phenomena is obtained by a coupling constant in the neighborhood of 0.07–0.08 and a cut-off energy near 6μ. These values are quite close to those used here to give best agreement for the low-energy n-p system. Thus, except possibly for peculiarities which may appear in higher orders, it seems that the low-energy n-p system can be predicted without recourse to adjustable parameters.

The application of these potentials to problems at higher energies has not yet been considered by the present author. In making the calculation here, nucleon kinetic energies have been consistently ignored relative to total meson energies. How large the nucleon energies may become before the calculation breaks down is a question which is difficult to answer without further work. Since the depth of the attractive part of the triplet central force is of the order 15 Mev, it is expected that the nucleon kinetic energies in the deuteron also are of this magnitude. The agreement obtained for the deuteron thus leads one to speculate that the potentials may be reliable for nucleon energies beyond the range of reliability of the effective range theory. Indeed, Fujii *et al.*[8] have applied their potentials to n-p scattering at 40 and 90 Mev and have concluded that the characteristic features of the high-energy nucleon-nucleon scattering are reproduced quite well. It seems reasonable, therefore, that the potentials derived here may also apply at these higher energies.

[8] S. Fujii *et al.*, Progr. Theoret. Phys. **11**, No. 1 (1954).

TABLE II. The singlet effective ranges obtained by adjusting f^2 to give the exact singlet scattering length.

f^2	$\omega_m(\mu)$	r_{0s} (10^{-13} cm)
0.10	6	2.54
0.11	5.6	2.65

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State-Vector Normalization in Formal Scattering Theory*

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The familiar problem of state-vector normalization and, in field theories, the related problem of charge renormalization are shown to arise in a natural manner in the formal scattering theory introduced by Lippmann and Schwinger. The mathematical arguments necessary for dealing with these problems are developed entirely within the framework of the formal theory and lead to the customary rules for the construction of the renormalized S-matrix and reactance operator, provided mass renormalization is simultaneously carried out and the one-to-one correspondence between perturbed and unperturbed eigenstates is set up in a "natural" fashion.

I. INTRODUCTION

LIPPMANN and Schwinger[1] have shown that the stationary states which describe scattering processes for a given system may be represented formally by

$$\psi^{\pm}=\lim_{\epsilon\to 0}\frac{\pm i\epsilon}{E-H\pm i\epsilon}\phi, \qquad (1)$$

where H is the Hamiltonian operator of the system, ϵ is a positive infinitesimal, E is the energy of the state in question, and the vector ϕ represents a plane wave of the same energy. The $+$ sign refers to the states with outgoing or "retarded" scattered waves, and the $-$ sign to the incoming or "advanced" wave states. That $\psi^{\pm}$ are indeed eigenvectors of H corresponding to the eigenvalue E is immediately seen by multiplying Eq. (1) on the left by $E-H\pm i\epsilon$ and then passing to the limit.

Although Eq. (1) was initially introduced in the limited context of simple scattering theory, its use as a method of constructing eigenvectors of an operator H has a much wider range of validity. For example, H may be a finite matrix, or an operator with discrete rather than continuous eigenvalues. In a review article (to be published) the author has used Eq. (1) as a starting point for a discussion of bound-state perturbation theory. It is the purpose of the present note, however,

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[1] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

to point out that Eq. (1) may also be used as a basis for a connected description of scattering theory as it appears in the complicated context of the remormalization program for quantized fields. It is easy to see, in fact, that the normalization (or renormalization) problem arises quite naturally in a formalism based on Eq. (1); for, when E is an eigenvalue of H then the operator $\pm i\epsilon(E-H\pm i\epsilon)^{-1}$ projects out only that portion of ϕ which lies in the corresponding eigen-subspace. Therefore $\psi^{\pm}$ as given by Eq. (1) is not generally normalized. The fact that it *is* normalized in simple scattering problems is due to special circumstances.

In dealing with discrete spectra one may use almost any vector ϕ on the right of Eq. (1) in order to construct an eigenvector of H. When continuous spectra are involved, however, it appears to be not only a convenience for physical interpretation but a practical mathematical necessity to choose ϕ in a special way, namely, to be an eigenvector of an "unperturbed" Hamiltonian operator H_0 which is obtained from H by subtracting a portion H_1 which has an obvious significance as a scattering potential, an interaction, or a "coupling":

$$H_0=H-H_1. \tag{2}$$

One then sets up a one-to-one correspondence between the eigenvectors $\psi_a^{\pm}$ of H and the eigenvectors ϕ_a of H_0 in such a way that the $\psi_a^{\pm}$ approach the ϕ_a as $H_1\to 0$, where the vanishing of H_1 is assumed to take place in a *linear* fashion; i.e., $g\to 0$ where $H_1=g\mathcal{H}$, g being often referred to as a "coupling constant." This, of course, involves some sort of assumption about quantities being analytic in g at the origin, which may or may not be justified but which nevertheless underlies all proposals for dealing with quantized field problems which have had any success to date. If the criterion of analyticity in g is satisfied, then, as will presently become apparent, Eq. (1) becomes an adequate vehicle with which to express the one-to-one correspondence between the $\psi_a^{\pm}$ and the ϕ_a.

We shall follow the accepted practice of rendering the continuous spectrum discrete by imposing periodic boundary conditions with respect to a fundamental volume $V=L^3$ upon the eigenfunctions of H_0 so that the uniform orthonormalization,

$$(\phi_b,\phi_a)=\delta_{ba}, \tag{3}$$

may be employed for all the eigenvectors ϕ_a. When V is very large we shall speak of the resulting spectrum as "quasi-continuous."

II. THE NORMALIZATION CONSTANTS

In order that normalization be properly taken into account in contexts more general than simple scattering theory, Eq. (1) must be modified by the insertion of an appropriate normalization constant. For compactness of notation let us introduce the Green's function[2]

$$G^{\pm}(E)=G^{\mp}(E)^{\dagger}=(E-H\pm i\epsilon)^{-1} \tag{4a}$$

$$=G_0^{\pm}(E)[1+H_1G^{\pm}(E)]$$
$$=[1+G^{\pm}(E)H_1]G_0^{\pm}(E) \tag{4b}$$

$$=G_0^{\pm}(E)[1-H_1G_0^{\pm}(E)]^{-1}$$
$$=[1-G_0^{\pm}(E)H_1]^{-1}G_0^{\pm}(E), \tag{4c}$$

where

$$G_0^{\pm}(E)=G_0^{\mp}(E)^{\dagger}=(E-H_0\pm i\epsilon)^{-1}, \tag{5}$$

and then write

$$Z_a^{\frac{1}{2}}\psi_a^{\pm}=\pm i\epsilon G^{\pm}(E_a)\phi_a, \tag{6}$$

where the Z_a are the normalization constants and the E_a are the eigenvalues of H. Here (and from now on) the limit $\epsilon\to 0$ is to be understood.

In the case of quantized fields, as in bound-state theory, the spectra of H and H_0 are not generally identical. We shall therefore redefine H_0 and H_1 according to the scheme[3]

$$\begin{aligned} H_0&\to H_0+\textstyle\sum_a \phi_a\rangle\Delta E_a\langle\phi_a,\\ H_1&\to H_1-\textstyle\sum_a \phi_a\rangle\Delta E_a\langle\phi_a,\end{aligned} \tag{7}$$

where the ΔE_a are the *level shifts* which may be computed later in the course of solving the problem (if desired). The ΔE_a evidently play a role in the definition of the one-to-one correspondence between the $\psi_a^{\pm}$ and the ϕ_a, for it must be assumed that $\Delta E_a\to 0$ as $g\to 0$. It is to be observed that the switching off of the "perturbation" ($H_1\to 0$) is no longer a linear process, since H_1 now involves not only $g\mathcal{H}$ but also the ΔE_a which are complicated functions of g assumed to be analytic at the origin.

With these modification one has

$$H\psi_a^{\pm}=E_a\psi_a^{\pm}, \quad H_0\phi_a=E_a\phi_a, \tag{8}$$

so that Eq. (6) may be rewritten in the form

$$Z_a^{\frac{1}{2}}\psi_a^{\pm}=G^{\pm}(E_a)[G_0^{\pm}(E_a)]^{-1}\phi_a$$
$$=[1+G_0^{\pm}(E_a)R^{\pm}(E_a)]\phi_a, \tag{9}$$

where

$$R^{\pm}(E)=R^{\mp}(E)^{\dagger}$$
$$=[G_0^{\pm}(E)]^{-1}G^{\pm}(E)H_1=H_1G^{\pm}(E)[G_0^{\pm}(E)]^{-1} \tag{10a}$$

$$=H_1+H_1G^{\pm}(E)H_1 \tag{10b}$$

$$=H_1[1+G_0^{\pm}(E)R^{\pm}(E)]$$
$$=[1+R^{\pm}(E)G_0^{\pm}(E)]H_1 \tag{10c}$$

$$=H_1[1-G_0^{\pm}(E)H_1]^{-1}=[1-H_1G_0^{\pm}(E)]^{-1}H_1. \tag{10d}$$

[2] The Fourier transform of $G^{\pm}(E)$, namely

$$G^{\pm}(t)=(2\pi)^{-1}\int_{-\infty}^{\infty}G^{\pm}(E)e^{-iEt}dE,$$

satisfies the differential equation $(i\partial/\partial t-H)G^{\pm}(t)=\delta(t)$ in the limit $\epsilon\to 0$.

[3] In relativistic field theories the customary redefinition of H_0 and H_1 is slightly different from this. Equation (7) amounts to a renormalization of energy, which is not relativistically invariant. The invariant *mass* renormalization, however, involves only a trivial modification.

Equation (9) now shows explicitly the correspondence $\psi_a^{\pm} \to \phi_a$, $Z_a \to 1$, as the perturbation is switched off, since $R^{\pm}(E) \to 0$ as $H_1 \to 0$.[4]

The $Z_a^{\frac{1}{2}}$, as has already been intimated, are projection coefficients and hence have a simple physical significance. To see this, take the scalar product of Eq. (6) with $\psi_a^{\pm}$ and impose the condition $(\psi_a^{\pm},\psi_a^{\pm})=1$, obtaining

$$Z_a^{\frac{1}{2}} = (\psi_a^{\pm}, \pm i\epsilon G^{\pm}(E_a)\phi_a) = (\mp i\epsilon G^{\mp}(E_a)\psi_a^{\pm}, \phi_a) = (\psi_a^{\pm},\phi_a). \quad (11)$$

The phases of the $\psi_a^{\pm}$ will be assumed to be defined by taking the $Z_a^{\frac{1}{2}}$ to be positive real numbers. Then Z_a *is the probability of finding the state* ϕ_a *in the state* $\psi_a^{\pm}$, and hence

$$0 \leq Z_a \leq 1. \quad (12)$$

The nondependence of the Z_a on the $\pm$ signs may be inferred by taking the scalar product of Eq. (6) with itself.

III. ORTHOGONALITY

We have now to investigate the orthogonality of the $\psi_a^{\pm}$. It will be convenient at this point to record three identities satisfied by the operators $R^{\pm}(E)$:

$$R^{\mp}(E_b)-R^{\pm}(E_a) \equiv (E_a-E_b\pm 2i\epsilon)R^{\mp}(E_b) \times G_0^{\mp}(E_b)G_0^{\pm}(E_a)R^{\pm}(E_a), \quad (13)$$

$$R^{\pm}(E_b)-R^{\pm}(E_a) \equiv (E_a-E_b)R^{\pm}(E_b) \times G_0^{\pm}(E_b)G_0^{\pm}(E_a)R^{\pm}(E_a), \quad (14)$$

$$\partial R^{\pm}(E)/\partial E \equiv -R^{\pm}(E)[G_0^{\pm}(E)]^2R^{\pm}(E). \quad (15)$$

The first two identities follow from Eqs. (10a, b) and the third is obtained from the second by passing to the limit $E_b \to E_a$.

Using Eqs. (3), (9), and (13), one may write[5]

$$\begin{aligned}(Z_bZ_a)^{\frac{1}{2}}&(\psi_b^{\pm},\psi_a^{\pm}) \\ &= (\phi_b, [1+R^{\mp}(E_b)G_0^{\mp}(E_b)][1+G_0^{\pm}(E_a)R^{\pm}(E_a)]\phi_a) \\ &= \delta_{ba}+\left(-\frac{1}{E_a-E_b\pm i\epsilon}+\frac{1}{E_a-E_b\pm 2i\epsilon}\right) \times (\phi_b, [R^{\mp}(E_b)-R^{\pm}(E_a)]\phi_a) \\ &\underset{\epsilon\to 0}{\to} \begin{cases} 0, & E_b \neq E_a, \\ \delta_{ba} \pm \epsilon^{-1}(\phi_b, \mathrm{Im}R^{\pm}\phi_a), & E_b = E_a, \end{cases} \end{aligned} \quad (16)$$

where

$$R^{\pm} = \sum_a R^{\pm}(E_a)(\phi_a)\langle\phi_a. \quad (17)$$

In simple scattering problems the operator H_0 is chosen in such a way that H_1 refers to a scattering force which is *confined to a limited region of space*. The vectors ϕ_a are usually taken to correspond to (1) plane waves, (2) spherical harmonics of plane waves, or (3) Coulomb wave functions if part of the scattering force is inverse-square. The matrix elements $(\phi_b,H_1\phi_a)$, and hence $(\phi_b,R^{\pm}\phi_a)$, are then of order V^{-1} in magnitude, owing to the normalization condition Eq. (3). Gell-Mann and Goldberger[5] have argued that the limiting process $\epsilon \to 0$ must be accompanied by a simultaneous limiting process $V \to \infty$ of such a nature that $(\epsilon V)^{-1} \to 0$, and hence that the term involving $\mathrm{Im}R^{\pm}$ in Eq. (16) vanishes, implying the orthonormality of the $\psi_a^{\pm}$ with $Z_a=1$. As the argument given by these authors is rather obscure we feel it advisable to restate it in a different form. The matter is really quite simple. With the imposition of the periodic boundary conditions the level separation in the quasi-continuous spectrum is of order L^{-1}. In order that the imaginary parts $\pm i\epsilon$ of the energy denominators of the Green's functions give the correct causal description of the scattering process (i.e., be able to make the distinction between retarded and advanced waves), ϵ must be much larger than the level spacing so that the summation over intermediate states will take on a *fine-grained* aspect with respect to ϵ and be representable as an integral over a contour which passes definitely to one side or the other of the energy pole. Therefore $L^{-1}/\epsilon \to 0$ and *a fortiori* $(\epsilon V)^{-1} \to 0$.[6]

In cases in which the Z_a are not simply equal to unity Eq. (16) may be used to compute their values. A problem evidently arises if the spectrum of H is degenerate. It is necessary to show that the quantity $(\phi_b, \mathrm{Im}R^{\pm}\phi_a)$ is diagonal in a and b when $E_b=E_a$, in order to insure the orthogonality of the $\psi_a^{\pm}$ within degenerate subsets. It is useful to consider first the situation as it occurs in simple bound-state (discrete spectrum) theory. If we exclude the possibility of accidental degeneracy, then the degeneracy in question persists for all values of g, and in particular when the perturbation is switched off. This type of "nonremovable" degeneracy has its origin in special symmetry properties possessed by H_1, and it is well known that the ϕ_a can then be chosen in such a way that $(\phi_b,\psi_a^{\pm})=0$ if $a \neq b$ with $E_a=E_b$. The combination of this result with Eqs. (9) and (11) implies $(\phi_b,R^{\pm}\phi_a)=\pm i\epsilon(Z_a-1)\delta_{ba}$ for $E_a=E_b$, and hence the orthogonality of the $\psi_a^{\pm}$ via Eq. (16). The nonremovability of the degeneracy also implies $Z_a=Z_b$ for $E_a=E_b$.

The situation in field theory is quite similar. Again

[4] If H possesses eigenstates (e.g., extra bound states) which have no counterparts among the ϕ_a, then these states and their corresponding eigenvalues are to be excluded from the present discussion. Also to be excluded are unstable eigenstates of H_0, since these have no counterparts among the eigenvectors of H. In passing from H_0 to H these latter states undergo a mathematical as well as physical decay, and their renormalization constants Z_a vanish rigorously.

[5] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

[6] A more physical argument involves the recognition that the use of ϵ is related to an adiabatic switching procedure in which the perturbation H_1 is "turned on" for a length of time of order ϵ^{-1}. If this mathematical trick is to provide an adequate substitute for an actual physical process involving a wave packet which moves unperturbed both before and after scattering, then ϵ^{-1} must be much shorter than the length of time L/v taken by the packet to traverse the fundamental volume, v being the packet velocity.

nonremovable degeneracy arises from the symmetry properties of H_1. Let us suppose that the ϕ_a represent free-particle momentum states, the operator H_0 describing the noninteracting fields. The closest analogs of the states of discrete-spectrum theory are the 1-particle states and the vacuum state. For these states the result $(\phi_b,R^{\pm}\phi_a)=\pm i\epsilon(Z_a-1)\delta_{ba}$ again holds. Here, moreover, on account of the symmetries of H_1 and the relativistic invariance of the theory, Z_a depends not just solely on the energy but solely on the particle in question. As in discrete-spectrum theory the constants Z_a, and hence $(\phi_b,R^{\pm}\phi_a)$ for these states, are independent of the normalization volume V.

For many-particle states the operator $R^{\pm}$ may be separated into two parts, one which is dependent on and the other which is independent of V. We shall call the latter the "singular part" and the former part the "remainder," writing

$$R^{\pm}=\mathrm{Sing}R^{\pm}+\mathrm{Rem}R^{\pm}. \tag{18}$$

The structure of these parts is easily visualized in terms of Feynman diagrams. $\mathrm{Rem}R^{\pm}$ corresponds to all those diagrams in which at least two real particles (as opposed to virtual particles) interact. The contribution to $R^{\pm}$ from such a diagram is of order $V^{-N/2}$ where N is the number of irreducible vertex parts leading to external real particles. Therefore $\mathrm{Rem}R^{\pm}$ makes no contribution to the normalization of the $\psi_a{}^{\pm}$. The normalization is completely determined by $\mathrm{Sing}R^{\pm}$, whereas the physically observed scattering is completely described by $\mathrm{Rem}R^{\pm}$.

$\mathrm{Sing}R^{\pm}$ corresponds to the diagrams in which the real particles involved (as well as the vacuum) undergo self-energy interactions with virtual quanta but otherwise suffer no change of state. One may therefore write generally

$$\mathrm{Sing}(\phi_b,\psi_a{}^{\pm})=Z_a{}^{\frac{1}{2}}\delta_{ba} \tag{19}$$

and

$$(\phi_b,\ \mathrm{Sing}R^{\pm}\phi_a)=\pm i\epsilon(Z_a-1)\delta_{ba}, \tag{20}$$

it being here not even necessary to specify $E_a=E_b$.[7] This completes the proof that

$$(\psi_b{}^{\pm},\psi_a{}^{\pm})=\delta_{ba}. \tag{21}$$

IV. MISCELLANEOUS RELATIONS

It is useful to have alternative statements of the facts expressed by Eqs. (19) and (20). We first introduce the convenient abbreviation

$$F_{ba}\equiv(\phi_b,F\phi_a) \tag{22}$$

for an arbitrary operator F, and then expand the Green's function of H in terms of the $\psi_a{}^{\pm}$:

$$G_{ba}{}^{\pm}(E)=\sum_c\frac{(\phi_b,\psi_c{}^{\pm})(\psi_c{}^{\pm},\phi_a)}{E-E_c\pm i\epsilon}. \tag{23}$$

[7] If H possesses any nonrelativistic features, such as in the problem where it describes a particle which is bound by a fixed potential but which scatters incident radiation, then the specification $E_a=E_b$ must be retained. Moreover Z_a will then generally vary from one stable bound level to another.

Taking the singular part of this equation one gets, with Eq. (19),

$$\mathrm{Sing}G_{ba}{}^{\pm}(E)=(E-E_a\pm i\epsilon)^{-1}Z_a\delta_{ba}. \tag{24}$$

This implies

$$\pm i\epsilon\ \mathrm{Sing}G_{ba}{}^{\pm}(E_a)=Z_a\delta_{ba}, \tag{25}$$

a result which could also be inferred from Eq. (6).

Using Eqs. (4b) and (10a), one may write

$$R^{\pm}(E)=[G_0{}^{\pm}(E)]^{-1}G^{\pm}(E)[G_0{}^{\pm}(E)]^{-1}-[G_0{}^{\pm}(E)]^{-1}, \tag{26}$$

and hence

$$R_{ba}{}^{\pm}(E)=(E-E_b\pm i\epsilon)(E-E_a\pm i\epsilon)\times\sum_c\frac{(\phi_b,\psi_c{}^{\pm})(\psi_c{}^{\pm},\phi_a)}{E-E_c\pm i\epsilon}-(E-E_c\pm i\epsilon)\delta_{ba}. \tag{27}$$

The singular part of this equation, with $E=E_a$, leads directly to Eq. (20). Differentiation of this equation with respect to E gives

$$\frac{\partial R_{ba}{}^{\pm}(E)}{\partial E}=\sum_c\left[1-\frac{(E_a-E_c)(E_b-E_c)}{(E-E_c\pm i\epsilon)^2}\right]\times(\phi_b,\psi_c{}^{\pm})(\psi_c{}^{\pm},\phi_a)-\delta_{ba} \tag{28}$$

and hence

$$\mathrm{Sing}[\partial R_{ba}{}^{\pm}(E)/\partial E]_{E=E_a}=(Z_a-1)\delta_{ba}, \tag{29}$$

a result which will be needed later.

Equations (25) and (29) also hold in discrete-spectrum theory provided the specification $E_a=E_b$ is added.

V. THE S-MATRIX

The elements of the S-matrix are defined by

$$(S^{\pm1})_{ba}=(\psi_b{}^{\mp},\psi_a{}^{\pm}). \tag{30}$$

Using Eqs. (9), (14), and (15), one may write

$$\begin{aligned}
&(Z_bZ_a)^{\frac{1}{2}}(S^{\pm1})_{ba}\\
&=(\phi_b,[1+R^{\pm}(E_b)G_0{}^{\pm}(E_b)][1+G_0{}^{\pm}(E_a)R^{\pm}(E_a)]\phi_a)\\
&=\delta_{ba}+(\phi_b,R^{\pm}(E_b)G_0{}^{\pm}(E_b)G_0{}^{\pm}(E_a)R^{\pm}(E_a)\phi_a)\\
&\quad+\frac{\frac{1}{2}}{E_b-E_a\pm i\epsilon}(\phi_b,[R^{\pm}(E_b)+R^{\pm}(E_a)\\
&\quad+(E_a-E_b)R^{\pm}(E_b)G_0{}^{\pm}(E_b)G_0{}^{\pm}(E_a)R^{\pm}(E_a)]\phi_a)\\
&\quad+\frac{\frac{1}{2}}{E_a-E_b\pm i\epsilon}(\phi_b,[R^{\pm}(E_a)+R^{\pm}(E_b)\\
&\quad+(E_b-E_a)R^{\pm}(E_b)G_0{}^{\pm}(E_b)G_0{}^{\pm}(E_a)R^{\pm}(E_a)]\phi_a)\\
&=\delta_{ba}\mp i\frac{\epsilon}{(E_b-E_a)^2+\epsilon^2}(\phi_b,[R^{\pm}(E_b)+R^{\pm}(E_a)]\phi_a)\\
&\quad+\frac{\epsilon^2}{(E_b-E_a)^2+\epsilon^2}(\phi_b,R^{\pm}(E_b)G_0{}^{\pm}(E_b)G_0{}^{\pm}(E_a)R^{\pm}(E_a)\phi_a)\\
&\underset{\epsilon\to0}{\longrightarrow}\delta_{ba}\mp2\pi i\delta(E_b-E_a)\\
&\qquad\times\{R_{ba}{}^{\pm}\mp\tfrac{1}{2}i\epsilon[\partial R_{ba}{}^{\pm}(E)/\partial E]_{E=E_a}\},
\end{aligned} \tag{31}$$

where the representation

$$\delta(E)=-\frac{1}{\pi}\frac{\epsilon}{E^2+\epsilon^2} \tag{32}$$

for the delta function has been used. Only the singular part of the term in $\partial R_{ba}^{\pm}(E)/\partial E$ contributes to the S-matrix in the limit $\epsilon\to 0$. Hence, use of Eqs. (20) and (29) and the formal identity $\delta(E_b-E_a)\delta_{ba}=(\pi\epsilon)^{-1}\delta_{ba}$ gives

$$\begin{aligned}(S^{\pm 1})_{ba}&=(Z_bZ_a)^{-\frac{1}{2}}\{[1\mp 2i\epsilon^{-1}(\pm i\epsilon\mp\tfrac{1}{2}i\epsilon)(Z_a-1)]\delta_{ba}\\&\qquad\mp 2\pi i\delta(E_b-E_a)\,\mathrm{Rem}R_{ba}^{\pm}\}\\&=\delta_{ba}\mp i\mathfrak{R}_{ba}^{\pm},\end{aligned} \tag{33}$$

where

$$\mathfrak{R}_{ba}^{\pm}=(Z_bZ_a)^{-\frac{1}{2}}\,\mathrm{Rem}\mathbf{R}_{ba}^{\pm} \tag{34}$$

and where boldface type is used to denote, for any operator, the energy shell operation

$$\mathbf{F}_{ba}\equiv 2\pi\delta(E_b-E_a)F_{ba}. \tag{35}$$

$\mathfrak{R}^{\pm}$ is the *renormalized* transition operator which describes physically observable scattering processes. Owing to the orthonormality conditions (21) the S-matrix, as defined by Eq. (30), is unitary.[8] $\mathfrak{R}^{\pm}$ therefore satisfies the probability conservation law

$$\mathfrak{R}^{\pm}\mathfrak{R}^{\mp}=\mathfrak{R}^{\mp}\mathfrak{R}^{\pm}=\pm i(\mathfrak{R}^{\pm}-\mathfrak{R}^{\mp}) \tag{36}$$

or, since $\mathfrak{R}^{\mp}=\mathfrak{R}^{\pm\dagger}$,

$$\mathfrak{R}^{\pm}\mathfrak{R}^{\pm\dagger}=\mathfrak{R}^{\pm\dagger}\mathfrak{R}^{\pm}=\mp 2\,\mathrm{Im}\mathfrak{R}^{\pm}. \tag{37}$$

In discrete-spectrum theory $R^{\pm}$ has no nonsingular part; hence $\mathfrak{R}^{\pm}=0$, $\psi_a{}^{+}=\psi_a{}^{-}$, and the S-matrix reduces to the triviality $S=1$.

VI. THE PARTICLE PROPAGATION FUNCTIONS AND THE RENORMALIZATION PROGRAM

The demonstration of the internal consistency of the renormalization program in quantum field theory is conveniently carried out in terms of irreducible diagrams, each of which is used as a replacement for an infinite class of diagrams.[9] In the computations for a given scattering process only irreducible diagrams need be considered, provided the products of diagonal one-particle matrix elements of the unperturbed Green's function $G_0^{\pm}(E)$ occurring in the expansion of Eq. (10d) are replaced by corresponding products of diagonal one-particle matrix elements of the perturbed Green's function $G^{\pm}(E)$:

$$G_{aa}^{\pm}(E)=\sum_c\frac{|(\phi_a,\psi_c^{\pm})|^2}{E-E_c\pm i\epsilon}. \tag{38}$$

When ϕ_a is a one-particle state the only terms which contribute to the above sum, other than $c=a$, are those for which $\psi_c^{\pm}$ involves two or more real particles with $E_c>E_a$. Expression (38) is then called the *modified propagation function* of the particle in question. Its structure may be displayed by writing it in the form

$$G_{aa}^{\pm}(E)=\frac{Z_a}{E-E_a\pm i\epsilon}+\sum_{c\atop E_c>E_a}\frac{|(\phi_a,\psi_c^{\pm})|^2}{E-E_c\pm i\epsilon}, \tag{39}$$

from which it is seen to have a behavior similar to Z_a times the unmodified propagation function $G_{0aa}^{\pm}(E)$ in the neighborhood of $E=E_a$. The *renormalized* propagation function is defined in such a way as to remove the factor Z_a:

$$\bar{G}_{aa}^{\pm}(E)\equiv Z_a{}^{-1}G_{aa}^{\pm}(E). \tag{40}$$

The practical computation of the modified propagation functions is conveniently carried out in terms of the *self-energy functions*. These functions are introduced by separating the right-hand side of Eq. (4b) into diagonal and off-diagonal parts:

$$\begin{aligned}&G_{ba}^{\pm}(E)\\&=\frac{1}{E-E_b\pm i\epsilon}[\delta_{ba}+H_{1ba}G_{aa}^{\pm}(E)+\textstyle\sum_{c\neq a}H_{1bc}G_{ca}^{\pm}(E)]\\&=\frac{1}{E-E_b\pm i\epsilon}[\delta_{ba}+\Sigma_{ba}^{\pm}(E)G_{aa}(E)],\end{aligned} \tag{41}$$

where

$$\Sigma_{ba}^{\pm}(E)=H_{1ba}+\sum_{c\neq a}H_{1bc}\frac{1}{E-E_c\pm i\epsilon}\Sigma_{ca}^{\pm}(E). \tag{42}$$

Iteration of Eq. (42) gives

$$\Sigma_{ba}^{\pm}(E)=(\phi_b,\,H_1[1-1_aG_0^{\pm}(E)H_1]^{-1}\phi_a), \tag{43}$$

where

$$1_a\equiv 1-\phi_a\rangle\langle\phi_a. \tag{44}$$

The diagonal one-particle elements $\Sigma_{aa}^{\pm}(E)$ are the self-energy functions, in terms of which one may write

$$G_{aa}^{\pm}(E)=\frac{1}{E-E_a-\Sigma_{aa}^{\pm}(E)\pm i\epsilon} \tag{45a}$$

$$=G_{0aa}^{\pm}(E)[1-\Sigma_{aa}^{\pm}(E)G_{0aa}^{\pm}(E)]^{-1}. \tag{45b}$$

Comparison of Eqs. (45a) and (39) allows one to infer[10]

$$[\Sigma_{aa}^{\pm}(E)]_{E=E_a\mp i\epsilon}=0, \tag{46}$$

$$[\partial\Sigma_{aa}^{\pm}(E)/\partial E]_{E=E_a\mp i\epsilon}=-\zeta_a, \tag{47}$$

$$Z_a=(1+\zeta_a)^{-1}. \tag{48}$$

An evaluation of the diagrams corresponding to Eq. (25) readily shows that the normalization constant Z_a associated with a many-particle state is simply the

[8] For the proof of unitarity when bound states are present see reference 5.

[9] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

[10] If the redefinition (7) of H_0 and H_1 had not been carried out initially then one would have obtained, in the limit $\epsilon\to 0$,

$$\Delta E_a=\Sigma_{aa}^{\pm}(E_a)=(\phi_a,\,H_1[1-1_aG_0^{\pm}(H_{0aa}+\Delta E_a)H_1]^{-1}\phi_a),$$

from which the level shifts can be computed.

product of the normalization constants associated with the individual particles in that state. Strictly speaking, a vacuum normalization constant should also be included, but since it is uniformly present in all states it is uniformly ignored. In a typical boson-fermion two-field theory in which the coupling is linear in the boson field and bilinear in the fermion field, the fermion and boson normalization constants are customarily called Z_2 and Z_3 respectively.

Let us consider the contribution to the operator $\mathfrak{R}^{\pm}$ from a given irreducible diagram. Suppose the numbers of initial fermions and bosons for this diagram are F_a and B_a respectively, and suppose the corresponding final numbers are F_b and B_b. Then

$$Z_a=Z_2{}^{F_a}Z_3{}^{B_a}, \quad Z_b=Z_2{}^{F_b}Z_3{}^{B_b}, \quad Z_bZ_a=Z_2{}^{F}Z_3{}^{B}, \tag{49}$$

where

$$F=F_a+F_b, \quad B=B_a+B_b. \tag{50}$$

Let V be the number of vertices in the diagram. Then the number of internal fermion lines is $V-\frac{1}{2}F$ (F must be an even number), and the number of internal boson lines is $\frac{1}{2}(V-B)$. Each internal fermion line contributes a modified fermion propagation function G_2, each internal boson line contributes a modified boson propagation function G_3, and each vertex contributes a modified vertex operator Γ (sum of all proper vertex diagrams) times the coupling constant g. Here we omit the $\pm$ signs and the energy dependence of these functions. Ignoring also the order in which these quantities must appear and the momentum-energy integrations in which they are involved, we may write the total contribution from the internal parts of the diagram schematically as $g^V\Gamma^V{G_2}^{V-\frac{1}{2}F}{G_3}^{\frac{1}{2}(V-B)}$.

The external lines do not contribute propagation functions since, as may be seen from the expansion of Eq. (10d), scattering diagrams must begin and end with vertices. Instead, each external line contributes a quantity

$$G_{cc}{}^{\pm}(E)[G_{0cc}{}^{\pm}(E)]^{-1}=[1-\Sigma_{cc}{}^{\pm}(E)G_{0cc}{}^{\pm}(E)]^{-1}, \tag{51}$$

in which the dangling unmodified propagator has been cancelled out by its inverse, and in which the subscript c refers to the particle in question. When this quantity is evaluated on the energy shell, with $E=E_c$, it reduces to Z_c. Therefore each external line contributes simply a normalization constant, the total external contribution being $Z_2{}^{F}Z_3{}^{B}$. Combining these results with Eqs. (34) and (49), one gets for the renormalized transition operator the schematic expression

$$\mathfrak{R}=T\sum(Z_2{}^{F}Z_3{}^{B})^{\frac{1}{2}}g^V\Gamma^V{G_2}^{V-\frac{1}{2}F}{G_3}^{\frac{1}{2}(V-B)}, \tag{52}$$

where the summation is over all irreducible diagrams and where T is a formal symbol replacing the weight factor $2\pi\delta(E_b-E_a)$ on the energy shell.

The well-known result that $\mathfrak{R}$ can be expressed entirely in terms of renormalized quantities[9] follows immediately. The renormalized particle propagation functions are

$$\bar{G}_2=Z_2{}^{-1}G_2, \quad \bar{G}_3=Z_3{}^{-1}G_3. \tag{53}$$

One may also include the vertex renormalization, although it plays no role in state-vector normalization:

$$\bar{\Gamma}=Z_1\Gamma \tag{54}$$

where

$$Z_1=(1+\zeta_1)^{-1}, \tag{55}$$

$$\zeta_1\gamma=g^{-1}[\partial\Sigma_2{}^{\pm}(E)/\partial A]_{E=E_2\mp i\epsilon,\,A=0}, \tag{56}$$

the latter quantity denoting a derivative of the fermion self-energy function with respect to a constant external boson field A, and γ denoting the unmodified vertex operator.[11] If now the renormalized coupling constant is introduced, namely

$$\bar{g}=Z_1{}^{-1}Z_2Z_3{}^{\frac{1}{2}}g, \tag{57}$$

one may write

$$\mathfrak{R}=T\sum\bar{g}^V\bar{\Gamma}^V{\bar{G}_2}^{V-\frac{1}{2}F}{\bar{G}_3}^{\frac{1}{2}(V-B)}. \tag{58}$$

VII. THE REACTANCE OPERATOR

In simple scattering theory the reactance operator is introduced by splitting up Eq. (10c) in the form

$$\left[1-H_1\mathcal{P}\frac{1}{E-H_0}\right]R^{\pm}(E) =H_1[1\mp\pi i\delta(E-H_0)R^{\pm}(E)] \tag{59}$$

with the use of the representation

$$1/(E\pm i\epsilon)=\mathcal{P}(1/E)\mp\pi i\delta(E), \tag{60}$$

the symbol $\mathcal{P}$ denoting the "principal value" when appearing in an integral. Equation (59) yields

$$R^{\pm}=K(1\mp\tfrac{1}{2}i\mathbf{R}^{\pm}), \tag{61}$$

where

$$K=\sum_a\left[1-H_1\mathcal{P}\frac{1}{E_a-H_0}\right]^{-1}H_1\phi_a\rangle\langle\phi_a. \tag{62}$$

The same procedure is valid in field theory provided a certain amount of caution is exercised in performing the energy-shell operation (35) which appears in Eq. (61). In simple scattering theory the matrix elements of the operator $R^{\pm}$ vary smoothly across the energy shell. In field theory, on the other hand, the singular part of this operator varies abruptly across the energy shell. Multiplying the singular part by $2\pi\delta(E_b-E_a)$ is equivalent to multiplication by $2/\epsilon$, but not so for the smooth nonsingular remainder. Since the quasi-continuous spec-

[11] As is well known, the modified vertex operator itself may be expressed as $\Gamma=\gamma+g^{-1}\delta\Sigma_2/\delta A$, where the derivative is now a variational one with respect to an arbitrary external field. Therefore the renormalized vertex operator reduces to γ in the neighborhood of $E=E_2$ and $k=0$, k being the momentum energy of the associated boson line. In a gauge-invariant theory, or in a theory in which the vertex operators commute, the differetiation in Eq. (56) becomes equivalent to the energy differentiation in Eq. (47) and $Z_1=Z_2$. [J. C. Ward, Phys. Rev. **78**, 182 (1950).]

trum is fine-grained with respect to ϵ there are many states lying on the energy shell, and not just a single state with E_b *exactly* equal to E_a. These closely packed states may be replaced by a single state with $E_b=E_a$ only if $2\pi\delta(E_b-E_a)$ is replaced in $\mathrm{Rem}\mathbf{R}^{\pm}$ by a factor much larger than $2/\epsilon$. This factor is the formal T-symbol of Eqs. (52) and (58).[12]

Using Eq. (20), one may write in the case of field theories

$$R_{ba}^{\pm}=\pm i\epsilon(Z_a-1)\delta_{ba}+\mathrm{Rem}R_{ba}^{\pm}, \tag{63}$$

$$\mathbf{R}_{ba}^{\pm}=\pm 2i(Z_a-1)\delta_{ba}+\mathrm{Rem}\mathbf{R}_{ba}^{\pm} \tag{64}$$

and hence, from Eq. (61),

$$\pm i\epsilon(Z_a-1)\delta_{ba}+\mathrm{Rem}R_{ba}^{\pm} =K_{ba}Z_a\mp\tfrac{1}{2}i\sum_c K_{bc}\,\mathrm{Rem}\mathbf{R}_{ca}^{\pm}. \tag{65}$$

The reactance operator is obtained by performing the energy-shell operation on K. It will appear presently that this operator has no singular part, and hence the limit $\epsilon\to 0$ may be taken at once in Eq. (65) with consequent elimination of the first term on the left. Multiplying the result by $(Z_bZ_a)^{-\frac{1}{2}}$ and performing the energy shell operation, one gets the renormalized Heitler integral equation

$$\mathfrak{R}^{\pm}=\mathfrak{K}(1\mp\tfrac{1}{2}i\mathfrak{R}^{\pm}), \tag{66}$$

where

$$\mathfrak{K}_{ba}=Z_b^{-\frac{1}{2}}Z_a^{\frac{1}{2}}\mathbf{K}_{ba}. \tag{67}$$

A contrast between Eqs. (34) and (67) is immediately apparent. The reactance operator $\mathbf{K}$ may be analyzed in terms of exactly the same diagrams as the transition operator $\mathbf{R}^{\pm}$, as comparison of Eqs. (10d) and (62) clearly shows. The propagation functions, modified vertex operators, and self-energy functions may all be again introduced, the only difference being that the imaginary parts $\pm i\epsilon$ are to be omitted and all integrals evaluated in the sense of the principal value. And yet the renormalization factors now appear with different exponents.

The explanation of the apparent contradiction is quite easy and has to do with the values to be assigned to the external lines associated with the particles in the *initial* state. These values are

$$G_{cc}(E_c)[G_{0cc}(E_c)]^{-1}=[1-\Sigma_{cc}(E_c)G_{0cc}(E_c)]^{-1}, \tag{68}$$

the $\pm$ signs being now omitted. Unlike the situation in Eq. (51), the energy evaluation is here taken directly at the pole of the modified propagation function rather than immediately above or below it. When $\pm$ signs are inserted expression (68) has the value Z_c. However, when they are omitted it is to be interpreted as having the value 1, because $\Sigma_{cc}(E_c)$ vanishes owing to the redefinition (7) of the unperturbed Hamiltonian. That is to say, the iterated self-energy diagrams which expression (68) evaluates are to be regarded as making no contribution since the particle self-energies (level shifts) have been adjusted to zero. For the same reason, the diagrams which contribute to the singular part of $\mathbf{R}^{\pm}$ make no contribution to $\mathbf{K}$.

The values to be assigned to the final particle lines, however, are determined by the requirement of smoothness for K across the energy shell, which demands an evaluation of the form

$$\lim_{E\to E_c} G_{cc}(E)[G_{0cc}(E)]^{-1}=Z_c. \tag{69}$$

The total external contribution in the present case is therefore $Z_2{}^{F_b}Z_3{}^{B_b}=Z_b$. Combination of this result with Eq. (67) leads once again to Eq. (52) with $\mathfrak{R}$ now replaced by $\mathfrak{K}$, and one sees that the renormalized reactance operator, like the renormalized transition operator, can be computed entirely in terms of renormalized quantities.

[12] The meaning of the T-symbol is expressed by the identity $2\pi=T\Delta E$, where ΔE is the level separation of the quasi-continuous spectrum. For a single particle $\Delta E=v\Delta p=2\pi v/L$, where $\Delta p=2\pi/L$ is the momentum interval and v is the velocity of the particle. Therefore T may be regarded as the time L/v for the particle to traverse the fundamental volume. (The units here are such that $\hbar=1$.) In the computation of cross sections the square of the absolute value of the transition operator is needed. The square of of the delta function is therefore encountered on the energy shell. Division by T cancels one delta function and gives the transition *rate*, while the remaining delta function has the effect of introducing a "density of final states."

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