states and the ground state where the shell model behavior is most apparent and the single-particle orbits best defined.

It is to be emphasized that the equivalent potential which we have determined is constructed to give the correct scattering of the neutron by the nucleus, i.e., to predict correctly the asymptotic behavior of the wave function. This does not imply that the actual wave function for the system is also given correctly everywhere by this interaction. The actual strongly correlated and highly mixed wave function is instead given in terms of the wave function determined by the equivalent uniform potential by an extremely complicated transformation [Eq. (3)]. Thus we expect that the Weisskopf complex potential can be used to predict only the asymptotic behavior of the scattering wave function, any more detailed information being available only if the transformation is explicitly constructed. This situation is analogous to that which exists in the shell model theory of the nucleus where again predictions of the detailed behavior of the nuclear ground state can be made only if the departures of the nuclear wave function from the shell model wave function due to the strong particle-particle forces are determined.

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Two-Nucleon Potential from the Cut-Off Yukawa Theory*

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Using the Yukawa theory with cutoff, the two-nucleon interaction is calculated up to fourth order in the coupling constant. The resulting potential at large distances ($\gtrsim 10^{-13}$ cm) is similar to the well-known potential with no cutoff. At small distances, however, the tensor potential approaches zero, and the central potential is strongly repulsive. The details are essentially determined by two parameters, the coupling constant and cutoff. The Schrödinger equation for the two-nucleon problem at low energies is solved numerically, and values are obtained for the binding energy, the quadrupole moment, and the four n-p scattering parameters. It is found that the triplet experimental values are reproduced quite well, and the singlet parameters reasonably well by the same coupling constant and cutoff required to explain pion-nucleon scattering and photoproduction.

I. INTRODUCTION

SINCE the time that Yukawa first proposed to explain nuclear forces as a manifestation of the exchange of field quanta by nucleons, numerous papers have been published on the predicted properties of nuclear systems due to this exchange of pions by nucleons.[1–3] To make any sort of progress, utilization has usually been made of perturbation expansions in the coupling constant as well as the so-called static approximation in which nuclear recoil is almost completely neglected. It has been found that with a local theory (no cutoff) the resulting two-nucleon potential has a strong singularity at small distances, and that this singularity is aggravated by including higher order terms in the expansion. The Schrödinger equation for a two nucleon system is not soluble with such a potential, and the tendency has been to replace the interaction at small distances with a phenomenological, infinite repulsive core. Thus, Brueckner and Watson[2] were able to fit most of the low-energy n-p data, treating the widths of the repulsive cores in the singlet and triplet states as adjustable parameters. Similarly, Taketani[1] utilizing the depth of the cores as an additional parameter, found that best agreement was obtained to the low energy n-p data with infinite repulsive cores. Neglect of nucleon recoil has been justified by the argument that one was interested primarily in low-energy phenomena, where the nucleon velocity is small. Attempts to include recoil terms,[2,3] have not led to conclusive results.

Recently, Chew[4] has shown that reasonable predictions for photo-meson production, pion-nucleon scattering etc. are obtained by the cut-off form of the Yukawa theory. There are two adjustable parameters in this theory, the renormalized coupling constant, f^2, and the cut-off energy of virtual mesons, ω_m. The purpose of the present paper is to show that all of the low energy n-p parameters (i.e., the binding energy, the quadrupole moment as well as the four scattering parameters) can be produced with reasonable accuracy by the above theory with the same values for f^2 and ω_m as are required by pion-nucleon scattering and photoproduction.

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[1] M. Taketani *et al.*, Progr. Theoret. Phys. **7**, 45 (1952).

[2] K. A. Brueckner and K. M. Watson, Phys. Rev. **92**, 1023 (1953).

[3] E. M. Henley and M. A. Ruderman, Phys. Rev. **92**, 1036 (1953).

[4] G. F. Chew, Phys. Rev. **95**, 1669 (1954).

Since the interaction Hamiltonian of reference 4 assumes that the motion of the nucleons can be neglected during the transit times of the pions, for purposes of computing the potential, we consider a physical situation in which two very slowly moving nucleons approach each other and are scattered. (By very "slowly" it is meant that the *kinetic* energy of the nucleons is small compared to the average *total* energy of an exchanged pion.) In the standard way, we separate out those diagrams of the transition matrix which do not have two simultaneous bare nucleon lines between two exchanged-meson lines. The sum of such diagrams is then called the potential V. The justification for this procedure is that if we use this V in the Schrödinger equation to solve the nucleon-nucleon scattering problem, we obtain exactly all of the diagrams included in the original transition matrix. The approximation considered here then, is that terms involving $\omega \pm k^2/M$ where ω is the meson energy and k^2/M is the nucleon energy, are consistently replaced by ω, while energy denominators, such as occur in the solution of the two nucleon Schrödinger equation, k^2/M, are kept intact. Further, we utilize the potential to calculate the deuteron properties on the assumption that the mean velocity of the nucleons in the deuteron is small.

II. DISCUSSION OF THE POTENTIAL

Using the above outlined technique, the calculation of the potentials is straightforward. For the interaction between pions and nucleons, we take[4]

$$H_{\text{int}}=(4\pi)^{\frac{1}{2}}\frac{f_0}{\mu}\sum_{N=1}^{2}\sum_{\lambda}\int d\mathbf{r}\rho(\mathbf{r}-\mathbf{r}_N)\tau_\lambda{}^N\boldsymbol{\sigma}^N\cdot\nabla\phi_\lambda(\mathbf{r}), \quad (1)$$

where f_0 is the dimensionless, unrationalized, unrenormalized coupling constant ($\hbar=c=1$), μ is the pion mass, $\rho(\mathbf{r})$ is the "source function," with the property $\int\rho(\mathbf{r})d\mathbf{r}=1$, $\boldsymbol{\sigma}^N$, $\boldsymbol{\tau}^N$ are the Pauli spin and isotopic spin operators for the Nth nucleon respectively, $\phi_\lambda(\mathbf{r})$ are the three real components of the pion field and $\mathbf{r}_N$ is the position vector of the Nth nucleon. It is useful to introduce the function $v(k)$ which is the Fourier transform of $\rho(\mathbf{r})$,

$$v(\mathbf{k})=\int e^{i\mathbf{k}\cdot\mathbf{r}}\rho(\mathbf{r})d\mathbf{r} \quad (2)$$

and the parameter k_m, defined by

$$\omega_m=(k_m{}^2+\mu^2)^{\frac{1}{2}},$$

where ω_m is the maximum energy for which $v(k)$ differs appreciably from zero. Using (1) we obtain for the second- and fourth-order potentials

$$V_2(r)=-\frac{4\pi}{(2\pi)^3}\left(\frac{f}{\mu}\right)^2\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2\int\frac{d\mathbf{k}v^2(k)e^{i\mathbf{k}\cdot\mathbf{r}}}{\omega_k{}^2}\boldsymbol{\sigma}^1\cdot\mathbf{k}\boldsymbol{\sigma}^2\cdot\mathbf{k}, \quad (3)$$

$$V_4(r)=-\frac{(4\pi)^2}{(2\pi)^6}\left(\frac{f}{\mu}\right)^4\int\int\frac{d\mathbf{k}_1d\mathbf{k}_2}{\omega_1{}^3\omega_2}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}$$
$$\times\left[\left(\frac{3}{\omega_2}+\frac{2\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2}{\omega_1+\omega_2}\right)(\mathbf{k}_1\cdot\mathbf{k}_2)^2\right.$$
$$+\boldsymbol{\sigma}^1\cdot(\mathbf{k}_1\times\mathbf{k}_2)\boldsymbol{\sigma}^2\cdot(\mathbf{k}_1\times\mathbf{k}_2)$$
$$\left.\times\left(\frac{3}{\omega_1+\omega_2}+\frac{2\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2}{\omega_2}\right)\right]v^2(k_1)v^2(k_2), \quad (4)$$

($\mathbf{r}$ is the internucleon distance) which except for the $v(k)$ factors are identical to the corresponding terms of formula (60) of reference 2. If $\rho(\mathbf{r})$ is spherically symmetric, it follows from Eq. (2) that $v(k)$ is also angle independent. The angular integrations are then elementary but tedious and ultimately yield:

$$V_2(r)=-\frac{2}{3\pi}\left(\frac{f}{\mu}\right)^2\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2\left\{\frac{\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2}{r}\int_0^\infty\frac{k^3dkv^2(k)}{\omega_k{}^2}\sin kr\right.$$
$$+3S_{12}\int_0^\infty\frac{kdkv^2(k)}{\omega_k{}^2}$$
$$\left.\times\left[\left(\frac{k^2}{3r}-\frac{1}{r^3}\right)\sin kr+\frac{k}{r^2}\cos kr\right]\right\}, \quad (5)$$

and

$$V_4(r)=-\frac{2}{\pi^2}\left(\frac{f}{\mu}\right)^4\int_0^\infty\int_0^\infty\frac{k_1k_2dk_1dk_2v^2(k_1)v^2(k_2)}{\omega_1\omega_2(\omega_1+\omega_2)}$$
$$\times\left\{\left[\frac{6}{\omega_1\omega_2}+(3+2\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2)\left(\frac{1}{\omega_1{}^2}+\frac{1}{\omega_2{}^2}\right)\right]\right.$$
$$\times\left[f_1''f_2''+\frac{2f_1'f_2'}{r^2}\right]$$
$$+\tfrac{2}{3}S_{12}\left[\frac{4\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2}{\omega_1\omega_2}+(3+2\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2)\right.$$
$$\times\left.\left(\frac{1}{\omega_1{}^2}+\frac{1}{\omega_2{}^2}\right)\right]\left[\frac{f_1'f_2'}{r^2}-\frac{f_1''f_2'}{r}\right]$$
$$+\tfrac{2}{3}\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2\left[\frac{4\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2}{\omega_1\omega_2}+(3+2\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2)\right.$$
$$\left.\left.\times\left(\frac{1}{\omega_1{}^2}+\frac{1}{\omega_2{}^2}\right)\right]\left[2\frac{f_1''f_2'}{r}+\frac{f_1'f_2'}{r^2}\right]\right\}, \quad (6)$$

where $f_i=\sin(k_i r)/r$; $i=1, 2$, the prime means differentiation with respect to r, and $S_{12}=(3\boldsymbol{\sigma}^1\cdot\mathbf{r}\boldsymbol{\sigma}^2\cdot\mathbf{r}/r^2)-\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2$ as usual.

In the formulas (3) to (6), the unrenormalized coupling constant, f_0, does not appear; instead the renormalized coupling constant, f, appears. This replacement, as is well known, has the effect of including many seemingly higher-order terms in the expansion of the potential. Thus, for example, if the coupling constant in the second order potential, Eq. (5), were f_0 instead of f, then this term would correspond to the exchange of a single meson by two "bare" nucleons. If one modifies this term to include also those diagrams in which various numbers of mesons are emitted and reabsorbed by the same nucleon while a single meson is exchanged, then the sum of this series is equal to the one-meson exchange term provided only that we replace f_0 by f. This in turn means that one may speak of a single meson being exchanged between two physical nucleons, i.e., nucleons which are modified by their associated mesonic clouds. Similarly, in the case of two-meson exchange, one can speak of the crossing of two mesons between two physical nucleons. In this situation, however, one may not so simply state that the effect of the nucleons not actually being bare is equivalent to replacing f_0 by f. For there are the well-known multiple-scattering corrections[2,3] which will not be included in the potential by this method. Thus, use of the renormalized coupling constant in the potential enables us to include many seemingly higher-order terms. However, radiative corrections, which go at worst like f^6, have been completely left out; this neglect is further discussed below.

The effect which the factor $v^2(k)$ has on the behavior of the above potentials is worthy of notice. To observe the small-r dependence one may expand the integrands of (5) and (6) in powers of r and keep only the first term. It is easily shown that there are no singularities in the potentials, that at small r the second- and fourth-order central potentials are strongly repulsive in singlet as well as in triplet even states, and that the second- and fourth-order tensor potentials approach zero with r. All the details in this region are completely determined by $v(k)$ and f^2. Further, it can be shown that for sufficiently small r the behavior of the potentials, to a good approximation, is given by a certain function of the argument, $k_m r$, which means that a change in k_m is equivalent to changing the scale of r in this region. For large r, on the other hand, the potentials, (5) and (6) coincide with the corresponding formulas of reference 2, showing that, as expected, $v(k)$ has little effect in this region.

To see the functional dependence of the potentials on $v(k)$ more clearly, let us examine (3) or (5) more closely in the absence and presence of $v(k)$. To be specific, let us concentrate on the central part of the force and in order to be able to evaluate integrals analytically, let us select

$$v(k)=k_m^2/(k^2+k_m^2). \tag{7}$$

The central part of Eq. (3) in even states can be written in general as

$$V_{2c}(r)=\frac{4\pi}{(2\pi)^3}\left(\frac{f}{\mu}\right)^2\int d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}v^2(k) - \frac{4\pi}{(2\pi)^3}\mu^2\left(\frac{f}{\mu}\right)^2\int\frac{d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}v^2(k)}{\omega_k^2}. \tag{8}$$

In the no-cutoff theory, $v^2(k)=1$ and (8) becomes

$$V_{2c}(r)=4\pi\left(\frac{f}{\mu}\right)^2\delta(\mathbf{r})-\mu f^2\frac{e^{-\mu r}}{\mu r}, \tag{9}$$

which corresponds to a delta-function repulsion at the origin and Yukawa attraction for $r\neq 0$. If now, however, we use $v(k)$ as given by (7), there results

$$V_{2c}(r)=\mu f^2\frac{1}{2}\left(\frac{k_m}{\mu}\right)^3 e^{-k_m r}-\frac{\mu f^2}{(1-\mu^2/k_m^2)^2} \times\left\{\frac{e^{-\mu r}}{\mu r}-e^{-k_m r}\left[\frac{1}{\mu r}+\frac{1}{2}\frac{k_m}{\mu}\left(1-\frac{\mu^2}{k_m^2}\right)\right]\right\}, \tag{10}$$

in which the first term on the right corresponds to the delta function in Eq. (9) and the second term to the Yukawa potential. In the cut-off theory, since $\omega_m\cong 6\mu$, (10) approaches the Yukawa potential for large $r(\gtrsim 1/\mu)$ but remains finite for $r=0$. Specifically in the limit of small r, equation (10) becomes

$$V_{2c}(0)=1+\frac{1}{2}\left(\frac{k_m}{\mu}\right)^3-\frac{3}{2}\left(\frac{k_m}{\mu}\right)$$

and approaches plus infinity as $(k_m/\mu)\to\infty$, which is identical to the result of (9). Also, it is clear from Eq. (10) that if for $r\neq 0$ one takes the limit as $(k_m/\mu)\to\infty$, then one obtains the usual result, namely, (9). However, in our case, where k_m is finite, the delta-function repulsion of (8) becomes spread out away from the origin and gives a repulsive core of nonzero width which is determined by the values of k_m and f^2.

In evaluating the potentials (5) and (6) one has substantial freedom in selecting the function, $v(k)$. A basic feature of the cut-off theory is that the precise functional form of $v(k)$ is irrelevant as far as predictions of the theory are concerned; the only requirements on $v(k)$ being that it be close to unity for values of k less than k_m and be effectively zero for k greater than k_m. This invariance of the theory does not mean, for example, that the potentials themselves are independent of the form of $v(k)$ but only that physical quantities which are predicted by these potentials are. The numerical value of k_m was obtained by the comparison of the

theory to pion-nucleon scattering data.[5] Specifically, the quantities which were compared to experiment were integrals of the approximate form:

$$\text{(a)} \quad \int_0^\infty v^2(k)d(k^2)$$

and

$$\text{(b)} \quad \int_0^\infty v^2(k)dk.$$

Selecting $v(k)$ as a square cutoff, the first of these integrals was defined to be k_m^2 and the second, k_m. If now a different functional form for $v(k)$ is selected we impose the condition on $v(k)$ that

$$\int_0^\infty v^2(k)d(k^2)=k_m^2$$

and check the consistency by seeing how close the integral (b) is to k_m.

The first evaluation of the potentials was done with a square cut-off function. This forcing of the integrands of (5) and (6) to become zero very abruptly, led to oscillations in the potentials for large values of r, and as such was not convenient for the numerical solution of the n-p differential equations. On the other hand, a $v(k)$ with a long tail makes evaluation of the potentials themselves difficult. As a compromise between these two extremes it was decided to select a Gaussian cut-off function,

$$v(k)=\exp(-k^2/2k_m^2).$$

This choice of $v(k)$ is satisfactory since with it the integral (a) becomes k_m^2 and the integral (b) is equal to $\frac{1}{2}(\pi)^{\frac{1}{2}}k_m$ which is within 10% of k_m. The potentials (5) and (6) were evaluated numerically with this choice of $v(k)$ and are plotted in Figs. 1 and 2.

Evaluation of (5) and (6) in the case of $\omega_m=6\mu$ and $f^2=0.089$, brings out the following characteristics. The second-order tensor force approaches zero at the origin, is then attractive all the way out, and has its maximum attraction of about 140 Mev near $r=0.6/\mu$. The fourth-order tensor potential also approaches zero with r, but on the other hand, is only about one-third as large and of the opposite sign, thus yielding an over-all attractive tensor force of about 100 Mev at max. As for the second-order central force, the possibility of subtracting out the delta function repulsion [Eq. (8)] leaves the attractive part of this potential smaller than "normal" by a factor of $\sim(\mu/k_m)^2$. Evaluation of the central part of Eq. (5) for $\omega_m=6\mu$, $f^2=0.089$, shows that although in close the repulsion is of order 1 Bev, the maximum attraction is only about 5 Mev; this is not nearly enough attraction to give the large singlet scattering length required by experiment. The fourth-order singlet potential, on the other hand, has a maximum attraction of about 70 Mev for the same choice of f^2 and k_m. Thus, in the singlet state, the second-order attractive force is effectively zero and the major contribution to it comes from the fourth and possibly higher orders. Similarly, in the triplet state, the fourth-order central potential is substantially larger than the second order. The large-

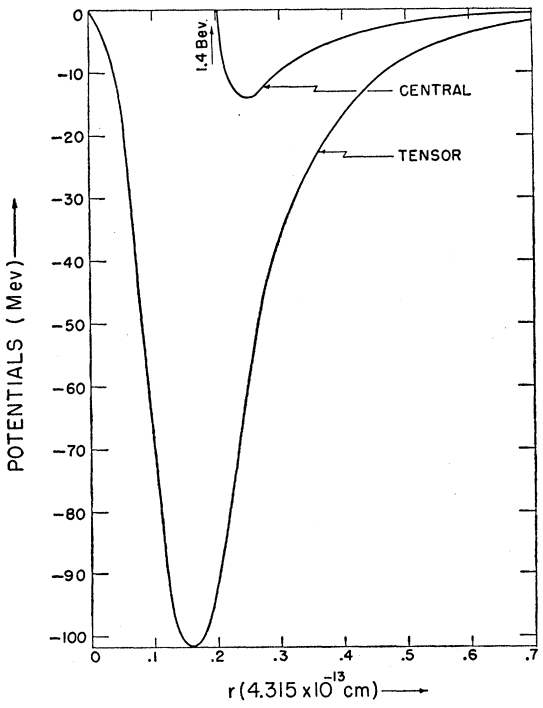

FIG. 1. Second- plus fourth-order central and tensor potentials in even angular momentum states for $f^2=0.089$ and $\omega_m=6\mu$.

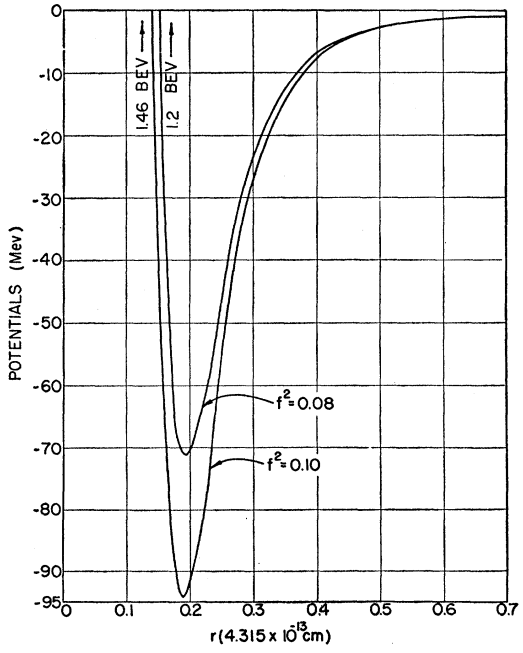

FIG. 2. Second- plus fourth-order singlet potentials in even angular momentum states for $\omega_m=6\mu$, $f^2=0.089$ and $f^2=0.10$.

[5] G. F. Chew, Phys. Rev. 95, 285 (1954).

TABLE I. Triplet parameters obtained with the potentials using the indicated values of f^2 and ω_m.

f^2	$\omega_m(\mu)$	P_D (%)	Q (10^{-27} cm^2)	r_{0t} (10^{-13} cm)	a_t (10^{-13} cm)
0.089	6.00	6.80	2.90	1.75	5.42
0.093	5.60	7.00	3.08	1.81	5.47
Experimental		2–8	2.74	1.70	5.39

ness of the tensor potential, however, leads one to feel that higher-order corrections to the triplet potentials will not be as important as those to the singlet.

The accuracy with which the second- and fourth-order terms alone represent the potential is difficult to gauge. As of now, no attempts have been made to derive the higher-order corrections to the potential within the cut-off theory. Indication from other considerations makes the author feel that these corrections will not radically change the potentials computed here. Brueckner and Watson[2] have concluded as a result of their studies that for $r \gtrsim 0.6/\mu$ the second- and fourth-order terms are good approximations to the potential, in that multiple scattering corrections are unimportant in this region, amounting to about 30% at maximum and decreasing very rapidly with r. In the present case, the maximum attraction of the triplet potential is very near to $r=0.6/\mu$ and thus the central potential computed here should be a reasonable approximation. The core of the singlet potential, on the other hand, occurs at about $r=0.5/\mu$ and thus may require some higher order corrections in this region. In any event, a very strong assumption made here is that higher-order corrections will not remove the repulsive core.

III. COMPARISON TO EXPERIMENT

The potentials (5) and (6) were evaluated numerically for even angular momentum states using

$$v(k)=\exp(-k^2/2k_m^2)$$

for $\omega_m=5.6\mu$ and 6μ. The deuteron characteristics and the n-p scattering parameters were obtained using the Blatt-Kalos routine on the Illinois electronic computer.[6] The characteristics of this code are such that it will multiply the given tensor force by a constant which is determined by the condition that the potential in question shall give the exact value of the binding energy. Subsequently, it utilizes the resulting potential to compute the other physical properties of the low-energy n-p system. Now in the present case, the value of f^2 was adjusted in this fashion to give the exact binding energy. This value of f^2 was then used in the remainder of the computation.

The results for the triplet state are tabulated in Table I for both values of ω_m. The corresponding S and D deuteron wave functions, $u(r)$ and $w(r)$ respectively are plotted in Fig. 3. Remarkably good results were obtained for $\omega_m=6\mu$ and $f^2=0.089$. The agreement for $\omega_m=5.6\mu$, although not as good as the other case, still gives results which probably lie within errors due to the approximations made in neglecting higher-order terms (e.g., the f^6 and higher terms are neglected). We conclude thus, that all the triplet parameters of the n-p system can be obtained, via the cut-off meson theory, provided only that higher-order corrections do not appreciably change the qualitative features of the potentials, which have been obtained here.

With the above values of ω_m and f^2, the singlet parameters were ostensibly not in as good agreement with experiment. The singlet potential, even though it has about three times the depth of the triplet central potential, does not have enough attraction with $f^2=0.089$, to give the large singlet scattering length which is observed. However, the conclusions of Brueckner and Watson,[2] that corrections to the potential due to multiple scatterings are *attractive* and of the order of 30% near $r=0.6/\mu$ tends to make the lack of agreement appear not too unreasonable. Since for a value of $f^2=0.089$, the maximum attraction in the singlet potential occurs at about $r=0.6/\mu$, it seems plausible to simulate some of these higher-order multiple scattering effects by an increase in the coupling constant. Thus it is found that for $\omega_m=6\mu$, a value of $f^2=0.10$ (this differs by 12%, from the value of f^2 required in the triplet state) gives the correct singlet scattering length and yields 2.54×10^{-13} cm for the value of the singlet effective range. Similarly, for $\omega_m=5.6\mu$, a value of $f^2=0.11$ was required to give the exact singlet scattering length and yield 2.65×10^{-13} cm for the effective range (see Table II). Thus, provided the higher-order corrections to the singlet potential do not appreciably change the sign and width of the core, it appears hopeful that the cut-off theory also predicts correctly the singlet behavior of the n-p system.

Chew and Low,[7] using an improved technique, have shown that agreement with the low-energy pion-

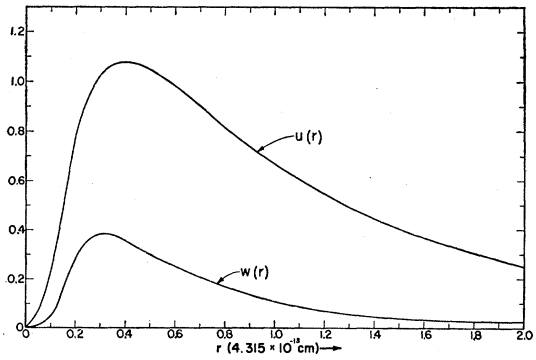

FIG. 3. The deuteron wave functions, normalized so that $\int_0^\infty (u^2+w^2)dr=1$, where $u(r)/r$ and $w(r)/r$ correspond to the S and D states respectively.

[6] M. H. Kalos and J. M. Blatt, Internal Report No. 50, University of Illinois, Digital Computer Laboratory, 1953 (unpublished).

[7] G. F. Chew and F. E. Low (to be published).

phenomena is obtained by a coupling constant in the neighborhood of 0.07–0.08 and a cut-off energy near 6μ. These values are quite close to those used here to give best agreement for the low-energy n-p system. Thus, except possibly for peculiarities which may appear in higher orders, it seems that the low-energy n-p system can be predicted without recourse to adjustable parameters.

The application of these potentials to problems at higher energies has not yet been considered by the present author. In making the calculation here, nucleon kinetic energies have been consistently ignored relative to total meson energies. How large the nucleon energies may become before the calculation breaks down is a question which is difficult to answer without further work. Since the depth of the attractive part of the triplet central force is of the order 15 Mev, it is expected that the nucleon kinetic energies in the deuteron also are of this magnitude. The agreement obtained for the deuteron thus leads one to speculate that the potentials may be reliable for nucleon energies beyond the range of reliability of the effective range theory. Indeed, Fujii *et al.*[8] have applied their potentials to n-p scattering at 40 and 90 Mev and have concluded that the characteristic features of the high-energy nucleon-nucleon scattering are reproduced quite well. It seems reasonable, therefore, that the potentials derived here may also apply at these higher energies.

[8] S. Fujii *et al.*, Progr. Theoret. Phys. **11**, No. 1 (1954).

TABLE II. The singlet effective ranges obtained by adjusting f^2 to give the exact singlet scattering length.

f^2	$\omega_m(\mu)$	r_{0s} (10^{-13} cm)
0.10	6	2.54
0.11	5.6	2.65

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The author would like to express his deepest gratitude to Professor G. F. Chew for his suggesting this problem and his constant help throughout. Without his guidance and encouragement this work would not have been possible. He would also like to thank Professor F. E. Low and Dr. Y. Yamaguchi for many helpful discussions. Also, thanks are due Professors R. E. Meagher and J. P. Nash at the Illinois computer laboratory for the liberal machine time allocated for this problem.

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State-Vector Normalization in Formal Scattering Theory*

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The familiar problem of state-vector normalization and, in field theories, the related problem of charge renormalization are shown to arise in a natural manner in the formal scattering theory introduced by Lippmann and Schwinger. The mathematical arguments necessary for dealing with these problems are developed entirely within the framework of the formal theory and lead to the customary rules for the construction of the renormalized S-matrix and reactance operator, provided mass renormalization is simultaneously carried out and the one-to-one correspondence between perturbed and unperturbed eigenstates is set up in a "natural" fashion.

I. INTRODUCTION

LIPPMANN and Schwinger[1] have shown that the stationary states which describe scattering processes for a given system may be represented formally by

$$\psi^{\pm}=\lim_{\epsilon\to 0}\frac{\pm i\epsilon}{E-H\pm i\epsilon}\phi, \qquad (1)$$

where H is the Hamiltonian operator of the system, ϵ is a positive infinitesimal, E is the energy of the state in question, and the vector ϕ represents a plane wave of the same energy. The $+$ sign refers to the states with outgoing or "retarded" scattered waves, and the $-$ sign to the incoming or "advanced" wave states. That $\psi^{\pm}$ are indeed eigenvectors of H corresponding to the eigenvalue E is immediately seen by multiplying Eq. (1) on the left by $E-H\pm i\epsilon$ and then passing to the limit.

Although Eq. (1) was initially introduced in the limited context of simple scattering theory, its use as a method of constructing eigenvectors of an operator H has a much wider range of validity. For example, H may be a finite matrix, or an operator with discrete rather than continuous eigenvalues. In a review article (to be published) the author has used Eq. (1) as a starting point for a discussion of bound-state perturbation theory. It is the purpose of the present note, however,

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[1] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).