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Internal Structure of Spinning Particles

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Elementary particle models with internal degrees of freedom have been investigated within the framework of special relativity and orthodox quantum mechanics. Classical arguments indicate that systems whose extensions are $\lesssim$ their Compton wavelength have spin excitation energies $\gtrsim$ their rest mass. The principal aim of this paper is enumeration and classification of particles with rigid internal structure and a useful classification of particle models is by their symmetry groups. In nonrelativistic mechanics this classification shows that there are only the three well-known types of rigid systems that might be labeled by number of degrees of freedom as [0], [2], and [3] and are exemplified by an ideal point, diatomic molecule and rotator, respectively; while of the three types, but one, [3], possesses a spin-$\frac{1}{2}$ state of the Pauli-election type. The corresponding analysis for relativistic mechanics shows there are nine types labeled here [0], [2], [3], [3′], [4], [4′], [4″], [5], and [6], and in addition two one-parameter infinities of types [3_f] and [5_f] ($0\leq f\leq\pi$). An algorithm exists for obtaining the spin-spectra of rigid structures from their symmetry groups. Of the $9+2\infty$ types, just three ([4], [5], and [6]) possess spin-$\frac{1}{2}$ states of the Dirac-electron type. The apparent rest mass depends upon the internal rotational state of the particle, as is shown by an unrealistic example of a Lagrangian which is an extension of that of the Klein-Gordon particle.

I. INTRODUCTION

THERE have been many attempts to bring order to the array of observed particles by calling some of them excited states of others. To do this is to attribute to the basic particles certain internal degrees of freedom[1] which, being capable of excitation, can account for the existence of various states with different properties. In many cases, this additional degree of freedom has described a motion outside of customary space-time—a rotation in isotopic spin-space, or a motion in six-dimensional relativity, etc. If only for the sake of completeness, we shall investigate here the possibilities within the present frameworks of quantum theory and special relativity.

Particles with spin present an interesting problem in this respect. First, of course, they cannot be completely structureless point masses whose states are represented in the original Schroedinger way by scalar wave functions $\psi=\psi(x,y,z,t)$ of position in space-time alone. Second, it is well known that even a composite system of such ideal point masses possesses no half-odd-integer spin states, so that an electron or a μ meson or any other particle of spin $\frac{1}{2}$ cannot be a tightly bound rotating aggregate of point masses with

$$\psi=\psi(x_1,\cdots,x_n). \tag{1}$$

Thus, it might appear that no mechanical model can possess spin $\frac{1}{2}$. Yet, ignoring for the moment the requirements that relativity places upon such theories, it is well known that a simple mechanical system is at hand which will exhibit spin $\frac{1}{2}$ as a result of ordinary rotation: the ideal rigid rotator, whose state is given by a wave function depending on the spatial coordinates x,y,z and three Euler angles θ,ϕ,χ:[2]

$$\psi=\psi(x,y,z;\theta,\phi,\chi). \tag{2}$$

The existence of such a mechanical model is not a contradiction: in quantum mechanics a system of ideal point masses, no matter how strongly bound, differs essentially from an *ideal* rigid rotator by the greater stringency of the regularity requirements on its wave functions.[1] It is easy to see that the simpler system of an ideal diatomic molecule—or ideal dipole, as it will be called—whose states have the form

$$\psi=\psi(x,y,z,\theta,\phi) \tag{3}$$

does not possess states of half-integer angular momentum; and thus the natural conclusion is that if a nonrelativistic mechanical model of a half-spin particle is to be sought, the simplest possible is the ideal rigid rotator.[2]

This conclusion is not at odds with the Pauli representation of half-spin particles; just as any atom in a p-state is necessarily represented by a vector wave function

$$\psi=\psi_m(x,y,z),\quad m=0,\pm1,\quad \text{or}\quad 1,2,3 \tag{4}$$

any structure whatever in a state of spin $\frac{1}{2}$ must be represented by a Pauli spinor $\psi=\psi_\alpha(x,y,z)$ ($\alpha=\pm\frac{1}{2}$, or 1,2). However, it will be extremely artificial to introduce, say, Euler angles into the wave function of a spinning particle to "explain" its spin, if its spin actually has but the single value $\frac{1}{2}$. In such a procedure the angles could have little physical significance, and it would be impossible to define any one of them with a precision better than

$$\Delta\theta>\hbar/2\Delta J\hbar\sim 1 \text{ radian}. \tag{5}$$

[1] F. Bopp, Z. Naturforsch. **3a**, 568 (1948); H. Honl, Z. Naturforsch. **3a**, 573 (1948); F. Bopp and R. Haag, Z. Naturforsch. **52**, 644 (1950); H. Yukawa, Phys. Rev. **77**, 219 (1950); R. L. Ingraham, Nuovo cimento **12**, 825 (1954); Hara, Marumori, Ohnuki, and Shimodaira, Progr. Theoret. Phys. (Japan) **12**, 177 (1954) and others.

[2] F. Bopp and R. Haag, Z. Naturforsch. **5a**, 644 (1950) discuss nonrelativistic spin models of this type.

The question raised here is whether, therefore, these particles truly exhibit but one spin, or whether it is possible for them to change their spin under the influence of external torques.

It is not permissible to answer this question within the theoretical framework of the Pauli spin formalism, which is constructed to permit the formulation only of interactions that commute with the magnitude of the spin; and it is not possible to answer it experimentally from the behavior of the particles at low energy. Such a hypothetical change in spin would not appear as a mere fine structure in spectral lines; it would be a catastrophic change in the nature of the particle. The magnitudes of the energies involved may be judged from relations familiar in the theory of rotational spectra of molecules. If E is the energy of rotation of a system and the "spin" J is on the order of unity,

$$E \sim \hbar^2/2I, \tag{6}$$

where I is a moment of inertia. To compare this energy with the rest mass M, we observe that $I \sim MR^2$ where R is a measure of the extension of the system, and that $\hbar/Mc = \lambda$ is the Compton wavelength. Then

$$E \sim (\lambda/R)^2 Mc^2. \tag{6'}$$

Thus if $\lambda \ll R$, $E \ll Mc^2$. For molecular systems, of course. R may be taken as an atomic size, say the Bohr radius, and λ as a proton Compton wavelength, so that $\lambda/R \sim (m/M)(1/137) \ll 1$ and $E \ll Mc^2$ ordinarily. However, for the "radius of gyration" R of an elementary particle it would be implausible to use a length much greater than its Compton wavelength λ or much less than its classical Coulomb radius $r_0 = e^2/Mc^2 = \lambda/137$ and in these opposite extremes the energy absorbed in a spin transition is, by (6'),

$$\begin{aligned} E &\sim Mc^2, \quad \text{if} \quad R \sim \lambda, \\ &\sim (137)^2 Mc^2, \quad \text{if} \quad R \sim r_0. \end{aligned} \tag{7}$$

Thus in any case the process is relativistic, and it is conceivable for large mass ratios to result in this manner.

Since the above estimates of E are large, it becomes necessary to put these considerations upon a relativistically invariant footing. This is the purpose of the present paper: to parallel the above heuristic discussion in a consistent relativistic theory, and to lay the groundwork for more detailed calculations.

It should be observed that to do this there are three points to be treated in order:

First, we shall enumerate all possible rigid structures (Parts II, III, IV). The only possibilities in nonrelativistic mechanics were the ideal point, the ideal dipole, and the ideal rotator; the enumeration in the covariant theory is not self-evident, and algebraic methods must be used. It turns out that there are not three but eleven possibilities, not all of which have been exploited in the literature.

Second, we shall investigate the spin spectrum of each model, to find whether it possesses states of spin $\frac{1}{2}$ (Part V). In nonrelativistic mechanics the answer was yes for but one of the three models, the rotator. In relativistic mechanics the answer is yes for three of the eleven models.

Third, we shall investigate the energy spectrum associated with the spin spectrum (Part VI). While the previous two points were questions of kinematics, requiring no knowledge of the Lagrangian, this is a question of dynamics, requiring a choice of Lagrangian, and the results must be strongly affected by interactions. Only a preliminary treatment of this question is presented in this paper, and the problem of interactions is not treated. Clearly a closer study of this question is necessary before a comparison of the theory with reality is permissible. The present paper is devoted rather to the determination of the systems under consideration and of their most permanent properties.

We shall, therefore, discuss covariant theories of localized systems with internal degrees of freedom, first from the point of view of classical mechanics and then from the point of view of quantum theory. To specify a point in the classical configuration space of such a system there must be given a point $x = (x^i)$ in space-time, and in addition, some kind of additional coordinates,

$$q = (q_1, \cdots, q_n), \tag{8}$$

specifying the internal configuration. We introduce a simple classification of all possible kinds of internal degrees of freedom q based on the symmetry properties of the quantity q. Unfortunately, the procedure may be summarized most concisely in the terminology of continuous groups of transformations, of which no knowledge will be assumed after the following summary. The central idea is mathematically elementary[3]: if a group (the Lorentz group now) acts transitively on a set, the set is in a 1–1 correspondence with a quotient space of the group. Now, if Q is the manifold over which q varies, it may be assumed—perhaps after sufficient subdivision of Q—that the Lorentz group G acts transitively upon Q. It follows that Q may be realized as a coset space G/G_0 of G modulo some subgroup G_0. G_0 determines the kinematics of the system. Indeed, if A_0 is the Lie algebra of G_0 and A of G, then any velocity $\dot{q}$ (thought of as determining an infinitesimal motion $q \to q + \dot{q}du$) may be represented by an element of the quotient space A/A_0, so classically the phase space is the direct sum

$$(G/G_0) \oplus (A/A_0); \tag{9}$$

while quantum mechanically, if $\Psi(Q)$ is the space of wave functions of the system (internal coordinates only), then $\Psi(Q)$ may be identified with the subspace of $\Psi(G)$ (complex functions on the Lorentz group)

[3] C. Chevalley, *The Theory of Lie Groups* (Princeton University Press, Princeton, 1946), p. 30.

consisting of those functions $\psi(g)$ invariant under G_0. Thus the first of the three points above, the enumeration of all kinematics, amounts to the enumeration-up-to-equivalence of all subgroups G_0 of the Lorentz group. The second point, the determination of the spin spectra, requires the decomposition of the subspace $\Psi(Q)$ into subspaces invariant under G. $\Psi(Q)$ is an invariant subspace of $\Psi(G)$ and its decomposition is accomplished, therefore, by the decomposition of $\Psi(G)$, i.e., by the well-known reduction of the regular representation, and can readily be computed. The particle can possess no states which are absent in this decomposition, and the dynamics may also exclude some of the terms actually present in this decomposition. The third point, the energy determination, requires at very least the choice of a Lagrangian for the free field. This can be done as follows. There is a unique (up to a constant factor) left-invariant quadratic form on G regarded as a differentiable manifold. Hence there is also one on $Q=G/G_0$, to be written $\|\dot{q}\|^2$. Thus, there is only one invariant quadratic Lagrangian function of $(q,\dot{q})$, again except for an arbitrary factor,

$$L=\tfrac{1}{2}\kappa^2\|\dot{q}\|^2. \tag{10}$$

Thus if the classical equations of motion are required to be completely invariant and linear in the second derivatives, their solutions are completely determined as geodesics on G/G_0. Similar requirements imposed upon the wave equation serve to determine the quantum field equations, for the metric on G/G_0 fixes a Laplacian operator Δ_q, the "Casimir operator" on Q.

We now carry out the procedure outlined above.

II. CLASSIFICATION OF INTERNAL STRUCTURES

Consider the effect of Lorentz transformations upon the internal coordinates q. (Terminology: the *Lorentz transformations* L are those $x\rightarrow bx+h$ preserving interval $|\Delta x|$. These include the *homogeneous Lorentz transformations*, where $h=0$, which in turn include the (space-time) rotations G where also $\det b>0$ and $b_0{}^0>0$.) The intent of the epithet "internal" is that a space-time translation is to leave the numbers q unaltered:

$$x\rightarrow x+h, \quad q\rightarrow q. \tag{11}$$

On the other hand, a space-time rotation $x\rightarrow bx$ may (or may not) have some effect on q; and in general q undergoes an arbitrary point transformation

$$x\rightarrow bx, \quad q\rightarrow boq \tag{12}$$

where boq will be written for the effect of b operating on q. The property of a group of transformations

$$(bb')oq=bo(b'oq) \tag{13}$$

must be required of the operation boq, where b and b' are arbitrary Lorentz transformations.

For any q, consider the collection $G(q)$ of all Lorentz rotations leaving q fixed: $G(q)oq=q$. From (13) it is evident that $G(q)$ is a subgroup of the Lorentz rotations G; it will be called the symmetry group (in G) of the point q. Referring again to the examples of a point particle and a dipole particle in Part I, $G(q)$ should be thought of as the group of space-time rotations "about" q. We observe that if q and q' are connected by a Lorentz transformation then their symmetry groups are equivalent: if

$$q'=boq,$$

then

$$G(q')=bG(q)b^{-1}. \tag{14}$$

The infinitesimal elements of G [or of $G(q)$] correspond to skew tensors $a_{ij}=-a_{ji}$ and form a collection which will be designated by A [or $A(q)$ respectively]. We will always assume that $G(q)$ is connected, so that it is generated by $A(q)$. $A(q)$ is closed under the operations of addition, multiplication by a real number, and commutation: the quantities

$$a+a',\ ca,\ [a,a'] \tag{15}$$

belong to $A(q)$ whenever a,a' do; and similarly for A. The classification of internal structures which we shall employ amounts to specifying $A(q)$ up to equivalence for each point q.

Special interest attaches to those cases where $A(q)$ is the same (up to equivalence) for all q, because of their simplicity and because all other possibilities can be built up out of such simple cases. In the following Part III, it will be shown that this type of structure may be regarded as rigid.

III. COVARIANT CONCEPT OF RIGIDITY

It is customary to call a body rigid in classical mechanics if it is made of particles whose mutual separations are fixed. This is not a satisfactory starting point for us, since such a system has only integer spins. It is especially unsatisfactory as a description of a supposedly primitive particle. However, such a system of points has several properties which can be expressed without reference to internal structure and which will not be incongruous in an elementary particle. We shall exhibit two such properties here:

First, such a system has only translational and rotational degrees of freedom. From any one configuration of the system it is possible to obtain any other by applying the purely geometric operations of translation and rotation to the system as a whole (or to the reference frame). This is usually expressed by saying that the Euclidean group acts *transitively* on the configuration space or that the configuration space is homogeneous (with respect to the Euclidean group).

Alternatively, for such a system there is no scalar invariant function of the coordinates except the trivial $f=\text{const}$. This expresses the fact that the interparticle separations and angles are constants, and it is not difficult to see that it is equivalent to transitivity.

Second, such a system possesses a weak sort of localizability in that its coordinates can be split into one part, "center-of-mass," which transforms as a position vector under the Euclidean group and another part, orientation, which is invariant under translation. This too can be expressed purely as a relation between the Euclidean group and the configuration space; it is equivalent to the assertion that no translation leaves any configuration fixed.

Thus there is a natural way to generalize the essential properties of the rotators of classical mechanics to a relativistic mechanics: (a) the configuration space in which the particle wanders in the course of proper time is homogeneous with respect to the group of space-time translations and rotations, and (b) no point of it is fixed under any space-time translation. It follows that a point of this configuration space has the form (x,q), where $x=(x^i)$ is a point in space-time we shall call the position of the particle and $q=(q_1,\cdots,q_n)$ is a collection of additional coordinates we shall call the orientation of the particle. It is clear that $0\leq n\leq 6$ and that q, which is invariant under translation, is transformed transitively by the space-time rotations alone. We shall call a system possessing these two properties (a) and (b) rigid for want of a better term.

Any two points of the configuration space of a rigid system have equivalent symmetry groups, by (14).

It is highly doubtful that forces exist capable of binding several particles into a system which will remain rigid under processes as energetic as the spin transitions considered here, so that it is of interest to investigate nonrigid systems. However, an elementary particle can be rigid in the above sense, and it is therefore fitting to survey all possibilities of this type.

IV. ENUMERATION OF RIGID STRUCTURES

By the kinematics of a particle we will understand the specification not only of its configuration space as a manifold but also of the manner goq in which a space-time rotation g transforms a point q of this space. It will now be shown that the kinematics of a rigid structure is completely defined by the symmetry algebra $A(q_0)$ for one point q_0. First, we have assumed that $G(q_0)$ is generated by its infinitesimals so $G(q_0)$ is determined. Now designate by $G(q,q_0)$ the set of space-time rotations transforming q_0 into q; $G(q,q_0)$ is a left coset of $G(q_0,q_0)=G(q_0)$, and it may be taken that the one-to-one correspondence

$$q\leftrightarrow G(q,q_0) \tag{16}$$

is an isomorphism. Thus we can identify the points q with the cosets of $G(q_0)$, $Q\sim G/G(q_0)$, and the law of transformation of Q is

$$goq\leftrightarrow gG(q,q_0) \tag{17}$$

where $gG(q,q_0)$ represents the result of multiplying each element of $G(q,q_0)$ in turn by g. This provides the entire kinematics.

To enumerate all possible rigid covariant kinematics it remains only to enumerate all the possibilities for $A(q_0)$ up to equivalence, since equivalent groups determine isomorphic configuration spaces by the above procedure. We shall separately consider the possible dimensions $0,1,\cdots,6$ for $A(q_0)$. The algebra of dimension $6-m$ leads to a particle of m internal degrees of freedom which will be designated by $[m]$ with further symbols where necessary.

[6]. The zero-dimensional algebra is the trivial one containing only 0, the group it generates is the identity 1 and the configuration space is $G/1$ or G itself. The particle may be thought of as a small *vierbein* or as four points not all in the same two-space. The nonquantum theory of this four-dimensional rotator was put forward by Klein[4] in a remarkable anticipation of the formalism of special relativity; his parametrization of this configuration space by Cayley-Klein parameters is the most convenient for may purposes, essentially associating with each point of the configuration space a Dirac spinor ξ subject to the conditions

$$\xi^+\xi=1,\quad \xi^+\gamma_5\xi=0. \tag{18}$$

(4 complex numbers $=8$ real numbers, less 2 constrainst, leaving 6 degrees of freedom.)

[5]. The infinitesimal Lorentz transformations (units: $\hbar=c=1$; metric: $+---$),

$$x^i\rightarrow x^i+w^i{}_jx^jd\tau,\quad (i,j=0,1,2,3)$$

correspond to skew-tensors $w_{ij}=-w_{ji}$, or equivalently to the spin operators

$$\omega=\tfrac{1}{4}iw^{ij}\sigma_{ij}=i\mathbf{w}\cdot\boldsymbol{\sigma}, \tag{19}$$

where $\sigma_{ij}=\sigma_k$, $\sigma_{i0}=i\sigma^i$ $(i,j,k=1,2,3$ cyclically) and the complex 3-vector

$$\mathbf{w}=\tfrac{1}{2}(w_{23}+iw_{10},\ w_{31}+iw_{20},\ w_{12}+iw_{30}) \tag{20}$$

is constructed from the tensor w_{ij} in much the same way that the Hertzian vector $\mathcal{E}+i\mathcal{H}$ is constructed from the electromagnetic field tensor F_{ij}. The correspondence (19) will be written

$$\omega=\Sigma(w_{ij})=\Sigma(w). \tag{21}$$

Any skew tensor a defines a one-dimensional algebra consisting of its real multiples, the closure requirements (15) being trivially satisfied. If $\boldsymbol{\alpha}$ is the 3-vector associated with a by (20) then the only independent scalars which can be constructed from a_{ij} are $\mathrm{Re}\mathbf{a}\cdot\mathbf{a}$ and $|\mathrm{Im}\mathbf{a}\cdot\mathbf{a}|$, and so the algebras can be separated into equivalence classes according to the values of the invariant ratio

$$\mathrm{Re}\mathbf{a}\cdot\mathbf{a}:|\mathrm{Im}\mathbf{a}\cdot\mathbf{a}|.$$

Either this ratio is 0:0, or it can be put in the form

$$\cos f:\sin f,\quad 0\leq f\leq\pi.$$

[4] F. Klein, *The Mathematical Theory of the Top* (Princeton University Press, Princeton, 1896); H. Goldstein, *Classical Mechanics* (Addison-Wesley Press, Cambridge, 1951), p. 180.

In the former case, the algebra can be transformed into

$$[5]=\{\sigma_+\}, \tag{22}$$

where the spin operator associated with a generator of [5] by (19) is used to define [5], and

$$\sigma_+=\frac{\sigma_1+i\sigma_2}{2}=\begin{pmatrix}0&1\\0&0\end{pmatrix}. \tag{23}$$

In the latter case, the algebra is equivalent to

$$[5_f]=\{e^{if/2}\sigma_3\},\ 0\leq f\leq\pi. \tag{24}$$

There are no others. There are thus "$1+\infty$" distinct kinematics of five degrees of freedom. This in sharp distinction to the situation in classical mechanics, where there are only three distinct rigid kinematics in all. Of all the structures exhibited in (22) and (24) it can be shown that only [5], [5_0], and [5_π] are exemplified by rigid sets of distinct points, as in Fig. 1. (This is shown by a study of the fixed points of the infinitesimal Lorentz transformations belonging to the associated algebra.)

[4]. Beyond this point the case analysis becomes quite tedious and only the results will be listed. The only rigid systems of four degrees of freedom are

$$\begin{aligned}[4]&=\{\sigma_3,\ i\sigma_3\},\\ [4']&=\{\sigma_+,\ i\sigma_3\},\\ [4'']&=\{\sigma_+,\ i\sigma_+\}.\end{aligned} \tag{25}$$

None of these possess point models, it can be shown. [4] will be of further interest due to its spin spectrum, and a parametrization is helpful: the points of the configuration space of [4] may be put in correspondence with skew tensors w_{ij} (or with 3-vectors $\mathbf{w}$ according to (20)) which obey the condition

$$\mathbf{w}\cdot\mathbf{w}=0. \tag{26}$$

A skew tensor and its associated three-vector obeying (26) will be called null; both its invariants vanish. The best known example of a null tensor, of course, is the field of a plane light wave $\mathbf{w}=\mathbf{H}+i\mathbf{E}$; $\mathbf{E}\cdot\mathbf{H}=0=E^2-H^2$. A suggestive description of the particle [4] other than a null six-vector is the associated two-spinor or Pauli spinor ξ_α: just as the particle [6] can be regarded as a point to which is attached a Dirac spinor, the particle [4] can be regarded as a point to which is attached a Pauli spinor.[5] This Pauli spinor, advantageously, is subject to no constraints.

A realization of the particle [4″] is a point to which is attached a skew tensor w_{ij} subject to the constraints

$$\mathbf{w}\cdot\mathbf{w}=1 \tag{27}$$

instead of (26). A particle of this structure was introduced by Frenkel[6] as a classical model for the electron. However, it will be shown that its canonically quantized theory cannot describe a spin-$\frac{1}{2}$ particle.

[3]. The rigid systems of three degrees of freedom are associated with the symmetry algebras

$$\begin{aligned}[3]&=\{\sigma_1,\sigma_2,\sigma_3\},\\ [3']&=\{\sigma_+,\sigma_-,i\sigma_3\},\\ [3_f]&=\{\sigma_+,i\sigma_+,e^{if/2}\sigma_3\}.\end{aligned} \tag{28}$$

There are no others. A convenient parametrization for the configuration spaces of some of these particles is by a four-vector r^i. (See Fig. 1.) This four-vector is subject to the constraints:

$$\begin{aligned}r^2&=\ \ 1\ \text{ for } [3],\\ &=\ \ 0\ \text{ for } [3_0],\\ &=-1\ \text{ for } [3'].\end{aligned} \tag{29}$$

A model for an elementary particle of this kinematic structure was proposed by Yukawa in his nonlocal meson theory, where the cases [3] and [3′] appear as particles of space-like and time-like extension respectively.[7] The remaining systems of three internal rigid degrees of freedom, namely the [3_f] with $0<f\leq\pi$, cannot be realized by models consisting of points in real space-time.

[2]. There is one structure of two degrees of freedom:

$$[2]=\{\sigma_+,i\sigma_+,\sigma_3,i\sigma_3\}.$$

[1]. There is none of one degree of freedom. This is also the case in nonrelativistic mechanics.

[0]. The structure [0] is, of course, the ideal point and its symmetry algebra is the entire algebra A.

Having enumerated all rigid structures, our next goal is the examination of their spin spectra.

V. SPIN SPECTRA

We shall obtain only partial information about the particle spectra without specifying a Lagrangian. Under the infinitesimal transformation,

$$x^i=\epsilon^i+\epsilon_j{}^ix^j, \tag{30}$$

the wave function $\psi(x,q)$ transforms according to

$$\delta\psi=-i(\epsilon^ip_i+\tfrac{1}{2}\epsilon^{ij}J_{ji}). \tag{31}$$

Here

$$p_i=-i\partial/\partial x^i$$

and

$$\begin{aligned}J_{ik}&=L_{ik}+S_{ik},\\ L_{ik}&=x_ip_k-x_kp_i,\end{aligned}$$

where S_{ik} is a differential operator with respect to the q alone and represents the contribution of the internal rotation to the angular momentum J_{ik} of the particle.

[5] The correspondence between null tensors and Pauli spinors is discussed by H. A. Kramers, *Quantentheorie des Elektrons und der Strahlung* (Akademische Verlagsgesellschaft, Leipzig, 1938), p. 260; E. T. Whittaker, Proc. Roy. Soc. (London) **A158**, 38 (1937); and E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave-Equations* (Hafner, New York, 1953), p. 31.

[6] J. Frenkel, Z. Physik **37**, 243 (1926); H. A. Kramers, reference 5, p. 229.

[7] H. Yukawa, Phys. Rev. **77**, 219 (1950).

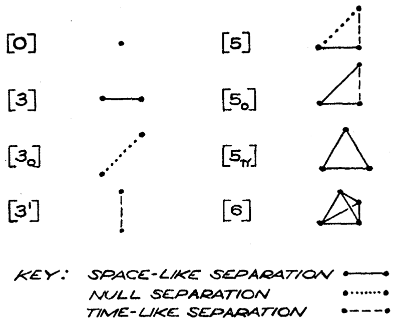

FIG. 1. Realizable rigid structures. These figures posses the same symmetry group as the indicated rigid structure. They may be regarded only as particular group-theoretic realizations of the corresponding internal structure. (In the diagram for [5], the dashed line should be solid.)

Let **S** be the 3-vector associated with S_{ik}, in the manner of (20), and let **S*** be the similar quantity obtained by replacing the explicit i by $-i$ in (20). Then it is well known that the commutation rules for the infinitesimal Lorentz transformation may be written simply

$$\begin{aligned} \mathbf{S}\times\mathbf{S}&=i\mathbf{S},\\ \mathbf{S}^*\times\mathbf{S}^*&=i\mathbf{S}^*,\\ [\mathbf{S},\mathbf{S}^*]&=0. \end{aligned} \tag{32}$$

Thus, the vector **S** has the same commutation rule as a three-dimensional angular momentum vector, although it is not formally Hermitian. **S*** commutes with **S** and possesses the same commutation rules and so a maximal commuting set of operators formed from the S_{ik} is

$$S^2, S_3;\ S^{*2}, S_3^*. \tag{33}$$

Any function of q can be expanded in terms of simultaneous eigenfunctions of the set (33). We shall now discuss these eigenfunctions.

The $\psi(q)$ are subject to a regularity condition employed by Bopp and Haag[1]: $G\psi$ shall be finite-dimensional. Here $G\psi$ represents the collection of all functions of q obtained by applying all the members of G to ψ, and taking linear combinations. As has been pointed out,[1] this condition contains the customary continuity requirements and implies for the systems of nonrelativistic mechanics the statements of Part I concerning spin-spectra from which the present discussion stems. This condition excludes two-valued functions of q in many cases but not all. In the relativistic case this condition has the additional consequence that the functions of q associated with the infinite-dimensional irreducible representations of G, unitary or otherwise,[8] are excluded. Given that $G\psi$ contains only a finite number of linearly independent functions, the commutation rules (32) imply by well-known methods[9] that the eigenvalues of the set (33) have the form

$$s(s+1), m;\ s^*(s^*+1), m^*$$

where

$$\begin{aligned} s&=0,\tfrac{1}{2},1,\cdots,\\ m&=-s,\ -s+1,\ \cdots,\ s-1,\ s;\\ s^*&=0,\tfrac{1}{2},1,\cdots,\\ m^*&=-s^*,\ -s^*+1,\ \cdots,\ s^*-1,\ s^*, \end{aligned} \tag{34}$$

respectively. A corresponding eigenfunction will be written

$$\psi(q)=\psi_{sms^*m^*}(q), \tag{35}$$

with additional labels in cases of degeneracy. S^2 and S^{*2} are scalars under G but are interchanged by time-reflection. Thus the invariant subspaces of the space $\psi(Q)$ of internal states under the (extended) Lorentz group are of the form

$$\Psi(s,s^*)=\Psi_{ss^*}\oplus\Psi_{s^*s} \quad \text{if} \quad s\neq s^* \tag{36}$$

(direct sum), and

$$\Psi(s,s)=\Psi_{ss},$$

where Ψ_{ss^*} represents an irreducible invariant subspace under G associated with the specified values of S^2 and S^{*2}. The intrinsic three-dimensional angular momentum is **S**+**S*** and its magnitude has the quantum numbers

$$|s-s^*|,\ |s-s^*|+1,\ \cdots,\ s+s^*$$

in the space $\Psi(s,s^*)$, by the usual rules for composition of two commuting angular momentum vectors. The wave function of a Dirac particle has the same transformation properties as a vector in the four-dimensional subspaces $\Psi(0,\frac{1}{2})$. We must inquire whether such a subspace appears in the decomposition of the function space $\Psi(Q)$ associated with each of the $9+2\infty$ possibilities enumerated in Part IV; and if so, how often. This can be answered by the most cursory inspection of the associated symmetry algebras $A(q_0)$:

[6]. The decomposition of $\Psi(Q)=\Psi(G)$ into irreducible invariant subspaces is well known[10] to contain $(2s+1)(2s^*+1)$ isomorphs of Ψ_{ss^*}, so that the Dirac spinor subspace $\Psi(0,\frac{1}{2})$ appears twice. For the further work a review of this decomposition is necessary.

With each Lorentz rotation $b=b^i{}_j$ is associated a 2×2 spin matrix $\beta=\Sigma(b)=\beta_\alpha{}^\gamma$ such that

$$\det\beta=1,\ \beta^H\sigma_i\beta=\sigma_k b^k{}_i(\sigma_0\equiv1). \tag{37}$$

(The sign of β is not determined.) For $Q=G$, each wave-function $\psi(q)$ is thus a function $\psi[\beta]$ of a 2×2 spin matrix β (and its complex conjugate β^*). We put

$$\begin{aligned} \beta_\alpha{}^1&=\xi_\alpha,\\ \beta_\alpha{}^2&=\eta_\alpha, \end{aligned} \tag{38}$$

so that

$$\psi[\beta]=\psi(\xi,\eta,\xi^*,\eta^*),$$

[8] V. Bargmann, Ann. Math. **48**, 568 (1947); P. A. M. Dirac, Proc. Roy. Soc. (London) **A183**, 284 (1945); I. M. Gelfand and M. Neumark, J. Phys. (U.S.S.R.) **10**, 93 (1946) and *Unitarnie Predstavleniia Klassiceskih Grupp*, Acad. Sci. U.S.S.R. (1950); Harish Chandra, Proc. Roy. Soc. (London) **A189**, 372 (1947).

[9] P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, Oxford, 1947), p. 140.

[10] H. Weyl, *Theory of Groups and Quantum Mechanics* (Dover Publications, New York, 1931), p. 316.

for it is more convenient to regard ψ as a function of the two spinors ξ and η. Because $\psi(q)$ is a scalar, for any Lorentz rotation g

$$g\psi(b)=\psi(g^{-1}b)=\psi[\gamma^{-1}\beta]=\psi(\gamma^{-1}\xi,\gamma^{-1}\eta,(\gamma^{-1}\xi)^*,(\gamma^{-1}\eta)^*)$$

$[\gamma=\sum(g)]$, so ξ and η transform separately. The functions ψ which are homogeneous of fixed degrees in ξ,η,ξ^*, and η^* clearly form an invariant subspace of $\Psi(G)$ and this subspace is irreducible. Designating this subspace by $\Psi_{sm's^*m^{*\prime}}$, the relation between the spin eigenvalues s, s^* and the degrees is indicated by

$$\Psi_{sm's^*m^{*\prime}}\sim\xi^{s+m'}\eta^{s-m'}(\xi^*)^{s^*+m^{*\prime}}(\eta^*)^{s^*+m^{*\prime}}, \quad (39)$$

where m' and $m^{*\prime}$ are auxiliary invariant quantum numbers. In the nonrelativistic rotator the quantum number corresponding to m' is the component of angular momentum along the body z-axis.[1] The range of the quantum number m' is from $-s$ to $+s$ in steps of unity, $2s+1$ values in all, and there are $2s^*+1$ values for $m^{*\prime}$. Therefore, there are in all $(2s+1)(2s^*+1)$ distinct subspaces belonging to the same values of s and s^*.

Thus the particle [6] is highly composite from the point of view of group representations.

[5]-[0]. The wave functions of [5]—or any of the other particles to be discussed—can be regarded in two distinct ways in their dependence on q. Up to now they have been regarded as assigning a complex value $\psi(q)$ to each coset q of a fixed subgroup of G, namely the symmetry group of the particle. Such a function $\psi(q)$ naturally defines a function on G itself, to be designated $\psi(g)$, where

$$\psi(g)=\psi(q), \quad \text{if} \quad goq_0=q. \quad (40)$$

Thus, the value of $\psi(g)$ is simply that associated with the coset to which g belongs. The resulting set of functions will be identified with $\Psi(Q)$ henceforth. Its characteristic property, of being constant on cosets of $G(q_0)$, is equivalent to invariance under right multiplication by $G(q_0)$:

$$\psi(g)=\psi(gg_0) \text{ for } \psi \text{ in } \Psi(Q),\ g_0 \text{ in } G(q_0). \quad (41)$$

It is enough to assert (41) for the infinitesimal g_0, which form the symmetry algebra $A(q_0)$:

$$\delta\psi(g)=0 \text{ for } \delta g=iga,\ \psi \text{ in } \Psi(Q),\ a \text{ in } A(q_0). \quad (42)$$

Now the criterion (42) can be used to see whether a given Ψ_{ss^*} is represented in $\Psi(Q)$ by ascertaining whether any $\Psi_{sm's^*m^{*\prime}}$ is included in $\Psi(Q)$. From the invariance of the subspace $\Psi_{sm's^*m^{*\prime}}$ under Lorentz transformation it is sufficient to verify (42) for $g=1$. Thus, $\Psi_{sm's^*m^{*\prime}}$ is in $\Psi(Q)$, if, and only if

$$\psi(1+ia)=\psi(1) \text{ for all } \psi \text{ in } \Psi_{sm's^*m^{*\prime}},\ a \text{ in } A(Q). \quad (43)$$

Together with (39) and the list of all $A(Q)$ in Table I, (43) provides an algorithm for getting the spin spectrum of each rigid structure. Again, terms present in this spin spectrum may be excluded subsequently by the dynamical equations of the system.

TABLE I. Under *Type* is given the number of degrees of internal freedom. When there is more than one type of a given number of degrees of freedom, primes or subscripts are added to distinguish. Under *Symmetry group* is given a set of infinitesimals generating the symmetry group, in the 2×2 spin-representation. Here $\sigma_+=\frac{1}{2}(\sigma_1+i\sigma_2)=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. Under *Parametrization* is given a coordinate system for some of the types, with the constraints if any.

Type	Symmetry group	Parametrization
[6]	0	4-spinor ξ; $\xi^+\xi=1$, $\xi^+\gamma_5\xi=0$
[5]	σ_+	
[5$_f$]	$e^{if/2}\sigma_3$	
[4]	$\sigma_+,i\sigma_+$	6-vector $(\mathbf{E},\mathbf{H})$; $(\mathbf{E}+i\mathbf{H})^2=0$
[4′]	$\sigma_+,i\sigma_3$	
[4″]	$\sigma_3,i\sigma_3$	6-vector $(\mathbf{E},\mathbf{H})$; $(\mathbf{E}+i\mathbf{H})^2=1$
[3]	$\sigma_1,\sigma_2,\sigma_3$	4-vector r^i; $r_ir^i=1$
[3′]	$i\sigma_1,i\sigma_2,\sigma_3$	4-vector r^i; $r_ir^i=-1$
[3$_f$]	$\sigma_+,i\sigma_+,e^{if/2}\sigma_3$	
[2]	$\sigma_+,i\sigma_+,\sigma_3,i\sigma_3$	
[0]	$\sigma_1,\sigma_2,\sigma_3,i\sigma_1,i\sigma_2,i\sigma_3$	

We will discuss only a particularly easy question: which structures possess Dirac states $\Psi(0,\frac{1}{2})$? The subspaces in point are by (39)

$$\Psi_{\frac{1}{2}\frac{1}{2}00}\sim\xi,\ \Psi_{\frac{1}{2}-\frac{1}{2}00}\sim\eta,\ \Psi_{00\frac{1}{2}\frac{1}{2}}\sim\xi^*,\ \Psi_{00\frac{1}{2}-\frac{1}{2}}\sim\eta^*.$$

It is readily seen now that $\Psi_{\frac{1}{2}\frac{1}{2}00}$ and $\Psi_{00\frac{1}{2}\frac{1}{2}}$ belong to $\Psi(Q)$ only if the first column $a_\alpha{}^1$ of each matrix in the 2×2 spin representation of $A(Q)$ given in Table I vanishes, while $\Psi_{\frac{1}{2}-\frac{1}{2}00}$ and $\Psi_{00\frac{1}{2}-\frac{1}{2}}$ are present only if the second column $a_\alpha{}^2$ vanishes. Inspection of Table I shows that only the structures [4], [5], and [6] possess either of these properties. We thus conclude: *only the structures [4], [5] and [6] can have Dirac-like states.*

Of these three structures, two can be realized by rigid systems of space-time points, [5] and [6]. (See Fig. 1.) If anything, this makes these two less appealing as models for an elementary particle. The simplest case [4] does not have this property and has some intriguing characteristics of its own. It can be described as a point in space-time to which is attached as internal coordinate a six-vector $(\mathbf{E},\mathbf{H})$ subject to

$$\mathbf{E}^2-\mathbf{H}^2=0,\ \mathbf{E}\cdot\mathbf{H}=0, \quad (44)$$

according to (25), where $\mathbf{w}=\mathbf{H}+i\mathbf{E}$. Such a six-vector, usually associated with a spin-one particle, can by its rotation *give rise to spin-$\frac{1}{2}$ states*, while a four-vector, or a six-vector not obeying (44), cannot. Of course, the wave-functions of these states are double-valued, and devoid of singularities. It seems that the structure [4] might be worthy of further study.

VI. ENERGY SPECTRA

In the present section, a covariant Lagrangian is selected for mathematical simplicity to show how the rotational energy of the structures considered can contribute to their rest mass. Since the Lagrangian chosen is very unrealistic only the briefest exposition is appropriate.

The form chosen for the Lagrangian density L is analogous to that of the Klein-Gordon particle, and is the sum of a scalar which is quadratic in ψ and a scalar which is quadratic in the first derivatives of ψ.

$$L=\tfrac{1}{2}[\partial_i\psi^*\partial^i\psi-\eta^2\partial_\alpha\psi^*\partial^\alpha\psi-m^2\psi^*\psi]. \tag{45}$$

Here $\partial_i\equiv\partial/\partial x^i$. Likewise, ∂_α represents differentiation with respect to the internal coordinates q^α, and the metric form $\gamma_{\alpha\beta}dq^\alpha dq^\beta=dq_\alpha dq^\alpha$ assumed for the internal coordinates q is discussed in a footnote.[11]

The action principle is then

$$\delta\int_V L(dx)(dq)=0,$$

where

$$(dx)=dx^0\cdots dx^3$$

and

$$(dq)=\gamma^{\frac{1}{2}}dq^1\cdots dq^N,\ (\gamma=\det\gamma_{\alpha\beta})$$

are invariant volume elements and V is a volume in $4+N$-dimensional configuration space. This leads to the wave equation

$$(\Box-\eta^2\Delta_q+m^2)\psi=0, \tag{46}$$

where $\Box=\partial_t^2-\nabla^2$ and $\Delta_q=\gamma^{-\frac{1}{2}}\partial_\alpha\gamma^{\frac{1}{2}}\partial^\alpha$ are the Laplacians with respect to the external and internal coordinates. Inasmuch as this is a second-order differential equation it is not of the Schrödinger form. Only for certain models is it possible to write an invariant equation of the first order in time differentiation. We present the above wave-equation only for its simplicity and generality. The second-order scalar differential operator Δ_q can be expressed, of course, in terms of the first-order differential operators $\mathbf{S}$ and $\mathbf{S}^*$ introduced in Part V:

$$\Delta_q=-\tfrac{1}{2}(S^2+S^{*2}).$$

Thus, the eigenfunctions and the eigenvalues of Δ_q have the form

$$\Delta_q Y_{ss^*}(q)=-\tfrac{1}{2}(s(s+1)+s^*(s^*+1))Y_{ss^*}(q),$$

and if ψ is expanded in terms of these "internal spherical harmonics", the coefficient $\psi_{ss^*}(x)$ of $Y_{ss^*}(q)$ in this expansion obeys the wave equation

$$[\Box+m^2+\tfrac{1}{2}\eta^2(s(s+1)+s^*(s^*+1))]\psi_{ss^*}(x)=0.$$

Defining the rest mass M by the relation $p_ip^i=M^2$, it is seen that the state ψ_{ss^*} has the rest mass

$$M_{ss}{}^*=m\left[1+\tfrac{1}{2}\left(\frac{\eta}{m}\right)^2[s(s+1)+s^*(s^*+1)]\right]^{\frac{1}{2}}. \tag{47}$$

If one puts $\lambda=m^{-1}$ for the Compton wavelength of the unrotating particle and $R=\eta^{-1}$, then by (47) the spin energy E when $\lambda\ll R$ is

$$E\sim(\lambda/R)^2mc^2.$$

Thus η^{-1} is like a radius of gyration. For $\lambda\sim R$ or $\lambda>R$, the result (7) of Part I is modified considerably: from (47) it follows that the rest mass M of an excited state of this particle is ~137 [instead of $(137)^2$] times the rest mass m of the state of spin zero, if the classical Coulomb radius of the particle is taken for its radius of gyration. In nature the situation appears reversed: $m_\pi\sim2\times137m_e$, the particle of spin zero is heavier than one of spin $\frac{1}{2}$.

This Lagrangian (45) is quite unsatisfactory, but shows the possibility of a covariant quantum theory for one particle. The many-particle or second-quantized theory will not be treated in this paper.

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[11] This form is completely determined throughout q-space up to a constant scale factor when its value is given for a single differential dq at any one point q, if it is assumed—as we do—that the form is invariant under Lorentz transformations. Because Q is realized as a quotient space of G, to fix scale factor it suffices to define a metric on the Lorentz group itself, as is required for the case [6] where $Q=G$. In that case, we take for the index α the couple $\alpha=(i,j)$ with $i<j=0,1,2,3$. Setting $dq^\alpha=\epsilon^{ij}$, where ϵ^{ij} is the skew-symmetric tensor associated with an infinitesimal Lorentz transformation in (29), we then define $dq_\alpha dq^\alpha=\frac{1}{2}\epsilon_{ij}\epsilon^{ij}$.