where μ is the reduced mass of the meson, α the fine structure constant, Z the atomic number, and n the principal quantum number.

The solution for the finite nuclear radius (nonpoint source) is obtained by a first-order perturbation calculation using the potential.[29]

$$V(r) = -(Ze^2/R)(\tfrac{3}{2} - \tfrac{1}{2}r^2/R^2) \quad \text{for} \quad r \leq R$$
$$= -Ze^2/r \quad \text{for} \quad r \geq R,$$

where $R = 1.2\times10^{-13}A^{\frac{1}{3}}$ cm (nuclear radius).

It suffices for low-Z materials to calculate the perturbation of the $1s$ level. The higher levels $n \geq 2$, except in the heaviest elements, satisfy the point nucleus solutions. The fractional energy shift due to the perturbation is[30]

$$\Delta E/E_{1s} = -\tfrac{4}{5}Z^2(R/a_0)^2,$$

TABLE IV. μ-mesonic atoms—lowest energy levels (in Mev).

Z	$1s_{\frac{1}{2}}$	$2p_{\frac{3}{2}}$	$2p_{\frac{1}{2}}$ $2s_{\frac{1}{2}}$	$2p_{\frac{3}{2}}-1s_{\frac{1}{2}}$	Computed from
6	−0.1018	−0.0254	⟶	0.0764	Schrödinger
	−0.1019	−0.0254	($2p_{\frac{3}{2}}$ −12 kev)	0.0765	Dirac
	−0.1015			0.0761	Perturbation of Dirac solution
8	−0.1823	−0.0456	⟶	0.1367	Schrödinger
	−0.1824	−0.0456	($2p_{\frac{3}{2}}$ −47 kev)	0.1368	Dirac
	−0.1807			0.1351	Perturbation of Dirac solution

where $a_0 = \hbar^2/\mu e^2 = 2.52\times10^{-11}$ cm (radius of the first Bohr orbit of the mesonic hydrogen atom).

The K and L shell energies obtained from the point nucleus solutions of the Schrödinger and the Dirac[31] wave equations are shown in Table IV. The perturbation correction has been applied to the $1s$ level.

[29] See reference 6, footnote on p. 173.

[30] This result is obtained also by L. N. Cooper and E. M. Henley, Phys. Rev. **92**, 801 (1953), their Eq. (11).

[31] $$E_{n,j} = \mu c^2\left[1 + \frac{(\alpha Z)^2}{\{n - |k| - (k^2 - \alpha^2 Z^2)^{\frac{1}{2}}\}^2}\right]^{-\frac{1}{2}}$$
$$\cong \mu c^2\left[1 - \frac{(\alpha Z)^2}{2n^2} - \frac{(\alpha Z)^4}{2n^4}\left(\frac{n}{|k|} - \frac{3}{4}\right)\right], \quad k = -\ell - 1 \text{ for } j = \ell + \tfrac{1}{2}, \quad k = \ell \text{ for } j = \ell - \tfrac{1}{2}.$$

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Angular Correlation Effects in V-Particle Decay

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In decay processes of the types: hyperon→nucleon+pion; or K-meson→pion+pion, the most general distribution function in η—the angle between the decay plane normal and a reference direction **N** normal to the line of flight of the unstable particle—has the form of a finite Fourier series. The degree of the highest harmonic is simply related to the spin of the unstable particle. The coefficients in the series depend on the state of polarization of the spin with respect to the reference direction **N**. It is possible however to set upper limits on the coefficients; this may prove useful in any attempt to analyze angular correlation data, particularly in the case of hyperon decay. The upper limits for various low spin values are computed, and other consequences of the angular momentum and parity conservation laws are discussed.

I. INTRODUCTION

ANGULAR correlation effects in V-particle decay have been investigated recently by a number of workers,[1–5] in an attempt to learn something about the spins of the new unstable particles. For reasons of parity and angular momentum conservation, a particle of spin zero or spin one-half must decay isotropically in its rest frame. In the case of particles which undergo two-body decay, this means that the distribution in η—the angle between the decay plane normal **n** and any reference direction **N** which is normal to the line of flight of the unstable particle and which is defined independently of **n**—must be uniform.[6] A nonuniform distribution in η would automatically imply spin greater than one-half; and from the form of the distribution (see below) one could set a lower limit to the value of the spin.

Even for particles of very large spin, however, angular correlation effects would show up only if the spins were somehow polarized (i.e., nonrandomly distributed) with respect to the reference direction **N**. This suggests that the effect, if it exists at all, would be most likely to manifest itself with unstable particles produced in low-energy interactions of elementary particles, e.g., the reaction $\pi^- + p \rightarrow$ hyperon $+ K$-meson observed at Brookhaven. Nevertheless, early Princeton work[1] on V^0-particles produced in generally complex

[1] Ballam, Hodson, Martin, Rau, Reynolds, and Treiman, Phys. Rev. **97**, 245 (1955); see also *Proceedings of the Fifth Annual Rochester Conference on High-Energy Physics* (Interscience Publishers, Inc., New York, 1955).

[2] Report by C. D. Anderson, *Proceedings of the Fifth Annual Rochester Conference on High-Energy Physics* (Interscience Publishers, Inc., New York, 1955).

[3] G. D. James and R. A. Salmeron, Phil. Mag. **46**, 571 (1955).

[4] Sreekantan, Pevsner, and Sandri, Phys. Rev. **98**, 642 (*A*) (1955).

[5] Fowler, Shutt, Thorndike, and Whittemore, Phys. Rev. **98**, 121 (1955).

[6] Treiman, Reynolds, and Hodson, Phys. Rev. **97**, 244 (1955).

cosmic-ray interactions gave some indication of angular correlation effects for events involving a pair of V^0-particles which have a common origin. However, other groups,[2–4] also working with cosmic-rays, found no appreciable effect. But recently, the Brookhaven workers[5] have reported results on the distribution in angle η between the decay and production planes of unstable particles, and they find a marked correlation effect for the hyperons, the angle η tending toward small values. They conclude that the spins of the hyperons are probably greater than one-half.

In view of the affirmative Brookhaven results, we consider it worthwhile to present a further discussion of the relationship of angular correlation effects and spin. The discussion does not involve any detailed model of the unstable particles but is instead based solely on the conservation laws of parity and angular momentum.

II. ANGULAR DISTRIBUTION FUNCTION

In a recent note it was shown that for a particle of given spin j, which undergoes two-body decay, the most general form of the angular distribution in η is given by[7]

$$p(\eta)=\frac{1}{2\pi}\times\left\{1+\sum_{M=2,4,\ldots}^{M_{\max}}(A_M\cos M\eta+B_M\sin M\eta)\right\};\quad (1)$$

where

$$M_{\max}=\begin{cases}2j, & \text{(integral } j)\\ 2j-1, & \text{(half-odd integral } j).\end{cases}\quad (1')$$

In the derivation of this equation, it is assumed that the decay plane normal **n** is a vector with a well-defined *sense* as well as direction and that the angle η between **n** and the reference vector **N** ranges from 0 to 2π. (One could, for example, take **n** to have the sense of the vector product of the two final particle momenta, taken in a prescribed order.) In practice, however, one has not thus far associated a sense with **n**, so that one in fact measures the smaller of the two angles between the directions of **n** and **N**. For the sake of clarity we denote by η' the angle so defined ($0\leq\eta'\leq\frac{1}{2}\pi$). The distribution $p'(\eta')$ in η' is then related to the distribution in η by

$$p'(\eta')=p(\eta')+p(\pi-\eta')+p(\pi+\eta')+p(2\pi-\eta')=\frac{2}{\pi}\left\{1+\sum_{M=2,4,\ldots}^{M_{\max}}A_M\cos M\eta'\right\},\quad \left(0\leq\eta'\leq\frac{\pi}{2}\right).\quad (2)$$

[7] See reference 6. The equations are written here in a slightly different notation.

The equation for $p'(\eta')$ is more restricted in form than that for $p(\eta)$ because of the lumping together of probabilities from the four quadrants of η. Some information is then necessarily lost.

The angular momentum and parity conservation laws not only prescribe the above forms for $p(\eta)$ and $p'(\eta')$, however, but also impose other restrictions on the angular distribution. For one thing, the conservation laws can be used to set upper limits on the magnitudes of the Fourier coefficients A_M and B_M; and in any attempt to fit data to expressions of the above form it will be very helpful to take these limits into account. This is especially true in the case of hyperon decay, where, as it turns out, the upper limits on the coefficients are fairly small in certain cases (see Table I). The main purpose of the present work is to compute the limits on the Fourier coefficients. We shall also briefly discuss other consequences of the conservation laws.

Equation (1) is derived by forming the expression $|\psi|^2$—where ψ is the final state wave function in the rest system of the unstable particle—and integrating over all variables except for the azimuthal angle η of the relative momentum vector of the decay products (the z-axis lies along the line of flight of the unstable particle and the y-axis along the reference direction **N**).[6] The Fourier coefficients A_M and B_M are evidently then the respective expectation values with respect to ψ of the operators $2\cos M\eta$ and $2\sin M\eta$[8]:

$$A_M=2\langle\psi|\cos M\eta|\psi\rangle,\qquad B_M=2\langle\psi|\sin M\eta|\psi\rangle.\quad (3)$$

The wave function ψ contains as a factor an angular momentum eigenfunction corresponding to j and having definite parity, determined by the parity of the unstable particle. This function can be expressed as a linear combination of the eigenfunctions φ_j^m of the operator $J_z(-j\leq m\leq j)$. We take the φ_j^m as our basis functions.

TABLE I. Maximum possible value of Fourier coefficient A_M (or B_M) in angular distribution $p(\eta)$ or $p'(\eta')$—for decay process of type: hyperon→nucleon+pion. The spin of the hyperon is denoted by j.

j	M	
3/2	2	0.577
5/2	2	0.775
	4	0.447
7/2	2	0.914
	4	0.577
	6	0.378
9/2	2	1.019
	4	0.655
	6	0.500
	8	0.333

[8] Angular correlations effects must properly be discussed in terms of mixtures of states ψ; and expectation values must correspondingly be taken with respect to density matrices. For the present purposes, however, it is enough to consider only pure states.

We can then form the matrix elements of $\cos M\eta$ and $\sin M\eta$ with respect to the φ_j^m; and it is then evident from Eq. (3) that the maximum possible value of A_M, for example, is just equal to the largest eigenvalue of the resulting matrix $2\langle jm'|\cos M\eta|jm\rangle$. (Since the upper limits on A_M and B_M are clearly identical, we discuss the former only.)

III. LIMITS ON THE FOURIER COEFFICIENTS

For decay processes of both types: (a) hyperon→nucleon+pion; and (b) K-meson→pion+pion, one can show that the matrix elements are given by[9]

$$2\langle jm'|\cos M\eta|jm\rangle = \begin{cases} (-1)^{\frac{1}{2}M}\left[\dfrac{(j-m)!(j+m)!}{(j-m')!(j+m')!}\right]^{\frac{1}{2}\epsilon}, & (m'=m\pm M) \\ 0, & \text{(otherwise)} \end{cases} \quad (4)$$

where

$$\epsilon=\pm 1 \quad \text{for} \quad |m'|\gtrless|m|.$$

As we have already seen, the maximum eigenvalue of the above matrix is just equal to the maximum possible value of the coefficient A_M in Eqs. (1) and (2). Under certain circumstances, mentioned below, it is possible to give a simple formula for the largest eigenvalue for given j and M. In other cases, we have determined the eigenvalues by numerical procedures. Results are set forth in Tables I and II, which refer to hyperon and K-meson decay, respectively.

In the case of hyperon decay (half-odd integral j), whenever $M\geq j+\frac{1}{2}$ the matrix takes the simple form in which nonvanishing elements occur only within the upper-right and lower-left quadrants. The maximum eigenvalue is then simply equal to the magnitude of the largest matrix element. For this, one finds from Eq. (4)

$$A_M^{\max}=\left(\frac{2j+1-M}{2j+1+M}\right)^{\frac{1}{2}}. \quad (5)$$

In particular, for the largest permissible value of M, $M=2j-1$, we have

$$A_{2j-1}^{\max}=(1/2j)^{\frac{1}{2}}. \quad (6)$$

It is apparent from these equations that for the larger values of M, the upper limits on the Fourier coefficients fall off with increasing j; whereas, from Table I we see that $A_M^{\max}$ increases with j for the smaller values of M.

In the case of K-meson decay (integral j), we note that the particular matrix element

$$2\langle j,\tfrac{1}{2}M|\cos M\eta|j,-\tfrac{1}{2}M\rangle$$

has absolute value unity. This means that there is always a state, namely $(1/\sqrt{2})(Y_j^{\frac{1}{2}M}\pm Y_j^{-\frac{1}{2}M})$, for which $A_M=1$. Thus, the upper limit on A_M is always at least unity. In fact, whenever $M\geq j$, the upper limit is precisely equal to unity. For $M<j$, the upper limit exceeds unity, although not by very much when the spin is fairly small (see Table II).

TABLE II. Maximum possible value of Fourier coefficient A_M (or B_M) in angular distribution $p(\eta)$ or $p'(\eta')$—for decay process of type: K-meson→pion+pion. The spin of the K-meson is denoted by j.

j	M	
1	2	1.00
2	2	1.00
	4	1.00
3	2	1.06
	4	1.00
	6	1.00
4	2	1.13
	4	1.00
	6	1.00
	8	1.00

IV. DISCUSSION AND FURTHER APPLICATIONS

It is perhaps worth emphasizing again that on the basis of the angular distribution data taken alone, there is no way to determine the exact value of the spin. As has already been pointed out, even if the spin were very large, the distribution in η might under the experimental conditions be uniform (unpolarized spins). The best one can do is determine a lower limit on the spin. There are many procedures which can be used for this purpose, all of them based on angular momentum and parity conservation and all, in part, independent of one another. The problem is to find the best procedure with respect to a given set of data, i.e., the procedure which leads to the largest lower bound on the spin j.

We have discussed two methods, which are partly independent of each other. The angular distribution $p'(\eta')$ can be expressed as a finite Fourier series. The degree of the highest harmonic required to fit a set of data automatically determines a lower limit on j. However, if any one of the Fourier coefficients A_M required to fit the data exceeds the limit imposed by angular momentum and parity conservation, one obtains a larger lower bound on j. For example, it could happen that the experimental data for a set of Λ^0-particle decay events is perfectly consistent with an angular distribution of the form $p'(\eta')=(2/\pi)(1+A_2\cos 2\eta')$, so that one has immediately $j\geq\frac{3}{2}$. But if the required value of A_2 is say 0.70, then from Table I it follows that in fact $j\geq 5/2$. Furthermore, it should be noted that there is no state for which two or more of the coefficients A_M have their maximum values simultaneously. Thus, when A_2 has its maximum value for given j, A_4 is certain to have a value less than its maximum.

[9] We would like to thank R. K. Adair and R. Dalitz for calling to our attention an error in our original expression for the matrix element.

TABLE III. Maximum possible value of $P'(\frac{1}{4}\pi)$, the probability that $\eta' \leq \frac{1}{4}\pi$—for decay process of type: hyperon→nucleon +pion.

j	
3/2	0.684
5/2	0.747
7/2	0.788
9/2	0.817

TABLE IV. Maximum possible value of $P'(\frac{1}{4}\pi)$, the probability that $\eta' \leq \frac{1}{4}\pi$—for decay process of type: K-meson→pion +pion.

j	
1	0.818
2	0.818
3	0.834
4	0.850

The number of Fourier terms, and the limits on their coefficients, by no means exhausts the limitations imposed by the conservation laws on the shape of $p'(\eta')$ for a given spin. We have emphasized these features because they are simple from a computational point of view and because they provide a useful basis for a first survey of experimental data. One could go on, however, and calculate for example the limits on the various moments of the angular distribution; or the limits on the actual value of $p'(\eta')$ itself for any specified angle η'. In such procedures one maximizes not the individual Fourier coefficients taken one at a time but rather a suitable sum of coefficients. Thus, the state which maximizes A_2, for example, is not the same as the state which maximizes the value of $p'(\eta')$ for η' equal to say 50°. It could happen that a given set of data is perfectly consistent with a certain j value on the basis of the tests described above but that the experimental value of $p'(50°)$, say, exceeds the limit for that value of j.

We mention one particular test which seems to be especially suited to analyzing quickly a limited amount of data and which occurred to us in connection with the Brookhaven results.[5] The Brookhaven workers find that in seven examples of hyperon decay ($4\Lambda^0$; $3Y^{\pm}$) the angle η' between the decay and production planes is less than ~45° in all cases. We therefore ask: for given spin of the hyperon, what is the maximum possible value for the fraction of cases in which $\eta' < \frac{1}{4}\pi$? We denote the integral distribution by $P'(\eta')$, where

$$P'(\eta') = \int_0^{\eta'} p'(\eta'')d\eta''. \tag{7}$$

From Eq. (2), it follows that

$$P'(\tfrac{1}{4}\pi) = \frac{1}{2} + \frac{1}{\pi} \times \sum_{M=2,4,\ldots}^{M_{\max}} \frac{1}{M} A_M (-1)^{\frac{1}{2}(M-2)} [1-(-1)^{\frac{1}{2}M}] = \frac{1}{2} + \frac{1}{\pi}\{A_2 - \tfrac{1}{3}A_6 + \cdots\}. \tag{8}$$

Just as before, we form the matrices A_M with respect to the basis function $\varphi_j{}^m$; and then, summing these matrices according to Eq. (8), we find the matrix for $P'(\frac{1}{4}\pi)$. The maximum eigenvalue gives the largest possible value of $P'(\frac{1}{4}\pi)$. We have carried out this procedure for the case of hyperon decay, for spins up to $j=9/2$ inclusive; and K-meson decay, for spins up to $j=4$ inclusive. The results are given in Tables III and IV, respectively. It should be noted, incidentally, that because of the symmetry about $\frac{1}{4}\pi$, these results also represent the upper limits on the probability that η' lies between $\frac{1}{4}\pi$ and $\frac{1}{2}\pi$.

Comparison of Tables I and III on the one hand with Tables II and IV on the other indicates that the maximum possible deviations from isotropy are much more restricted in the case of hyperon decay than in the case of K-meson decay. Pronounced angular correlations in the case of hyperon decay would immediately force one to a high spin assignment for the hyperon. Although the experimental situation is far from clear as yet, the results reported from Brookhaven seem to point in this direction.