Theory of Neutron Reactions with Nuclei at Low Energy*

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A theory of the interaction of low-energy neutrons with nuclei has been developed using methods previously applied to the study of the nuclear ground state. It is show that an average potential is predicted by the theory which very closely resembles that used by Feshbach, Porter, and Weisskopf in their studies of the neutron cross sections. The calculated parameters for scattering at a few Mev are a real depth of 41 Mev and an imaginary part somewhat less than 1 Mev. The theory also predicts the appearance of characteristic narrow compound-state resonances in the cross section; these are the result of sharp fluctuations in the real and imaginary part of the equivalent potential acting on a neutron which are the result of appreciable coupling between the single-particle and compound-nucleus states. An estimate of the level width is given for a simple class of compound-state levels and is of the order of typical widths observed experimentally.

A comparison of these results is made with a theory of Wigner, Lane, and Thomas; the principal difference is in the much smaller imaginary part of the potential determined by this theory.

I. INTRODUCTION

FESHBACH, Porter, and Weisskopf[1] have shown that certain average cross sections for neutron-nuclear reactions can be represented by a model consisting of a single neutron scattering on a complex neutron scattering on a complex square well. It is the primary aim of this paper to show how this model may be derived and the complex potential calculated from the Schrodinger equation for many nucleons in strong interaction with each other. The secondary aim is to examine the finer detail of the resonance structure which will be seen if one does not average cross sections over finite energy ranges.

The basis of the method is the construction of a transformation operator which relates the independent particle model of the nucleus to the actual nuclear state. This operator has to be constructed so that not only the stationary states of the nucleus are accurately described but also so that in the model a single neutron scatters on a potential well in the same way as it would on the actual nucleus. It is found that the methods developed for discussing the gross properties[2–5] and also the detailed properties[6–8] of the nuclear ground-state wave function can be extended so as to apply to the present problem in the energy range under discussion (incident neutron 0 to 5 Mev). In Sec. II, an equivalent one particle potential for the incident neutron is constructed and is found to be complex and strongly energy dependent at certain energies. An approximation related to smoothing out this energy dependence enables us to calculate this potential which is found to have a real part of about 41 Mev and an imaginary part of about 1 Mev which increases slowly with energy.

The method can be described also in terms of shell model language which gives a useful interpretation of the collision process and approximations though of course it is a picture which must not be taken too literally. If the neutron was bound it would appear to move simply in the shell-model potential which is real and can be calculated[2–4] from two-body interactions within the nuclear medium. An incident neutron above zero energy sees a formally similar potential well, however, there are two important differences: (1) the incident wave when matched on to the single-particle wave functions in the real well leads to a broad resonance; (2) at these energies there are, in addition to excited states of one independent particle in the well, also excited states of the same energy of two (or more) independent particle modes of motion. The potential seen by the incident neutron contains an imaginary part which is the result of coupling between the single particle and two (or more) particle levels. These two particle levels themselves are also broadened by similar coupling. The imaginary part of the potential appears in a term containing a sum over discrete states; if this sum is replaced by an integration one no longer obtains the strong effects of the individual levels as the energy is varied but rather obtains an average effective potential; it is this which corresponds to the imaginary term in the model of Feshbach, Porter, and Weisskopf.

Certain simple aspects of the resonance fine structure are examined in Sec. III using a different approximation for the potential in which the neutron scatters. We consider a somewhat idealized case in that we suppose that at neutron energies of two or three Mev, where the single particle levels are very broad and overlap many more complicated compound nucleus levels, we can fix our attention on the sharp resonance resulting from a particularly simple type of excitation. This is possible

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[1] Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954).
[2] Brueckner, Levinson, and Mahmoud, Phys. Rev. **95**, 217 (1954).
[3] K. A. Brueckner, Phys. Rev. **96**, 508 (1954).
[4] K. A. Brueckner, Phys. Rev. **97**, 1353 (1955).
[5] K. A. Brueckner and C. A. Levinson, Phys. Rev. **97**, 1344 (1955).
[6] R. J. Eden and N. C. Francis, Phys. Rev. **97**, 1366 (1955).
[7] Brueckner, Eden, and Francis, Phys. Rev. **98**, 1445 (1955).
[8] Brueckner, Eden, and Francis, Phys. Rev. **99**, 76 (1955).

only if we assume that the overlap in energy of the compound nucleus levels can be neglected; a consequence is that broadening of the levels occurs only because of transitions to the single-particle level. It is found that if two particles are in excited states, there is an imaginary part to the effective potential acting on this state which arises from tranisitions to single-particle excited states. On the evarage the imaginary part is of the order of 10 kev at excitations corresponding to the incident neutron energies of a few Mev; it corresponds to the width of a type of sharp "compound state" resonance. It is also found that the real part of the equivalent potential well seen by the neutron changes rapidly in this narrow energy range. The fluctuation in the real potential is the same order of magnitude ($\sim$2 Mev) as the width of the giant neutron resonance. It is this fluctuation which is responsible for the narrow resonance may give either a peak or a peak followed by a sharp dip in the cross section as a function of energy.

In Sec. IV, the results of Secs. II and III are summarized and a comparison is made with a theory recently developed by Wigner, Lane, and Thomas.[9] Finally some concluding remarks are made in Sec. V.

II. EQUIVALENT SCATTERING POTENTIAL

A. Real Part of the Potential

We consider the matrix element for elastic scattering of a neutron on a nucleus. The wave function of the particles in the nucleus is antisymmetrized, and we will also later introduce a device which takes account of the indistinguishability of the incoming neutron and the neutrons in the nucleus.

The equivalent potential for elastic scattering has been derived by Francis and Watson,[10] neglecting exchange effects and assuming knowledge of the nuclear wave functions. Adopting their formula to the notation used in references 2–8, this potential is

$$V_0=\sum_{i=1}^{A}(\Psi(A),t_{1i}F_{1i}\Psi(A)), \tag{1}$$

where $\Psi(A)$ is the ground-state wave function of the target nucleus (which contains A nucleons), t_{1i} is the two-body scattering matrix evaluated in the nuclear medium, F_{1i} is an operator which allows for the effects of incoherent scattering in the nuclear medium.

The difficulty of using the formula (1) for V_0 lies in the fact that the ground-state wave function $\Psi(A)$ has a very complicated form, and further this formula does not take account of the identity of the scattered neutron with the other neutrons in the system so it is not directly applicable at the energies (0 to 5 Mev) which we wish to consider. We must therefore modify the form (1) of the potential to overcome these difficulties. This we shall do by making use of the extensive knowledge of the nuclear ground-state wave function given us by the results of references 2–8. We could in principle do this by introducing a second transformation of the wave function $\Psi(A)$ of Eq. (1); this method, however, is inconvenient since the treatment of the $A+1$ nucleons is unsymmetrical and the effects of the identity of the particles, for example, cannot easily be taken into account. Consequently we shall use a method closely related to that of references 2–6 which treats all nucleons symmetrically.[11] This we do by solving not Eq. (1) for the potential, but instead by considering the problem of determining the potential acting on a shell model particle in a stationary state with the same energy as the energy of the actual problem. That is we shall neglect the fact that we are actually dealing with a scattering problem and consider instead a stationary problem with boundary conditions such that the wave functions vanish at the nuclear surface. This clearly is a good approximation if in the actual scattering problem the reflection coefficient at the nuclear surface is sufficiently close to unity so that the nonstationary character of the state due to flux loss through the nuclear surface can be treated as a perturbation. In this approximation we have replaced the scattering problem by a stationary problem which we can solve using the technique developed by dealing with the ground-state problem.

An additional advantage of this method is that it allows us easily to take into account the identity of the scattered particle with the nuclear particles since the methods we have developed[5] treat all particles in an equivalent way using the powerful and convenient formalism of second quantization. Here as in other papers we shall use the somewhat more compact and intuitive language of particle description but keep in mind that in actual quantitative evaluation, the quite simple translation to the language of second quantization must be made.

In this approximation we determine V_0 by considering the excited states of the nucleus containing $A+1$ nucleus. It has been shown in previous papers[2–6] that these states can be related to a model of equal energy which describes independent particle motion. The nuclear wave function $\Psi(A+1)$ is given in terms of the model wave function $\Phi(A+1)$ by the relation

$$\Psi(A+1)=F\Phi(A+1), \tag{2}$$

where

$$F=1+\frac{1}{e}\sum_{ij}{}' I_{ij}F_{ij},$$
$$F_{ij}=1+\frac{1}{e}\sum_{lm\neq ij}{}' I_{lm}F_{lm}. \tag{3}$$

[9] Lane, Thomas, and Wigner, Phys. Rev. **98**, 693 (1955).
[10] N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953).
[11] The same method was used in reference 2 and a result nearly identical with the following was obtained.

The operators here are labeled by the independent-particle states of the model. The two-body operator t_{ij} is defined in terms of the two-body potential v_{ij} by the integral equation

$$t_{ij}=v_{ij}+v_{ij}\frac{1}{e}{}'t_{ij}. \tag{4}$$

The operator t_{ij} is then separated into

$$t_{cij}=\text{diagonal part of } t_{ij}, \tag{5}$$

and

$$I_{ij}=\text{nondiagonal part of } t_{ij}. \tag{6}$$

The propagator e^{-1} is the reciprocal of

$$e=E-\sum_{i=1}^{A+1}T_i-\sum_{i<j=1}^{A}t_{cij}. \tag{7}$$

In these expressions a prime to the left of an operator means that matrix elements to the state $\Phi(A+1)$ must be omitted. The independent particle or model wave function $\Phi(A+1)$ is a product of single-particle wave functions which are eigenfunctions of the kinetic energy operator satisfying boundary conditions appropriate to the model. The energy of the system is in the first approximation characteristic of this method:

$$E=\sum_{i=1}^{A+1}T_i+\sum_{i<j=1}^{A}t_{cij}. \tag{8}$$

It will be seen that the Eqs. (4) and (7) for t_{ij} and e form a coupled set for which a self-consistent solution has to be sought. This has been done by one of us (K.A.B.)[4] and leads to an energy-dependent potential $V_c(k)$ where the wave number k specifies the state of the particle. The higher approximations to this equivalent potential involve higher order terms in the incoherent reaction matrices I_{ij} and can be disregarded[1,8] since they have a very small effect on the energy.

Next we determine the potential energy of a particle moving with such a wave number that the total energy of the nucleus plus one neutron is equal to the energy of the nucleus in its ground state together with a separated neutron of energy E_n. In the present approximation, the energy of the excited nucleus of $A+1$ nucleons differs from the ground-state energy by

$$E(A+1)=E(A)+k^2/2M+V_c(k), \tag{9}$$

where k is the wave number of the particle moving in the nucleus. Thus, to determine the wave number and hence the potential, we have

$$k^2/2M+V_c(k)=E_n. \tag{10}$$

In the energy range of interest, a good approximation to the potential is

$$V_c(k)=-68\text{ Mev}+(1/2\sigma)k^2, \tag{11}$$

where $(1/M+1/\sigma)^{-1}=M^*\cong 0.60M$. Using this result, we find

$$k^2=2M^*(E_n+68\text{ Mev}), \tag{12}$$

and

$$V_c(k)=-\frac{M^*}{M}(68\text{ Mev})+\left(1-\frac{M^*}{M}\right)E_n. \tag{13}$$

Using $M^*/M=0.60$, which is an approximate value valid at energies near the Fermi energy, we find

$$V=-41\text{ Mev}+0.40E_n. \tag{14}$$

Consequently the potential is slowly energy-dependent decreasing in strength by a few percent as the neutron energy is increased from 0 to 5 Mev. This is the equivalent well depth experienced by a particle having neutron mass, and determines the wave function which has to be matched on to the incoming neutron wave.

B. Imaginary Part of the Potential

The real well depth 41 Mev has been calculated on the assumption that the independent particle states are stable. This assumption will not be valid if the incident neutron has energy greater than zero. However, it is reasonable that the variation of the real potential V_1 with energy will be correctly given by Eq. (14) for low neutron energies where as mentioned before the neutron wave function inside the nucleus can still be approximately represented by that of a stationary state. The main change will come from the appearance of an imaginary part in the equivalent potential encountered by the neutron. Physically this imaginary part of the potential comes from the existence of states of the compound $(A+1)$ nucleus which are coupled to and have a larger lifetime than the single-particle (positive energy) states. Mathematically it comes from the change in the equation for t_{1j} where the subscript 1 refers to the particle excited above zero energy. In deriving the real part of the potential, we have not fully taken into account the change from discrete states to the continuum scattering states. The principal effect is the appearance of excited states of the system with the same energy as the initial state. Consequently the singularity in the propagator can no longer be specified by eliminating matrix elements to the ground state. We must instead change the nature of the singularity by introducing a small imaginary part to correspond to the fact that the scattering wave function contains outgoing scattered waves.[12] This we do in the usual way by replacing Eq. (4) for the reaction matrix by the modified equation for the scattering operator

$$t_{ij}'=v_{ij}+v_{ij}\frac{1}{e+i\eta}t_{ij}'. \tag{15}$$

[12] B. Lippman and J. Schwinger, Phys. Rev. **79**, 469 (1950).

We see now that t_{ij}' must have an imaginary part, hence

$$V_c'=\sum_i \sum_j t_{ci}{}_j' \tag{16}$$

must also have an imaginary part which in turn must be inserted in the propagator e^{-1}. This means that the self consistent solution which we are seeking is constructed to correspond to a set of states within the nuclear volume which are moving in a complex potential.

We introduce a device here to allow us to treat the scattering single-particle state in a manner symmetrical with the compound-nucleus states. The stationary single-particle state which would exist if the boundary conditions appropriate to a stationary state were imposed at the nuclear surface is in fact considerably broadened by the possibility of actual penetration of the nuclear surface and particle loss. At low energies the state is quite long-lived, however, since the reflection factor at the surface is nearly unity; consequently it is a good approximation to replace the effects of particle loss through the nuclear surface by the effects of a uniform imaginary part to the potential. Thus we use an approximate wave function for the single-particle state inside the nucleus which corresponds to a potential with imaginary part

$$W=W_a+W_b, \tag{17}$$

where W_a is the result of transitions to other compound nucleus states and W_b is adjusted to give the same attenuation of the state that would result from surface penetration. We shall use this approximation only in the derivation of the imaginary part of the potential associated with absorption processes and in actually solving the scattering problem we will need to work with a potential having only iW_a for its imaginary part. In this approximation we can write for the potential acting on particle i:

$$V_{ci}'=V_i+iW_i. \tag{18}$$

Both the real and imaginary parts of the potential will in general depend on the state of the particle, particularly the imaginary part which will increase rapidly with the excitation energy. The solution of Eq. (15) for all of the relevant t's will determine the V_c's; consequently a problem of self-consistency arises. Although in determining the real part of the potential we have solved the problem following a self-consistent procedure, we shall work with less rigor in determining the imaginary part of the potential. This is possible since the states and energies are only slightly perturbed by the transition processes and the imaginary part of the well is very small. This we shall determine only approximately; the result will then be self-consistent in first order. The precise approximations to be made will be evident as we proceed.

The imaginary part of the potential due to absorption for particle 1 is given by (writing the diagonal matrix element explicitly)

$$W_a=\mathrm{Im}\sum_{j=1}^{A}(\Phi_0(A+1),\, t_{1j}'\Phi_0(A+1)), \tag{19}$$

where by Im { } we mean the imaginary part of { }. We will estimate this by replacing the repulsive core potential V_{ij} in the equation for t_{ij}' by a regular potential[13] and work to second order in this potential. This gives

$$W_a=\mathrm{Im}\sum_{j=1}^{A}\sum_{\lambda}(\Phi_0(1,j),\, v_{1j}\Phi_\lambda(1,j))\times(E-T_\lambda-V_\lambda-iW_\lambda)^{-1}(\Phi_\lambda(1,j),\, v_{1j}\Phi_0(1,j)), \tag{20}$$

where we have made the matrix product explicit by inserting the intermediate states $\Phi_\lambda(1,\gamma)$. T_λ is the kinetic energy in this state, $V_\lambda+iW_\lambda$ is the complex potential acting on this state. We next write this sum in more compact form, combining the summations over j and λ,

$$W_a=\sum_\lambda (0|v|\lambda)^2\frac{W_\lambda}{(E-E_\lambda)^2+W_\lambda{}^2}, \tag{21}$$

where $E_\lambda=T_\lambda+V_\lambda$ is the real energy which determines the positions of the energy levels of the compound nucleus and

$$(0|v|\lambda)=(\lambda|v|0)=(\Phi_0(1,j),v_{1j}\Phi_\lambda(1,j)). \tag{22}$$

C. Evaluation of the Imaginary Part of the Potential

We shall determine the value of W_a by two quite different methods, both of which give only an average value and neglect fluctuation due to the discreteness of the sum over states. We shall not attempt to give a precise determination of W_a although such an evaluation is in principle straightforward in the approximation of Eq. (21).

We first evaluate the average value of W_a over an energy interval Δ. Such an average is related to the averaging process used by Barschall *et al.*[14] in their analysis of the neutron scattering data. The average is

$$\bar{W}_a{}^{(1)}=\frac{1}{\Delta}\int_{E_0-\frac{1}{2}\Delta}^{E_0+\frac{1}{2}\Delta}\sum_\lambda(\lambda|v|0)^2\frac{W_\lambda}{(E-E_\lambda)^2+W_\lambda{}^2}dE. \tag{23}$$

This we evaluate making use of several simplifying approximations. We will assume that Δ is large enough so that the number of states λ with energies E_λ in Δ always includes many levels, i.e.,

$$\Delta\gg D, \tag{24}$$

[13] This approximation derives its validity from the fact that for low momentum values the correct t_{ij}' closely resemble the scattering from a regular well of correct depth and range.

[14] H. H. Barschall, Phys. Rev. **86**, 431 (1952); Am. J. Phys. **22**, 517 (1954); N. Nereson and S. Darden, Phys. Rev. **94**, 1678 (1954); M. Walt and H. H. Barschall, Phys. Rev. **93**, 1062 (1954).

where D is the mean spacing of the levels E_λ. We then can approximate to the results of the integration of Eq. (23) by

$$\bar{W}_a^{(1)}=\pi\sum_\lambda(\lambda|v|0)^2\Gamma_\Delta(E_0,E_\lambda), \tag{25}$$

where

$$\Gamma_\Delta(E_0,E_\lambda)=\frac{1}{\pi\Delta}\int_{E_0-\Delta/2}^{E_0+\Delta/2}\frac{W_\lambda dE}{(E-E_\lambda)^2+W_\lambda^2}$$
$$\cong 1/\Delta \text{ if } |E_0-E_\lambda|<\tfrac{1}{2}\Delta$$
$$\cong 0 \quad \text{if } |E_0-E_\lambda|>\tfrac{1}{2}\Delta. \tag{26}$$

This function Γ_Δ is analogous to a spread-out delta function; it is independent of the level widths W_λ as long as $\Delta\gg W_\lambda$ which will be satisfied if

$$D>W_\lambda. \tag{27}$$

Using this approximation to the integral, we have

$$\bar{W}_a^{(1)}=\pi\sum_{E_\lambda \text{ in } \Delta}\frac{1}{\Delta}|(\lambda|v|0)|^2. \tag{28}$$

Now making use of our assumption that many levels λ lie in Δ, it is reasonable to replace $(\lambda|v|0)^2$ by its average value, giving us

$$\bar{W}_a^{(1)}=\pi(\lambda|v|0)_{Av}^2\frac{1}{\Delta}\sum_{E_\lambda \text{ in } \Delta}'=\frac{\pi}{D}(\lambda|v|0)_{Av}^2, \tag{29}$$

where Δ/D is the number of levels in Δ. Thus we have a simple relationship between the level width, the matrix elements for transitions between the states 0 and λ, and the level spacing. Before evaluating this, we shall give another way of estimating $\bar{W}_a$.

We make another approximation suggested by the result we have just derived. This is to make use of the fact that $\bar{W}_a$ does not depend on the level widths or on the value of E but rather on the density of levels. Consequently, it is reasonable to estimate $\bar{W}_a$ by replacing the discrete states λ by continuum states and letting $\Delta\gg W_\lambda$ approach zero. In this case we find the result of ordinary perturbation theory[15]

$$\bar{W}_a^{(2)}=\pi\int d\lambda(\lambda|v|0)^2\delta(E_0-E_\lambda), \tag{30}$$

where the integral over λ has replaced the discrete state sum.

For simplicity we shall in evaluating the two values of W_a given by Eqs. (29) and (30) use a simple representation of the nuclear states, i.e., we shall represent the model ground state by a degenerate Fermi gas and the excited states by plane waves of definite momenta. This approximation is accurate for large nuclei as long as we are not interested in precise details such as the exact positions and widths of resonances, for example. It is to be noted that this approximation is not equivalent to assuming that the actual nuclear wave function Ψ is a Fermi gas, Ψ being related to the model wave function Φ by the transformation operator F (Eq. 3).

We first evaluate $(0|v|\lambda)_{Av}^2$ in this representation. For v we use a Yukawa well with Serber exchange mixture. Introducing the momentum states explicitly, we find

$$(\mathbf{k}_1'\mathbf{k}_2'|v|\mathbf{k}_1\mathbf{k}_2)=\left(\frac{2\pi V_0}{\mu v}\right)\left[\frac{1}{\mu^2+(\mathbf{k}_1-\mathbf{k}_1')^2}+\frac{1}{\mu^2+(\mathbf{k}_1+\mathbf{k}_1')^2}\right]\delta_{\mathbf{k}_1+\mathbf{k}_2,\,\mathbf{k}_1'+\mathbf{k}_2'}. \tag{31}$$

The average over states λ is equivalent to averaging over states of excitation E_λ in Δ and also to averaging over the angles of $\mathbf{k}_1'$. Making use of the fact that the states which interest us are of low excitation so that the vectors $\mathbf{k}_1$ and $\mathbf{k}_1'$ are both approximately equal to k_F in magnitude, the average gives

$$(0|v|\lambda)_{Av}^2\cong\left(\frac{2\pi V_0}{\mu v}\right)^2\frac{1}{\mu^2k_F^2} \tag{32}$$

where we have dropped a small term using $k_F^2/\mu^2\gg 1$. Taking $V_0=0.252\mu$, $v=(4/3)\pi A/\mu^3$, $k_F\simeq 1.8\mu$, and inserting the results in Eq. (29) for W_a, we find

$$\bar{W}_a(1)\cong(200/A)^2\times 68 \text{ Mev}/D(\text{kev}). \tag{33}$$

Thus, for a typical heavy nucleus ($A=200$) and for a level spacing $D=100$ kev, this gives $W_a=0.68$ Mev. It will be noted that the level spacing D will decrease with energy so that the predicted imaginary potential W_a will increase as the energy of the incident neutron increases.

To evaluate the continuum approximation given by Eq. (30) for the level width, we re-insert explicitly the sums over the relevant momentum indicies and replace the summations by integrals. The result

$$\bar{W}_a^{(2)}=\frac{\pi}{(2\pi)^6}\left(\frac{2\pi V_0}{\mu}\right)^2\int d\mathbf{k}_i\int d\mathbf{k}_i'\left\{\frac{1}{\mu^2+(\mathbf{k}_1-\mathbf{k}_i')^2}+\frac{1}{\mu^2+(\mathbf{k}_1+\mathbf{k}_1')^2}\right\}^2\delta(E_1+E_i-E_1'-E_i'). \tag{34}$$

This integration must be carried out in such a way that the exclusion principle is not violated; the requirement is that

$$|\mathbf{k}_1'|\geqq k_F, \quad |\mathbf{k}_i'|=|\mathbf{k}_1+\mathbf{k}_i-\mathbf{k}_1'|\geqq k_F. \tag{35}$$

In addition the integration over $\mathbf{k}_i$ runs only over the unexcited states of the core, i.e., over the Fermi gas.

[15] This approximation to Eq. (19) for the imaginary part of the potential is formally similar to an approximation used by Lane (unpublished). His method differs in that he replaces the correct combination $v(1/e)t$ of Eq. (4) by $t(1/e)t$ instead of our replacement by $v(1/e)v$. He also supposes that the actual nuclear wave function can be represented as a degenerate Fermi gas; we regard this approximation as valid only in describing the model wave function (see also the remarks at the beginning of Sec. III).

To remove the delta function on momentum, we make use of the reduced mass approximation[4] for the energies

$$E=k^2/2M^*. \tag{36}$$

Consequently the integral over $\mathbf{k}_1'$ becomes

$$\int d\mathbf{k}_1'\delta(E_1+E_i-E_1'-E_i')=M^*\int\frac{k_1'dk_1'}{|\mathbf{k}_1+\mathbf{k}_i|}, \tag{37}$$

where we have used

$$\frac{\partial E}{\partial\cos\theta_1'}=\frac{\partial}{\partial\cos\theta_1'}\left[\frac{k_1'^2}{2M^*}+\frac{(\mathbf{k}_1+\mathbf{k}_i-\mathbf{k}_1')^2}{2M^*}\right]$$
$$=|\mathbf{k}_1+\mathbf{k}_i|k_1'/M^*, \tag{38}$$

where θ_1' is the angle between $\mathbf{k}_1+\mathbf{k}_i$ and $\mathbf{k}_1'$.

To evaluate the integral we shall use the fact that the restrictions due to the exclusion principle severely limit the range of interaction, and make an expansion valid for k_1 near k_F. That is, we let

$$\mathbf{k}_1=\mathbf{k}_F+\mathbf{x},\quad \mathbf{k}_1'=\mathbf{k}_F'+\mathbf{x}',\quad \mathbf{k}_i=\mathbf{k}_F''-\mathbf{x}_i, \tag{39}$$

and keep only terms linear in the x's. The integral then becomes

$$\left(\frac{2\pi V_0}{\mu}\right)^2\frac{1}{(2\pi)^3}\frac{k_F^3M^*}{\mu^4}\int\frac{dx_idx_idx_1'}{|\mathbf{k}_1+\mathbf{k}_i|}, \tag{40}$$

where μ_i is the cosine of the angle between $\mathbf{k}_1$ and $\mathbf{k}_i$. We have simplified the result by taking terms in the square of the potential which are large when the approximations $\mathbf{k}_1\cong\mathbf{k}_1'$ and $\mathbf{k}_1\cong-\mathbf{k}_1'$ are made, these being the values of $\mathbf{k}_1'$ which give the principal contribution for transitions near the Fermi momentum. We also correct for our neglect of the exclusion principle by multiplying by a factor of $\frac{3}{4}$ which corrects for the apriori probability for interactions in even states to occur. The restrictions on the integration now are that

$$x_1'\geqq 0,\quad x_1+x_i-x_1'\geqq 0, \tag{41}$$

where we have made use of the energy conservation condition in simplifying the second of these. We shall finally make another approximation in relaxing the conditions of energy conservation on the integration. These have only the effect of somewhat decreasing the range of integration so that we shall in neglecting them overestimate the integral. Estimates of the error made in this approximation indicate that it is rather small. Using these restrictions, we find

$$\bar{W}_a^{(2)}=\frac{3}{8\pi}\left(\frac{V_0}{\mu}\right)^2\frac{k_F^3M^*}{\mu^4}\int_0^{x_1}dx_i\times\int_0^{x_1-x_i}dx_1'\int_{-1}^{1}\frac{d\mu}{(2k_F^2-2k_F^2\mu)^{\frac{1}{2}}}$$
$$=\frac{3}{8\pi}\left(\frac{V_0}{\mu}\right)^2\left(\frac{k_F}{\mu}\right)^2\frac{M^*}{\mu}\frac{x_1^2}{\mu}. \tag{42}$$

This we bring to final form by using the approximation

$$x=k_1-k_F\cong(k_1^2-k_F^2)/2k_F=(E_1-E_F)M^*/K_F. \tag{43}$$

The result is

$$\bar{W}_a^{(2)}=\frac{3}{8\pi}\left(\frac{V_0}{\mu}\right)^2\left(\frac{M^*}{\mu}\right)^3\left(\frac{E_1-E_F}{\mu}\right)^2\mu. \tag{44}$$

This is an overestimate of the original integral since both approximations made have increased the result. $\bar{W}_a^{(2)}$ shows a quadratic dependence on the excitation energy measured relative to the energy E_F of the particle in the highest Fermi level. The magnitude predicted is

$$\bar{W}_a^{(2)}=\left(\frac{M^*}{M}\right)^3\left(\frac{E_1-E_F}{8.4}\right)^2\text{ Mev}, \tag{45}$$

where E_1 and E_F are measured in Mev. Thus, at excitations of 10 Mev corresponding to scattering states of low energy, W_a is a few tenths of a Mev (taking $M^*/M\sim 0.6$). This agrees with the previous estimate; we conclude that the imaginary part of the potential for a single particle is roughly one Mev and probably somewhat less.[16] However, it should be noted that there will in addition be (smaller) contributions from more complicated forms of compound states which will tend to increase the imaginary part of the mean potential which has been estimated here.

III. RESONANCE FINE STRUCTURE

In the previous section, we have been primarily concerned with an approximation appropriate to determining the average effective potential for the scattering of a neutron on a nucleus. The average was defined over an energy range which was required to contain many two-particle excited states for the equivalent shell-model problem. It is to be emphasized that although these shell-model states are used to define corresponding nuclear states and are used also in computation of the average potentials, the method does not assume weak interactions between nucleons. The apparent paradox is explained by considering the complicated relation between the actual nuclear wave function and the model wave function [see for example Eqs. (2) and (3)]. It will be noted that the transformation operator which changes a shell model state into a nuclear state introduces strong correlations into the nuclear wave function which correspond to strong forces between nucleons. The great value of the shell model lies in its use in determining observables which do not depend strongly on short-distance correlations between nucleons, and it is in this connotation that we have used it in the previous section.

In the present section we shall investigate not the average equivalent potential acting on the neutron over

[16] The result obtained by Lane (see reference 15) is larger than this, primarily due to the appearance of $(M^*/M)^3\sim\frac{1}{4}$ in our formula [Eq. (45)]. This is a consequence of the strong momentum dependence of the self-consistent potential predicted by our methods.[4]

a fairly wide energy interval, but we will consider instead the detailed behavior of the equivalent potential in the neighborhood of a two-particle excited state of the nuclear model. It is clear that such an excited state will correspond to a much more complicated compound state of the system, target nucleus plus neutron, and the fluctuation in the equivalent potential which we will derive will correspond to some complicated change with energy of the compound nuclear state. There are two aspects of the problem; the first we shall consider is the level width of the two-particle excited states, and the second is the manner in which the resonances arise in terms of the scattering of a neutron on the equivalent potential.

A. Compound Nucleus Level Widths

In Sec. II, we have seen that there are terms in the equivalent potential which result from the incoming neutron interacting with a single shell-model particle in the target nucleus and forming a two-particle excited state. We will examine the broadening of these two-particle levels which arises because they are energetically capable of decaying by the inverse process. This process can be represented as a transition to a single-particle excited state of the nucleus followed by a decay of this state through absorption to other compound states or through penetration of the nuclear surface. In this evaluation we will neglect direct transitions to nearby two- and many-particle excited states, assuming that these levels in contrast to the broad single-particle levels do not overlap appreciably with the two-particle level we are considering.

To evaluate the broadening of the two-particle level, we use a generalization of the result for the imaginary part of the potential given in Eq. (21), i.e.,

$$W_\lambda = \mathrm{Im}\sum_{\lambda'}(\lambda|v|\lambda')\frac{1}{E_\lambda - T_{\lambda'} - V_{\lambda'} - iW_{\lambda'}}(\lambda'|v|\lambda), \quad (46)$$

where λ is the excited two-particle state and the sum over λ' is over all other excited states of the system. We have in Eq. (46) taken the energy E to be the energy E_λ which determines the position of the level. We make use of the approximation just described which assumes that all of the level broadening is due to transitions to the single-particle excited state. Thus in the sum over λ' we include only the state "0". Furthermore we write for this state $T_{\lambda'}+V_{\lambda'}=E_1$ and $W_{\lambda'}=W$. The result is

$$W_\lambda = (\lambda|v|0)^2\frac{W}{(E_1-E_\lambda)^2+W^2}. \quad (47)$$

To estimate $(\lambda|v|0)$, we use the expression for v used before [Eq. (31)] and find

$$W_\lambda \approx \left(\frac{2\pi V_0}{\mu v}\right)^2 \frac{1}{\mu^4 W}\frac{1}{(1+\Delta k^2/\mu^2)^2}\frac{1}{[1+(\Delta E/W)^2]}, \quad (48)$$

where Δk is the momentum transfer and ΔE is the energy difference. Taking the nuclear volume

$$v=(4/3)\pi A/\mu^3$$

and the Yukawa well strength $V_0=0.25\mu$, this expression becomes

$$W_\lambda = 0.070\ \mathrm{Mev}\left(\frac{200}{A}\right)^2\frac{1}{W\ (\mathrm{Mev})} \times\frac{1}{(1+\Delta k^2/\mu^2)^2}\frac{1}{[1+(\Delta E/W)^2]}. \quad (49)$$

For a typical case, with good energy and momentum overlap ($\Delta k\sim\Delta E\sim 0$), if one takes $A=200$ and $W=3$ Mev, this gives $W_\lambda=23$ kev.

There obviously can be great fluctuations in the widths of these two-particle levels, particularly as a result of the great possible variations in Δk, the momentum transfer. Since the momentum in a low excited state is quite large ($k\sim k_F\sim 1.8\mu$), $(\Delta k^2/\mu^2)$ can range from zero to about four, leading to variations in width from 23 kev to about 1 kev. In spite of the roughness of our approximations, it is apparent that level broadening of these two-particle excited states of the model (which correspond to excited compound states of the nucleus) is comparable with the broadening of the detailed resonances which are observed in neutron scattering at energies of one or two Mev.[17]

In addition to these two-particle excited states, there will be more complicated excited states corresponding to three or more particles not in the lowest unoccupied levels. For the initial single-particle state to couple to a state with three particles excited, the incoherent scattering operator I_{ij} [Eq. (6)] will have to act twice on successive pairs of particles to bring about the necessary changes in excitation. This more indirect coupling will tend to make the levels narrower than the simple two-particle excited states. In order to investigate them in the present formalism, it is necessary to use the more exact form for the equivalent potential given by

$$(\Phi_0, \textstyle\sum_{ij} t_{ij}F_{ij}\Phi_0). \quad (50)$$

To see explicitly how a resonance associated with a three-particle excited state can arise, consider a typical term of third order in the incoherent scattering matrices I_{ij}. This is

$$\sum_{i\neq j\neq k}\left(\Phi_0, I_{ij}\frac{1}{e}I_{jk}\frac{1}{e}I_{ki}\Phi_0\right). \quad (51)$$

[17] It should be emphasized that the determination of resonance widths which we have made here is valid only when the single-particle levels are sufficiently broad to overlap in energy the compound nucleus levels. This condition is not well satisfied at low energies where all levels become very narrow as the metastable states of the bound system are approached. This restriction is evident in Eq. (49) since if $\Delta E \gg W$ as is the case at very low energy, the level widths become very small.

To evaluate a typical matrix element, let the operator act on an initial system with particles a,b in the ground state and particle c in the single-particle excited state. Under the action of the incoherent operators, the following transitions can occur in succession:

$$\begin{aligned} &ab(c) \rightarrow a'b'(c), \\ &a'(b')c \rightarrow a(b')c', \\ &(a)b'c' \rightarrow (a)bc, \end{aligned} \tag{52}$$

where we have indicated the particle unaffected in a transition by parentheses. For such a possible series of transitions, the contribution is

$$I_{bc,b'c'}(E_b+E_c-E_{b'}-E_{c'})^{-1} \times I_{ac',a'c}(E_a+E_b-E_{a'}-E_{b'})^{-1}I_{a'b',ab}, \tag{53}$$

where we write explicitly the matrix elements of e^{-1}. The three-particle excited state reached in this process are a', b', c and a, b', c'; if either of these states has the same energy as the initial state, then the matrix element of Eq. (53) will be markedly increased and a fluctuation in the potential strength will occur. This is of course a consequence of transitions occurring between the single-particle and the three-particle states. These terms are expected to be considerable smaller than those resulting from simpler types of excitation since they are of higher order in the interaction operators t_{ij} or I_{ij}. We shall not proceed farther in considering these more complicated excitations.

Before considering the way in which the resonances arise there is one further point to be noted about the width of the two particle resonances. According to the formula of Eq. (49), the width W_λ depends on the energy of the levels relative to the single-particle levels in a simple way, namely

$$W_\lambda \sim [1+(\Delta E/W)^2]^{-1}. \tag{54}$$

This dependence will be considerably masked by the strong fluctuations in the matrix elements of v and also by the fact that the experiments do not serve to distinguish whether a particular resonance corresponds to a two- or three- (or more-) particle excitation.

B. Resonances

In order to show how it is that resonances in the scattering cross section occur at energies which correspond to the two-particle excited states we consider the behavior of the equivalent potential near these energies. In the approximation of Eqs. (8) the equivalent potential is

$$V_c(1)=\sum_i(\Phi_0,t_{1i}\Phi_0)=\sum_i\Big[(\Phi_0,v_{1i}\Phi_0) +\sum_\lambda(\Phi_0,v_{1i}\Phi_\lambda)\frac{1}{E-E_\lambda-iW_\lambda}(\Phi_\lambda,t_{1i}\Phi_0)\Big]. \tag{55}$$

We wish to exhibit the fluctuation in $V_c(1)$ which occurs for energies near the energy of a particular two-particle excited state Φ_λ; thus we break the expression for $V_c(1)$ up into two terms:

$$V_c(1)=V_c'(1)+(\Phi_0,v_{1i}\Phi_\lambda)\frac{1}{E-E_\lambda-iW_\lambda}(\Phi_\lambda,t_{1i}\Phi_0). \tag{56}$$

We can now expect that if E is close to E_λ the term $V_c'(1)$ will be slowly varying, partly because it is mainly determined by contributions from far off states and partly because the rapidly varying term depending on the state Φ_λ has been omitted. The remaining term will be rapidly varying and we will consider its variation by approximating v and t with a Yukawa well. This gives a term:

$$\Delta V_c(1)=(0|v|\lambda)^2\frac{1}{E-E_\lambda-iW_\lambda}. \tag{57}$$

It is clear that the real and imaginary parts of this part of the equivalent potential will vary rapidly as E is varied near E_λ. The variation can be estimated by using the relation of Eq. (47) for the width of a level λ. Substituting the potential matrix element $(\lambda|v|0)^2$ given by this equation into Eq. (57) we obtain

$$\Delta V_c(1)=\frac{W_\lambda}{W}[(E_\lambda-E_1)^2+W^2]\frac{E-E_\lambda+iW_\lambda}{(E-E_\lambda)^2+W_\lambda^2}. \tag{58}$$

For E_λ near E_1 so that $(E_\lambda-E_1)<W$, this simplifies to

$$\Delta V_c(1)\cong WW_\lambda\frac{E-E_\lambda+iW_\lambda}{(E-E_\lambda)^2+W_\lambda^2}. \tag{59}$$

We see that both the real and imaginary parts of ΔV_c change by amounts of order W, W being the width of the single-particle level (absorption plus surface penetration). Thus W is the width of the giant resonance and is of order 3 Mev. The sharp variation in V_c occurs over energy interval determined by W_λ which we have shown in the first part of this section to be of the order of tens of kev.

A variation in the real part of the equivalent potential V_c of the order of 3 Mev is of course quite sufficient to lead to a pronounced resonance in the scattering cross section, particularly if the energy E is separated from the giant (single particle) resonance by an energy less than the variation. Although it is rather an oversimplification to regard this change in real well depth as equivalent to a change in the incident energy, this comparison is certainly adequate to show that either a true resonance or a resonance followed by a sharp dip in the cross section can arise from a fluctuation in V_c of the order of 3 Mev.

IV. DISCUSSION AND COMPARISON WITH THE THEORY OF WIGNER, LANE, AND THOMAS

The results we have obtained can be described qualitatively in the following way: the average scattering of a neutron by a nucleus can be represented by the scattering by a complex well, the real part being given by Eq. (14) and the imaginary part by Eqs. (33) and (45). Such an interaction leads to giant resonances in the cross section at the position of the independent-particle states; the general behavior of the cross section is due almost entirely to the real part of the potential, the imaginary part giving only a slight broadening of the resonances and a general smoothing of the cross section. Consequently, the appearance of the giant resonances is a manifestation primarily of the representation of the neutron-nucleus interaction by a uniform-potential-well small imaginary part.

It is possible to derive an interesting relationship between the average level width of the compound nucleus and the parameters of the single-particle levels. Combining Eqs. (29) and (47) we find

$$\frac{(W_\lambda)_{Av}}{D}\cong\frac{W_aW}{(E-E_1)^2+W^2}, \tag{60}$$

where W_a is that part of the single-particle level width due to absorption, W is the total single-particle level width, and E_1 is the energy of the single-particle state. This result shows similarity to a result used by Wigner *et al.*,[9]

$$(\gamma_\lambda^2)_{Av}/D=S(E), \tag{61}$$

where $S(E)$ is a "strength function" which is assumed to have a dependence on energy similar to Eq. (60). Consequently, both in this work and in Wigner's, the average compound-state level width is governed by its proximity to the single-particle state. It is to be noted, however, that very great fluctuations of the widths of individual levels are to be expected due to the range of variation of the matrix elements of the operator connecting the single particle and compound nucleus states. In addition, on our work the strength function $S(E)$ does not play a dominant role in the determination of the scattering. We instead transfer our emphasis to the determination of an equivalent uniform and constant potential which will automatically lead to the giant resonances.

Another departure of our results from those of Wigner *et al.*, is in the determination of the real and imaginary part of the single particle equivalent potential. The most important qualitative departure is in the imaginary part which in our formalism is given by Eq. (21), repeated here for convenience,

$$W_a=\mathrm{Im}\sum_\lambda\sum_j(0|v_{1j}|\lambda)\frac{1}{E-E_\lambda-iW_\lambda}(\lambda|v_{1j}|0). \tag{62}$$

This formula, as we have shown, predicts correctly the approximate magnitude of the average imaginary well strength. It is possible to show that Wigner's result[18]

$$W_a^2=\sum_j(0|v_{1j}^2|0) \tag{63}$$

corresponds to a certain approximation to Eq. (62). Suppose that it is assumed that W_λ is constant and equal to W_a, i.e., that the imaginary part of the potential is the same for all states. Further suppose that the important contributions to the sum over λ come from such states that $|E-E_\lambda|\ll W_\lambda$. In this approximation

$$W_a=\mathrm{Im}\sum_\lambda\sum_\lambda(0|v_{1j}|\lambda)\frac{i}{W_a}(\lambda|v_{1j}|0), \tag{64}$$

or

$$W_a^2=\sum_\lambda\sum_j(0|v_{1j}|\lambda)(\lambda|v_{1j}|0)=\sum_j(0|v_{1j}^2|0), \tag{65}$$

which is Wigner's result. This approximation, however, is not a good representation of the summation of Eq. (62) since the widths W_λ are not constant and, more important, the summation over λ includes states very different in energy from E so that, over most of the sum, $|E_\lambda-E|\gg W_\lambda$. Consequently, a different approximation such as we have used in Sec. II must be used and as shown there gives a much smaller value for W_a than Eq. (65).

V. CONCLUSIONS

We have applied to the neutron reaction problem methods[2–8] which are based on a detailed study of the problem of many nucleons in strong interaction. These methods have in other applications given detailed and quantitative insight into many aspects of the ground and low excited states of nuclei. In extending them to the study of the neutron reactions at low energy, we have shown that the "cloudy crystal ball" model of Weisskopf *et al.*,[1] appears in a natural way and that the predicted parameters of the interaction agree closely with those determined empirically.[19] The methods also show the origin of the sharp compound-nucleus resonances as a manifestation of coupling between the single-particle states and the two- (or more-) particle excited states of the compound nucleus.

It is apparent from these results that the simplicities of nuclear structure which are evidenced in the shell model also persist strongly in the behavior of the nucleus in low-energy scattering. The very small imaginary part of the equivalent potential acting on a particle in a scattering state at low energy reflects the close relationship which exists between the low excited

[18] We have not inserted the effects of the exclusion principle explicitly; these automatically included if, as we have remarked in the introduction, we interpret the v's as appropriately second quantized operators.

[19] The calculated imaginary part of the potential is somewhat smaller than that determined in the empirical analysis; the discrepancy is perhaps due to our omission of the effects of transitions to more complicated (than two-particle) states of the compound nucleus.

states and the ground state where the shell model behavior is most apparent and the single-particle orbits best defined.

It is to be emphasized that the equivalent potential which we have determined is constructed to give the correct scattering of the neutron by the nucleus, i.e., to predict correctly the asymptotic behavior of the wave function. This does not imply that the actual wave function for the system is also given correctly everywhere by this interaction. The actual strongly correlated and highly mixed wave function is instead given in terms of the wave function determined by the equivalent uniform potential by an extremely complicated transformation [Eq. (3)]. Thus we expect that the Weisskopf complex potential can be used to predict only the asymptotic behavior of the scattering wave function, any more detailed information being available only if the transformation is explicitly constructed. This situation is analogous to that which exists in the shell model theory of the nucleus where again predictions of the detailed behavior of the nuclear ground state can be made only if the departures of the nuclear wave function from the shell model wave function due to the strong particle-particle forces are determined.

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Two-Nucleon Potential from the Cut-Off Yukawa Theory*

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Using the Yukawa theory with cutoff, the two-nucleon interaction is calculated up to fourth order in the coupling constant. The resulting potential at large distances ($\gtrsim 10^{-13}$ cm) is similar to the well-known potential with no cutoff. At small distances, however, the tensor potential approaches zero, and the central potential is strongly repulsive. The details are essentially determined by two parameters, the coupling constant and cutoff. The Schrödinger equation for the two-nucleon problem at low energies is solved numerically, and values are obtained for the binding energy, the quadrupole moment, and the four n-p scattering parameters. It is found that the triplet experimental values are reproduced quite well, and the singlet parameters reasonably well by the same coupling constant and cutoff required to explain pion-nucleon scattering and photoproduction.

I. INTRODUCTION

SINCE the time that Yukawa first proposed to explain nuclear forces as a manifestation of the exchange of field quanta by nucleons, numerous papers have been published on the predicted properties of nuclear systems due to this exchange of pions by nucleons.[1–3] To make any sort of progress, utilization has usually been made of perturbation expansions in the coupling constant as well as the so-called static approximation in which nuclear recoil is almost completely neglected. It has been found that with a local theory (no cutoff) the resulting two-nucleon potential has a strong singularity at small distances, and that this singularity is aggravated by including higher order terms in the expansion. The Schrödinger equation for a two nucleon system is not soluble with such a potential, and the tendency has been to replace the interaction at small distances with a phenomenological, infinite repulsive core. Thus, Brueckner and Watson[2] were able to fit most of the low-energy n-p data, treating the widths of the repulsive cores in the singlet and triplet states as adjustable parameters. Similarly, Taketani[1] utilizing the depth of the cores as an additional parameter, found that best agreement was obtained to the low energy n-p data with infinite repulsive cores. Neglect of nucleon recoil has been justified by the argument that one was interested primarily in low-energy phenomena, where the nucleon velocity is small. Attempts to include recoil terms,[2,3] have not led to conclusive results.

Recently, Chew[4] has shown that reasonable predictions for photo-meson production, pion-nucleon scattering etc. are obtained by the cut-off form of the Yukawa theory. There are two adjustable parameters in this theory, the renormalized coupling constant, f^2, and the cut-off energy of virtual mesons, ω_m. The purpose of the present paper is to show that all of the low energy n-p parameters (i.e., the binding energy, the quadrupole moment as well as the four scattering parameters) can be produced with reasonable accuracy by the above theory with the same values for f^2 and ω_m as are required by pion-nucleon scattering and photoproduction.

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