]

The conclusion drawn from this calculation is that the Lévy potential supplemented by the minimal amount of three-body repulsion, of the form Eq. (10) as suggested by pseudoscalar meson theory, demanded for saturation also fails to account for the radius difference R_N-R_C. The reason for this is that the two-body attraction and the three-body repulsion have closely the same range as felt by an incident nucleon outside of the nucleus, and consequently balance each other to a large extent. This conclusion is unchanged by refining the above calculation with the actual charge density in Fig. 2 and is verified for smaller R_0 and A values.

As another example we calculate the interaction energy, Eq. (6), assuming a simple phenomenological Yukawa potential

$$V(r)=-50\text{ Mev }e^{-\mu r}/\mu r.\quad(16)$$

We consider this case in order to show specifically the importance of a long-range tail in the nuclear force as a possible source of explanation of the difference between R_N and R_C. The potential in Eq. (16) gives reasonable deuteron binding and together with various combinations of short-range many-body forces or of repulsive cores and exchange mixtures can be adjusted to meet saturation requirements. In fact, a qualitatively reasonable saturation results from the combination of Eq. (16) with the three-body repulsion of Eq. (10), with $\beta=450$ Mev. The calculations on the nuclear interaction energy, Eq. (6), using Eq. (16) are presented in Fig. 3 and Table I along with the preceding results. It is evident that with the longer range potential of Eq. (16) it is possible to account for the radius difference R_N-R_C.

Finally we consider briefly two phenomenological nuclear force models with long range potentials of the Yukawa type which have achieved some success in fitting data on low energy interactions of several nucleons. The potential used by Pease and Feshbach,[33]

$$V_F=-40\text{ Mev }e^{-1.2\mu r}/\mu r-28.6\text{ Mev }S_{12}e^{-0.81\mu r}/\mu r,\quad(17)$$

fits the observed properties of two-nucleon systems for energies up to ~15 Mev, as well as the binding energy of H^3. The first, or direct, term in Eq. (17) contributes an attractive interaction energy of -28 Mev for a nucleon at $R_0=1.35A^{\frac{1}{3}}$, with $A^{\frac{1}{3}}=5.6$. It is evident from this that an interaction of the type in Eq. (17) can account for the radius difference R_N-R_C. However, this is not a saturating force since the strong Wigner attraction would operate in heavy nuclei to collapse all of the A nucleons to within one meson Compton wavelength of one another. It is possible to counter this tendency with very short-range three-body repulsions which would not appreciably diminish the above calculated 28-Mev attraction and which at the same time would not affect the success of Eq. (17) in matching properties of the two nucleon system at low energy. However, the effect of such a modification of Eq. (17) on the calculated binding energy of H^3 remains to be studied.

[32] This result is established on the basis of the variation calculation discussed in reference 24. Brueckner, Levinson, and Mahmoud (reference 28) suggest a much smaller value of the constant, β, for saturation. However, their statement was based on the step functioned not on the trial function calculations of reference 24.

[33] R. L. Pease and H. Feshbach, Phys. Rev. **88**, 945 (1952).

Another two-body interaction which has met with noteworthy success in fitting data on two-body systems at low energies is the Rosenfeld mixture[34]

$$V_R = +40 \text{ Mev}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2[0.10+0.23\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2]e^{-\mu r}/\mu r. \quad (18)$$

This is a saturating interaction. With this interaction, Inglis[35] and others have had considerable success in fitting observed level structures, moments, and transition matrix elements in their intermediate coupling studies in the p-shells of light nuclei. However, we see that there is no direct interaction in Eq. (18), so that the first-order contribution of the Rosenfeld mixture to the interaction energy vanishes. To this order Eq. (18) thus fails to account for the observed difference, R_N-R_C. This result is extended through second order interactions in the next section.

In summing up the calculations in this section we conclude that the inter-nucleon potential must contain a sizeable amount of long-range direct interaction, $V_d(r)$, if nuclear forces alone are to be held to account for R_N-R_C. The saturating character of the forces would then be a consequence of short range repulsions, due either to hard-core or many-body effects. Neither of the two meson-theoretically inspired potentials of Brueckner-Watson and of Lévy (with three-body repulsions for saturation) can explain the observed difference in radii unless it is assumed that the neutron distribution extends beyond the proton one. It must be added that the presently studied and applied versions of the pseudoscalar meson theory give no hint as to a possible source of any simple, central, Yukawa type potential with a range of roughly one meson Compton wavelength, corresponding to single meson interchange. The above analysis on the Lévy potential showed that a strong direct attraction of half that range, arising from two-meson exchange, can yield a sufficient energy contribution if saturation is achieved by shorter-range many-body repulsions than considered there. In this connection we note that Bonneveay[36] has shown that the meson-meson interaction, which the renormalization procedures introduce into meson theory, can serve as a source of central direct interactions of range $\sim 1/2\mu$. It is not established at present to what extent such a term is actually present, or whether a sufficient amount of this interaction to account for R_N-R_C can be made consistent with nuclear saturation and with the information on two-body systems at low energy.

IV. INDUCED POLARIZATIONS

In this section, we calculate the second-order corrections to the interaction energy between the nucleus and the incident nucleon. These corrections result from the spin, charge, and matter polarizations which are induced in the nuclear distribution due to its interaction with the incident nucleon. Besides the second order effects due to the direct interaction term, V_d, which in general serve as small corrections to the first-order calculations of the previous paragraph, there are the more important contributions from the various exchange and tensor force terms of Eq. (2). These latter, as we have seen, contribute no first-order effects.

First we develop the techniques to calculate these effects. We wish to evaluate the change in the nuclear distribution resulting from interactions with the incident nucleon and then to determine the contribution of this change to the interaction energy. This is the same as evaluating the second order energy contribution

$$En_{(2)} = \sum_{(X)} \frac{|\langle X|V|g\rangle|^2}{E_g - E_X}, \quad (19)$$

where $|g\rangle$ and E_g are the ground-state eigenfunction and eigenvalue of the unperturbed nucleus and the sum extends over the excited states (X). Here again we treat the external nucleon as a distinguishable particle. The nucleus is idealized as a degenerate Fermi-Dirac gas. The excited states which contribute to the sum in Eq. (19) all correspond to a nucleon jumping out of a state, below the Fermi energy $E_F = k_F^2/2M$, which has momentum and energy k_l and $E_l = k_l^2/2M$, and up into a state above the Fermi level with k_u and $E_u = k_u^2/2M$, with appropriate spins and isotopic spins, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$. Thus, introducing Eq. (2) into (19) and carrying out the elementary spin sums we obtain:

$$En_{(2)} = -2M\cdot 4\cdot \sum_{k_l<k_F}\sum_{k_u>k_F} \cdot\left[\frac{V_d^2(\mathbf{k}_l,\mathbf{k}_\mu)+3V_\tau^2(\mathbf{k}_l,\mathbf{k}_\mu) +3V_\sigma^2(\mathbf{k}_l,\mathbf{k}_\mu)+9V_{\sigma\tau}^2(\mathbf{k}_l,\mathbf{k}_\mu)}{(k_\mu^2-k_l^2)}\right]. \quad (20)$$

For the equivalent uniform nucleus of radius $1.2A^{\frac{1}{3}}$, the Fermi energy corresponds to[37]

$$k_F = 1.77\mu. \quad (21)$$

In Eq. (20), we have omitted the tensor force terms, which we shall discuss separately below, and have introduced the notation

$$V(\mathbf{k}_l,\mathbf{k}_\mu) \equiv v^{-1}\int \rho(r)\exp[i(\mathbf{k}_l-\mathbf{k}_\mu)\cdot\mathbf{r}]V(\mathbf{r}-\mathbf{R}_0)d\mathbf{r} = V^*(\mathbf{k}_l,\mathbf{k}_\mu).$$

The numerical factors are computed for a "standard heavy nucleus" with equal numbers of up and down $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ states.

Before carrying out the coordinate integrations in Eq. (20), we effect the momentum sums by going over to the continuous limit of a momentum integral and

[34] L. Rosenfeld, *Nuclear Forces* (North Holland Publishing Company, Amsterdam, 1948), p. 234.
[35] D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953); J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) (to be published).
[36] G. Bonnevay, Compt. rend. **16**, 1641 (1954).
[37] See, for example, reference 1, p. 143.

by replacing the integral over the unfilled states by the difference

$$\sum_{k_u>k_F}\rightarrow v(2\pi)^{-3}\int_{k>k_F}d\mathbf{k}$$

$$=v(2\pi)^{-3}\left[P\int_{\text{all }k}d\mathbf{k}-P\int_{k<k_F}d\mathbf{k}\right],\quad(22)$$

where P denotes the principal value of the integral to the right. We note next that the second term in the right member of Eq. (22) contributes nothing to the energy, Eq. (20), due to the oddness of the integrand, i.e.,

$$\int_{k'<k_F}d\mathbf{k}'P\int_{k<k_F}d\mathbf{k}\frac{\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{\Delta}]}{k^2-k'^2}=0.$$

The momentum integrals then yield directly

$$\sum_{k_l<k_F}\sum_{k_u>k_F}\frac{\exp[i(\mathbf{k}_l-\mathbf{k}_\mu)\cdot\mathbf{\Delta}]}{k_l^2-k_\mu^2}$$

$$=\left(\frac{v}{8\pi^3}\right)^2\int_{k'<k_F}d\mathbf{k}'P\int_{\text{all }k}d\mathbf{k}\frac{\exp i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{\Delta}}{k'^2-k^2}$$

$$=-\left(\frac{v}{8\pi^3}\right)^2\int_{k'<k_F}d\mathbf{k}'\frac{2\pi^2}{\Delta}\cos k'\Delta\cdot\exp i\mathbf{k}'\cdot\mathbf{\Delta}$$

$$=-(v^2k_F^2/16\pi^3\Delta^2)j_1(k_F\Delta),$$

where j_1 is the regular spherical Bessel function of order unity. Collecting these results into Eq. (20), we have

$$En_{(2)}=-\frac{2Mk_F^4}{\pi^3}\int d\mathbf{r}\int d\mathbf{r}'\rho(r)\rho(r')\frac{j_1(2k_F|\mathbf{r}-\mathbf{r}'|)}{(2k_F|\mathbf{r}-\mathbf{r}'|)^2}$$
$$\times[V_d(\mathbf{r}-\mathbf{R}_0)V_d(\mathbf{r}'-\mathbf{R}_0)$$
$$+3V_\tau(\mathbf{r}-\mathbf{R}_0)V_\tau(\mathbf{r}'-\mathbf{R}_0)$$
$$+3V_\sigma(\mathbf{r}-\mathbf{R}_0)V_\sigma(\mathbf{r}'-\mathbf{R}_0)$$
$$+9V_{\sigma\tau}(\mathbf{r}-\mathbf{R}_0)V_{\sigma\tau}(\mathbf{r}'-\mathbf{R}_0)].\quad(23)$$

It requires a "moderate" amount of work to carry through these integrals. The angular integrals can be achieved with the aid of the Poisson integral representation[31]

$$\frac{j_1(z)}{z^2}=\frac{1}{2}\int_0^1(1-u^2)du\frac{\cos uz}{z},\quad(24)$$

and of the usual addition theorems. Equation (23) is then reduced to a manageable form for elementary and graphical integrations.

We discuss first, the application of Eqs. (23) and (24) to the Brueckner-Watson and Lévy potentials. Since the calculations are effected with standard techniques we present only results here. The two meson exchange part of the Brueckner-Watson potential is largely a direct interaction; the relative amounts of exchange terms are seen in Fig. 2. They contribute to Eq. (23) all told an energy of attraction which is about 0.1 Mev, or 3 percent of the first order contribution for the nucleon at $R_0=1.35A^{\frac{1}{3}}$, ($A^{\frac{1}{3}}=5.6$). Roughly half of this contribution comes from the direct term and half from the exchange terms, the contributions from which are enhanced by the spin factors. The two-meson part of the Lévy potential (the pair term) has no exchange terms and gives a second order correction of about 0.7 Mev or less then 5 percent of the first order contribution for the nucleon at $R_0=1.35A^{\frac{1}{3}}$, ($A^{\frac{1}{3}}=5.6$). For larger distances their energy contributions are, of course, relatively smaller and indeed entirely negligible. The smallness of these second-order contributions is a direct consequence of the extreme short range of these forces. We can qualitatively understand their magnitude by the following rough approximations in Eq. (23):

Because of the short range character of the potentials most of the energy contribution results when the coordinates, r and r' in Eq. (23), lie near the surface of the nucleus facing R_0. The factor depending on the particle separation within the nucleus, is replaced by its value at an average distance $|\mathbf{r}-\mathbf{r}'|\sim 1/\mu$, since this is the range of important coordinate values in the integral. Equation (23) then approximates to

$$En_{(2)}\approx-\frac{2Mk_F^4}{\pi^3}\frac{j_1(2k_F/\mu)}{(2k_F/\mu)^2}\left(\frac{v}{A}\right)^2$$
$$\times[U_d^2+3U_\tau^2+3U_\sigma^2+9U_{\sigma\tau}^2],\quad(25)$$

where $U_i=(A/v)\int\rho(r)V_i(\mathbf{r}-\mathbf{R}_0)d\mathbf{r}$ is the first-order interaction energy for a potential with coordinate dependence specified by $V_i(\mathbf{r}-\mathbf{R}_0)$, as in Eq. (4). Using the relation for the nuclear volume above Eq. (6) and Eq. (21), we simplify Eq. (25) to

$$En_{(2)}\approx-\frac{28}{\mu}\frac{j_1(2k_F/\mu)}{(2k_F/\mu)^2}[U_d^2+3U_\tau^2+3U_\sigma^2+9U_{\sigma\tau}^2].\quad(26)$$

Equation (26) serves only to show the rough form and variation of the interaction energy. It is indeed untrustworthy for actual numerical comparison because the Bessel function is a sensitive function for arguments in the region of four as occur here.

Of greater significance in the discussion of the second order effects is the role of the tensor force terms. This is because pseudoscalar meson theory yields a long range (μ^{-1}) tensor force interaction as a consequence of single meson exchange. There are additional tensor contributions resulting from two-meson exchange in the Brueckner-Watson version, but these are of lesser importance due to their shorter range.[21] We note that there is very little by way of spin-dependent central forces in both the Brueckner-Watson and Lévy potentials, and it is the large tensor force term of long range, μ^{-1}, that is responsible for the difference in behavior in the singlet and triplet states of the two-body system. This is of significance because it indicates that both of these potential models are making full use of the long-range tensor term (which appears from pseudoscalar meson theory in the characteristic combination with a spin-spin term)[38]

$$V=\frac{1}{3}\left(\frac{g^2}{4\pi}\right)\left(\frac{\mu}{2M}\right)^2\mu\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\left(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+S_{12}\frac{3+3x+x^2}{x^2}\right)\frac{e^{-x}}{x};$$

$$x\equiv\mu r \quad (27)$$

for their spin dependence.

This term[39] contributes in second order an attractive interaction energy of $0.016(g^2/4\pi)^2$ Mev for $R_0=1.35A^{\frac{1}{3}}$; $A^{\frac{1}{3}}=5.6$. This amounts to 3.6 Mev for $g^2/4\pi=15$ as in the Brueckner-Watson potential, and 1.6 Mev for $g^2/4\pi=10$ as in the Lévy model. When the two-meson exchange parts of the tensor forces in the Brueckner-Watson model are included, the attraction decreases to 2.6 Mev for the same choice of R_0 and A. At a distance of $R_0=1.4A^{\frac{1}{3}}$, these interaction energies decrease by 40% to values of 1.5 Mev and 0.9 Mev, respectively, for the Brueckner-Watson and Lévy potentials. Henley and Ruderman[40] have shown that higher-order contributions due to multiple meson scattering at a source serve only to increase the tensor force by 7 percent, so there is at present no known way of appreciably increasing the effect of this term on a field theoretic basis.

In the cases of both of the nuclear potentials which are rooted in the pseudoscalar meson theory, it thus seems clear that one cannot attribute the difference (R_N-R_C) to the finite range of nuclear forces, alone. In the Brueckner-Watson model, the sum of the effects of the short range direct forces of attraction and of the longer range tensor forces add up to 7 Mev at a distance of $1.35A^{\frac{1}{3}}$, with $A=5.6$, and thus serve merely to cut down the Coulomb barrier for protons from a height of 14 Mev to +7 Mev at this distance. For the Lévy potential the total nuclear attraction is only $\sim$2 Mev.

[38] See, for example, G. Wentzel, *Quantum Theory of Wave Fields* (Interscience Press, New York, 1950).

[39] The calculational problems encountered in evaluating the second order energy, Eq. (19), with a potential of the form in Eq. (27) are most readily surmounted by expressing the bracket in Eq. (27) in terms of differentials

$$(\quad)\rightarrow\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}$$

and by carrying out the indicated differentiations only at the very end. In place of Eq. (23), one has an expression of the form

$$\lim_{R'\to R_0}(\boldsymbol{\nabla}_{R'}\cdot\boldsymbol{\nabla}_{R_0})^2\int d\mathbf{r}\int d\mathbf{r}'\rho(r)\rho(r')\frac{j_1(2k_F|\mathbf{r}-\mathbf{r}'|)}{(2k_F|\mathbf{r}-\mathbf{r}'|)^2}\times V(\mathbf{r}-\mathbf{R}_0)V(\mathbf{r}'-\mathbf{R}').$$

[40] E. M. Henley and M. A. Ruderman, Phys. Rev. 92, 1036 (1953).

Finally, we note briefly the results of calculations on the second-order interaction energies using the various phenomenological potentials discussed in the preceding section. The results are presented as before for $R_0=1.35A^{\frac{1}{3}}$ and $A^{\frac{1}{3}}=5.6$. For the Yukawa potential, Eq. (16), the second-order energy gives an 8 percent correction to the first-order result, serving to increase the attraction by 3.5 Mev. The second order contributions for the Pease-Feshbach potential, [Eq. (17)], come to -2.5 Mev, or a 10 percent increase in the first-order result. The total second-order contributions in the case of the Rosenfeld force amount to an attraction of -1.4 Mev only.

These results give an idea of the role of the second-order contributions in this work.[41] Of special importance here are their general orders of magnitude, which are small, and the feature that they do not provide the necessary attractions in the field theory cases to account for R_N-R_C.

V. ANTISYMMETRIZATION

We analyze here the effects which are introduced when we modify our work to obey the requirement of the Pauli principle that the wave function of the system, incident nucleon plus nucleus, be antisymmetric in all coordinates.

These effects do not manifest themselves directly as simple potential terms but as exchange modifications of the scattering amplitudes and cross sections. We shall show, however, that these modifications are of quite minor importance for our work, and that their effects can be simulated by slight alterations of the energies calculated in the preceding sections.

Before displaying the relevant formalism for effecting the antisymmetrization, we note the following general features:

1. The first-order interaction energy calculated in Sec. III for the direct term, $V_d(r)$, of Eq. (2), is decreased in magnitude in general. This is because the Pauli principle inhibits the close approach to two fermions. The magnitude of the reduction factor is roughly given by

$$1-(1/4)\exp[-(R_0-R_C)/\lambdabar], \quad (28)$$

where the factor (1/4) represents the fraction of possible exchanges, since states with different spin and/or isotopic spin are mutually orthogonal. The exponential factor serves as a rough estimate of the overlap in the exchange of a nuclear particle at the nuclear surface, R_C, with the external nucleon at R_0; the length $\lambdabar\sim1/\mu$

[41] These second-order calculations indicate the nature of the limitations on the conclusions drawn below Eq. (7). Thus a potential of the form $(a+b\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau})\exp(-\mu r)/\mu r$ is a saturating interaction provided $a\geqslant0$ and $a+b\geqslant0$, and can account for R_N-R_C if $b\sim50$ Mev and $a\lesssim b/10$. However such a pathological potential would predict among other things much too large a deuteron binding energy.

corresponds to the barrier penetration distance of a particle bound by $\sim$10 Mev in a square well. In that, we assign the maximum penetration length, $\lambda\!\!\!^{-}$, to all nuclear particles including those bound deeply in the well, we overestimate the exchange correction in Eq. (28). For $R_0=1.35A^{\frac{1}{3}}$, with $A^{\frac{1}{3}}=5.6$, the factor expressed by Eq. (28) equals $\sim$0.9 and serves to decrease the energies in Sec. III by roughly 10%. It is thus a small effect and contributes only in the direction of strengthening our conclusions.

2. The second-order interaction energies due to polarization as calculated in Sec. IV are all quite small. The exchange corrections to these are negligible since, aside from other factors, they are reduced by the same exponential as in Eq. (28).

3. The one important effect of the antisymmetrization procedure is to give rise to *first*-order contributions to the interaction energy from the exchange potentials in Eq. (2). In particular, for the Brueckner-Watson and Lévy potentials, we need consider only the long-range $V_{\sigma\tau}$ term of Eq. (2) as contained in Eq. (27) since this is the only term of any significance. However, we find that this term contributes an attraction of less than one Mev for $R_0=1.35A^{\frac{1}{3}}$; $A^{\frac{1}{3}}=5.6$.

In order to outline a development of this result, we formulate the problem of the scattering of a nucleon by a nucleus of A particles. If we take as the nuclear Hamiltonian

$$h_A=\sum_{i=1}^{A} t_i+\sum_{i>j=1}^{A} V_{ij},$$

with eigenvalues ϵ_i and eigenfunctions $g_i(1\cdots A)$ which are antisymmetric under particle interchanges, the Schrödinger equation for the nucleus plus an incident nucleon reads

$$\left(h_A+t_s+\sum_{i=1}^{A} V_{is}-E\right)\Psi=0. \tag{29}$$

Let us assume first of all that the incident nucleon is distinguishable so that we can write the formal scattering solution of Eq. (29) as[42]

$$\Psi=g(1\cdots A)\varphi(s)+\frac{1}{E-h_A-t_s}\sum_{i=1}^{A} V_{is}\Psi, \tag{30}$$

for the nucleus initially in the ground state, g, and the incident nucleon initially in the plane wave state, $\varphi(s)$. The outgoing wave contour is understood in the propagator in Eq. (30). By projecting Eq. (30) onto appropriate nuclear eigenstates we obtain the scattering amplitudes for different processes.

[42] This is similar to the development in N. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953).

In particular, for elastic scattering we have

$$\begin{aligned}(g,\Psi)&\equiv\int g^*(1\cdots A)\Psi(s;1\cdots A)d\mathbf{1}\cdots d\mathbf{A}\\ &=\varphi(s)+(E-\epsilon-t_s)^{-1}\left(g,\sum_{i=1}^{A} V_{is}\Psi\right)\\ &=\varphi(s)+(E-\epsilon-t_s)^{-1}\left[\left(g,\sum_{i=1}^{A} V_{is}g\right)\right.\\ &\quad+\sum_{n,n'}{}'\left(g,\sum_{i=1}^{A} V_{is}g_n\right)\\ &\quad\cdot\frac{1}{(E-\epsilon_n-t_s)\delta_{n,n'}-\left(g_n,\sum_{i=1}^{A} V_{is}g_{n'}\right)}\\ &\quad\left.\cdot\left(g_{n'},\sum_{i=1}^{A} V_{is}g\right)\right](g,\Psi),\end{aligned} \tag{31}$$

with the sum on n and n' extending over all nuclear states except the ground state. The first term in the brackets in the right member of Eq. (31) corresponds to the first order interaction energy calculations of Sec. III. It contributes the potential scattering[43] amplitude

$$f_D=\int\varphi_f^*(s)\left(g,\sum_{i=1}^{A} V_{is}g\right)(g,\Psi)d\mathbf{S}. \tag{32}$$

We investigate this term further in order to calculate the Pauli exchange corrections to it. Due to the equiva-

[43] The second term gives all of the compound elastic effects in the terminology of reference 2. The induced polarization contributions which we calculated in Sec. IV correspond to a second order approximation to the second term which neglects the potential term in the intermediate energy denominator and replaces the kinetic energy operator, t_0, by its value for the incident particle before (or after) scattering in state $\varphi(s)$. The first of these two approximations is judged to be good in consequence of the smallness of the second order relative to the first order interaction energies for the direct terms, $V_d(r)$. The second approximation corresponds to making the optical approximation that a given potential can represent the scattering. It is rigorous in the adiabatic limit of slow incident particles (<50 kev). It also applies to calculations for the incident nucleon "outside" of the nucleus so that the nucleon can be considered to be moving freely with its initial kinetic energy. Incident nucleons of kinetic energy $\gtrsim$20 Mev can be localized to within 10^{-13} cm so that this approximation should be valid in the energy range above 20 Mev for calculations of interaction energies at distances up to within 10^{-13} cm of the surface; i.e., to within $1.4A^{\frac{1}{3}}$ for $A^{\frac{1}{3}}=5.6$. As the initial nuclear effect will be one of attraction, the incident nucleon will be speeded up, thereby increasing the magnitude of the energy denominator and decreasing the contribution of this term. We have no quantitative criterion for the validity of this second approximation at lower energies. Here we must rely on the smallness of the calculated second order corrections and on the results of the cloudy crystal ball analysis which indicate that the potential scattering predominates for elastic processes (except in the immediate vicinity of a resonance). The average elastic cross section is primarily due to potential scattering from an impenetrable sphere whose radius coincides quite closely to the radius of the potential well. (See Fig. 1 of reference 2.)

lence of the nuclear particles we can replace the sum, $\sum_{i=1}^{A} V_{is}$, by one term, AV_{is}. Representing the nuclear ground-state function by a Slater determinant of independent particle orbitals and spin functions in Eq. (32), we have

$$f_D=\sum_b \int \varphi_f^*(s)\varphi_b^*(1)V_{1s}\varphi_b(1)(g,\Psi)d\mathbf{S}d\mathbf{1}, \quad (33)$$

where the sum on b extends over the A filled single particle states of the nucleus. Now we can simply include the exchange scattering by replacing the product $\varphi_f^*(s)\varphi_b^*(1)$ by the antisymmetrized product[44] $[\varphi_f^*(s)\varphi_b^*(1)]-[\varphi_f^*(1)\varphi_b^*(s)]$ so that the scattering amplitude, including exchange, becomes

$$f=\sum_b \int [\varphi_f^*(s)\varphi_b^*(1) - \varphi_f^*(1)\varphi_b^*(s)]V_{1s}\varphi_b(1)(g,\Psi)d\mathbf{S}d\mathbf{1}.$$

In terms of an effective potential for the scattering of the incident nucleon, we have then

$$\mathcal{V}_{\text{eff}}(s)=\sum_b \int [\varphi_b^*(1)V_{1s}\varphi_b(1) - (\varphi_f^*(s))^{-1}\varphi_f^*(1)\varphi_b^*(s)V_{1s}\varphi_b(1)]d\mathbf{1}. \quad (34)$$

We calculate $\mathcal{V}_{\text{eff}}(s)$ now for the term of interest

$$V_{1s}=\tfrac{1}{3}(g^2/4\pi)(\mu/2M)^2\mu\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_s\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_s \times(\exp|\mathbf{x}_1-\mathbf{x}_s|)/|\mathbf{x}_1-\mathbf{x}_s| \quad (35)$$

in Eq. (27). The tensor term averages to zero and, as we have seen, the first, or non-exchange, term in Eq. (34) gives no first-order energy contribution for exchange potentials. For elastic exchange scattering, the spin factors in Eq. (35) give $\frac{1}{4}$ and the effective potential is from Eq. (34)

$$\mathcal{V}_{\text{eff}}(s)=-0.95\text{ Mev}\sum_b \int e^{i\mathbf{k}\cdot(\mathbf{r}_s-\mathbf{r}_1)}\left(\frac{u_b^*(s)}{u_b^*(1)}\right) \times\left(u_b^*(1)\frac{e^{-\mu|\mathbf{r}_s-\mathbf{r}_1|}}{\mu|\mathbf{r}_s-\mathbf{r}_1|}u_b(1)\right)d\mathbf{1}, \quad (36)$$

where u_b is the spatial part of the wave function of a particle bound in the nucleus. The numerical constant is calculated for a coupling constant of $g^2/4\pi=15$ in Eq. (35) as in the Brueckner-Watson potential; it is replaced by 0.64 Mev for the parameters in the Lévy potential. The following series of approximations overestimate the magnitude of $\mathcal{V}_{\text{eff}}(s)$ and thereby provide an upper bound for the effective exchange potential of Eq. (36):

1. replace $e^{i\mathbf{k}\cdot(\mathbf{r}_s-\mathbf{r}_1)}$ by unity;
2. replace the ratio of bound state space functions, $u_b(s)/u_b(1)$ by its maximum value, corresponding to the particle in the highest filled nuclear level, bound by an energy E_{BE}. The function $u_b(s)$ in this state has the largest penetration distance outside of the nucleus. In the denominator, we insert for the wave function inside the nucleus, $u_b(1)$, its value near the nuclear surface where the density is low.

Introducing these approximations into Eq. (36) and summing over the bound states, we obtain then for the effective interaction energy

$$\mathcal{V}_{\text{eff}}(s)=-0.95\text{ Mev}(A/v)\int \rho(r)\frac{e^{-\mu|\mathbf{r}_s-\mathbf{r}|}}{\mu|\mathbf{r}_s-\mathbf{r}|} \times\exp[-(2ME_{BE}/\mu^2)^{\frac{1}{2}}\mu|\mathbf{r}_s-\mathbf{r}|]d\mathbf{r}.$$

Integrating over the nuclear distribution we obtain an exchange attraction of 0.3 Mev for the incident nucleon at $r_s=R_0=1.35A^{\frac{1}{3}}$, with $A^{\frac{1}{3}}=5.6$ and for a binding energy $E_{BE}=10$ Mev. This result corresponds closely to that obtained below Eq. (28). If we include inelastic spin-flip scatterings as well as elastic processes, the spin factor in Eq. (35) gives $\frac{3}{4}$ and the effective exchange attraction increases to 0.9 Mev. These numbers are given for the coupling constant $g^2/4\pi=15$ used in the Brueckner-Watson potential and are reduced by one third for the Lévy potential.

This rough approximation to the exchange energy is an overestimate of its effect. It is clear, however, that it is of little qualitative importance in our development.

VI. CONCLUSION

To summarize, we have studied the difference between the nuclear force radius, R_N, and the radius of the charge distribution, R_C, for heavy nuclei, $A>100$. From the analyses of neutron[2–4] and proton[5,8] cross sections we take $R_N\simeq1.4A^{\frac{1}{3}}$ as the distance at which an incident nuclear particle feels an appreciable nuclear attraction, $\gtrsim14$ Mev. From the electron scattering work at Stanford[10,13] we take the charge distribution for gold as drawn in Fig. 1, which decreases to half-value at $R_C=1.12A^{\frac{1}{3}}$. The surfaces of the nuclear potential and charge distributions are seen to be similar in Fig. 1 but with the nuclear potential extending roughly 25 percent beyond the charge distribution.

The aim of this work has been to infer some properties of nuclear forces from the difference, R_N-R_C, if it is *assumed* that the nuclear charge and matter distributions coincide. We have considered various nuclear force models which are motivated by meson theory or by phenomenological analyses to see if they can simultaneously satisfy the saturation requirements in heavy nuclei and can account for the radius difference, R_N-R_C. This radius difference is to a large extent a measure of the long range behavior of the direct ($\boldsymbol{\sigma}$- and $\boldsymbol{\tau}$-independent) part of the central interaction

[44] S. Altshuler, Phys. Rev. **91**, 5 (1953).

potential between nucleons. Our calculations show that the nuclear forces must contain long range direct central attractions with short range repulsive interactions if they are to account for both $R_N - R_C$ and saturation.

Such forces can be constructed phenomenologically. However, the two nuclear force models based on pseudoscalar meson theory, which have met with success in fitting low energy data on two-body systems, fail to meet these requirements. The Brueckner-Watson[21] potential (gradient coupling theory; no pair term) has too little direct central attraction to account for $R_N - R_C$. Using this potential and calculating the effects of induced polarizations and of antisymmetrization as well as the direct interaction [Eq. (6)] we have obtained a total nucleon-nucleus energy of attraction of less than 8 Mev, or roughly half of what we require to make the nuclear radius appear to be only as large as $1.35A^{\frac{1}{3}}$, for $A=176$. The difficulties in handling the repulsive core behavior of the potential for small distances and the calculations in Sec. IV of the second order energies arising from the induced polarizations caution us not to extend these calculations to smaller separations between the nucleon and nucleus. It is thus not possible for us to give a quantitative answer to the question: how far beyond the charge distribution does the neutron distribution have to extend if we use the Brueckner-Watson potential and require $\sim$14 Mev of attraction at $R_N = 1.4A^{\frac{1}{3}}$ in order to explain the difference in radii, $R_N - R_C$? However, if we disregard the above warnings to caution, we can carry out the calculations and estimate that the neutron distribution must extend at least 10% beyond the charge distribution,[45] corresponding to a neutron surface layer 0.7–1.0 thick for $A^{\frac{1}{3}} = 5.6$.

These calculations with the Brueckner-Watson potential were also applied to lighter nuclei in order to test the dependence of our results on atomic number. For a mass number $A=120$, we took the matter distribution to be the same as the Stanford charge distribution shown in Fig. 1; i.e., we assumed that the matter and charge distributions scaled together between $A=176$ and $A=120$. The nuclear attraction in this case was evaluated to be $\sim$9 Mev at $R_0 = 1.35A^{\frac{1}{3}}$ in comparison with the less than 8 Mev for $A=176$. The reason for this increase of roughly 1 Mev in the attraction is simply this: the separation between the nuclear matter and an incident nucleon at $R_0 = 1.35A^{\frac{1}{3}}$ decreases with $A^{\frac{1}{3}}$. For a singular inter-nucleon potential the accompanying increase in interaction energy more than compensates for the decrease due to the fact that there are fewer particles in the nucleus which interact with an incident nucleon. The above figure of 9 Mev is qualitatively but not quantitatively ($\pm 15\%$) reliable because one begins to encounter the same difficulties as mentioned in the preceding paragraph with decreasing A values. It is still significantly less than the 14 Mev of attraction required at $R_N = 1.4A^{\frac{1}{3}}$ to account for the observed radius and indicates the validity of our conclusions for $A \gtrsim 100$.

Further extrapolating the Stanford charge distribution (Fig. 1 for $A=197$) down to $A\sim 60$ and calculating at $R_0 = 1.4A^{\frac{1}{3}}$, because of the aforementioned difficulties in handling the repulsive cores for smaller distances, we obtain with the Brueckner-Watson potential less than half of the 14 Mev required to account for the observed difference, $R_N - R_C$.

The Lévy[20] potential has a strong, direct attraction due to the "pair" term which is sufficient in strength and range to explain the radius difference. However, when the three-body repulsions resulting from the "pair" term, and required for nuclear stability[24] are included, the attraction is balanced at large distances and it is no longer possible to account for $R_N - R_C$. In addition there is strong evidence especially from the work of Klein[30] for the damping of the pair term which is the source of the strong direct interaction in the Lévy potential. Without the pair term, there is only a weak direct central attraction of range $\sim 1/2\mu$ corresponding to double meson exchange in the gradient coupling version of pseudoscalar meson theory.[21,40] We note that the only clue at present as to a possible source of additional central attractions (also of range $\sim 1/2\mu$) in presently studied and applied versions of pseudoscalar meson theory comes from the meson-meson interaction introduced into meson theory by the renormalization program.[36] However, the role of this term has yet to be fully explored.

An alternative explanation of the radius difference assumes that the nuclear charge and matter distributions do not coincide, but that the radius of the neutron distribution extends beyond the radius, R_C, of the proton distribution.[15,16] The difficulties which face this explanation were pointed out in Sec. I. In particular, the optical analysis by Williams[19] of the reaction cross sections (total minus coherent elastic) for 1.4-Bev neutrons and 860-Mev protons incident on various medium and heavy nuclei indicates nuclear matter distributions in close agreement with the Stanford charge distribution. His results cannot be understood if the neutron radius is appreciably larger than the radius, R_C, of the charge distribution. They can, however, be explained, as Williams points out, in terms of a decrease in the effective range of the nucleon-nucleon interaction at high energies.[46]

[45] The charge dependent contribution from the second term in Eq. (2) is relatively unimportant in this consideration since V_τ is reduced in magnitude relative to V_d by a factor of $\sim$4, as seen in Fig. 2. As mentioned in footnote 26, Eq. (3) is not valid for neutrons which form a surface layer since it is derived on the basis of a uniform proton-neutron mixture. For the surface neutrons Eq. (2) is replaced by $V_d \pm V_\tau$ with the $+$ ($-$) sign obtaining for incident neutrons (protons). However, because of the small ratio of V_τ to V_d in the Brueckner-Watson potential, the charge dependent contribution is unimportant here.

[46] This point will be discussed more fully in a note now in preparation. For large impact-parameter collisions in which the incident nucleon is only slightly deflected, one can show that

We wish to emphasize, however, that the aim of this work is not to argue the equality or inequality of the size of the neutron and proton distributions in nuclei but to show that, if one *assumes* their identity, then a special class of nuclear forces with long range attractions and short range repulsions is required to account for both the radius difference, $R_N - R_C$, and nuclear stability. Nuclear forces deduced from current meson theory and in accord with saturation requirements are not of this type.

the effective interaction potential decreases at high energy in proportion to $(1-v^2/c^2)^{\frac{1}{2}} = M/E$ if the interaction results from a (pseudo) scalar field. This is in contrast with the electromagnetic interaction in which the so-called "scalar potential" is the time component of a Lorentz four-vector and in which the above factor does not appear. This effect seems capable of qualitatively accounting for Williams' results if the large nuclear radius at low energies is interpreted as a consequence of the nuclear force attractions.

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Gamma-Ray Branching in Sr^{89}

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The decay of strontium-89 has been found to proceed through Y^{89m} to the extent of 0.02%.

STRONTIUM–89 (53 day) has been reported as a pure beta emitter in the tabulated nuclear data.[1] Various investigators have attempted to find gamma radiation accompanying Sr^{89} with no success. With the advent of NaI gamma-ray spectrometers, however, the limits of detection for gamma rays have been decreased sufficiently to enable many previously undetected small gamma-ray branchings to be observed. Since a 14-sec Y^{89m} isomeric state is known, it was of interest to see what fraction, if any, of the Sr^{89} decay proceeds through this level.

Approximately 1 Mc of strontium-89 was obtained from the Operations Division, Oak Ridge National Laboratory and put through chemical purification.[2] Strontium and barium carriers were added and the mixed barium-strontium nitrates precipitated with fuming nitric acid. The nitrates were dissolved in a few ml of H_2O and reprecipitated by the addition of fuming nitric acid. Upon dissolution of the precipitate, Fe^{+3} carrier was added and $Fe(OH)_3$ was precipitated by the addition of NH_4OH. The supernate from this scavenging was neutralized with HNO_3, acidified with acetic acid, buffered with ammonium acetate, and $BaCrO_4$ precipitated by addition of Na_2CrO_4. The $BaCrO_4$ was filtered off, the supernate was made basic with NH_4OH and the strontium was precipitated as the oxalate by addition of ammonium oxalate. The precipitate was filtered, washed, and mounted on a card for counting.

The Sr^{89} source was examined on a 3 in.×3 in. NaI(Tl) gamma-ray spectrometer, and a photopeak corresponding to a gamma ray of energy 0.91 Mev was observed. The total number of gamma-ray processes in the source was obtained by integration of the gamma peak and dividing by an appropriate efficiency factor in the usual manner.[3]

The total number of Sr^{89} disintegrations in the source was obtained by dissolving the strontium oxalate and absolute beta counting an aliquot of this solution. The ratio γ/total β^- was found to be 2×10^{-4}. Using this fractional branching, the log ft value for the 0.55-Mev beta group which must precede the gamma ray was calculated[4] to be 12.7. This corresponds to a $\Delta l=2$, no, transition and is in agreement with spin and parity changes deduced from tabulated nuclear data,[1] i.e., a $5/2+$ level in Sr^{89} and a $9/2+$ level in the Y^{89m} isomer.

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