deuteron energy of 3.33 Mev. The yield curves shown in Figs. 3 and 6 were normalized to the absolute values at this energy. The curves have not been corrected for the variation of the sensitivity of the modified long counter with neutron energy. The efficiency of the modified long counter in its position behind the "slow counter," is a maximum for neutrons with an energy of about 1.5 to 2 Mev and it has a lower efficiency for both lower and higher energy neutrons. The decrease in efficiency is about 40% for 5-Mev neutrons and is slightly greater for neutrons of energy less than about 0.3 Mev. Since the correction is complicated by the existence of more than a single neutron energy group, such a correction has not been attempted.

The estimated accuracy to which the absolute cross sections have been determined is $\pm 50\%$.

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# Isotopic Spin Impurity in Light Nuclei. I. Core Impurity

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The introduction of isotopic spin impurity by the Coulomb mixing of different nuclear eigenstates of $T^2$ can occur both through the perturbation of the wave function for nucleons in a $J, T=0$ core by the Coulomb interaction of nucleons in the core, and through the perturbation of the wave function for nucleons outside the core by their Coulomb interaction with nucleons in the core and with each other. In this paper the core impurity, the sum of the squared amplitudes of higher isotopic spin eigenstates ($T\neq 0$), is calculated for the ground state of $N=Z$ even-even nuclei on the Fermi gas model. The core impurity is found to exceed by a large factor the isotopic spin impurity in the wave function for nucleons outside the core.

## I. INTRODUCTION

THE total isotopic spin quantum number $T^2$ exists under the assumption of a "charge-independent" interaction between nucleons of the form $\sum'(a+b\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)$, where $a$ and $b$ are functions of space and spin and $\boldsymbol{\tau}_i$ is the isotopic spin vector for the $i$th nucleon. The primed summation indicates that one sums over $i\neq j$. The present active interest in the isotopic spin quantum number for nuclei began primarily among the experimentalists and among those interested in cataloging and understanding the large amount of experimental information on light nuclei which is being accumulated at an increasing rate. For this purpose the isotopic spin quantum number provides selection rules on each of three types of nuclear reactions: (1) reactions involving absorption and emission of heavy particles, (2) $\beta$-decay, and (3) isomeric transitions. Selection rules for processes of type (1) are usually simple and forbid such reactions as $(d,\alpha)$ going from the ground state of an $N=Z$ nucleus to the $T=1$ states of the final $N=Z$ nucleus. Selection rules for the second process were given by Wigner[1] and are different for the Fermi and Gamow-Teller matrix elements.

$$\begin{aligned}&\text{Fermi:} && \Delta T=0,\\ &\text{Gamow-Teller:} && \Delta T=0,\pm 1,\quad 0\not\rightarrow 0.\end{aligned}$$

Finally selection rules for electric dipole transitions were recently derived by Trainor[2] in supermultiplet theory and more generally by Christy,[3] Radicati,[4] and Gell-Mann and Telegdi.[5] We have discussed this selection rule in some detail and have shown it to be a sensitive test of the validity of the isotopic spin quantum number.[6]

The validity of the selection rules is affected only by a nuclear interaction which does not commute with $T^2$; i.e., by a "charge-dependent" nuclear potential, or by the Coulomb interaction. Consequently, before any conclusions can be drawn about the nuclear potential, the quantitative effect of the Coulomb force on the isotopic spin quantum number must be determined. The possibility of accounting for any observed violations of the above selection rules by ascribing them to the Coulomb potential would strongly suggest a nuclear interaction of the form $\sum'(a+b\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)$. Conversely, the observation of large departures from the isotopic spin selection rules which could not be explained by the Coulomb force would certainly imply the existence of charge-dependent nuclear interactions. Of course, in case Coulomb forces should be shown to give rise to considerable mixing of the states of different isotopic spin, the usefulness of the isotopic spin quantum number would be destroyed.

We are therefore interested in the extent to which one can assign a total isotopic spin quantum number $T$ to the states of light nuclei for which $A\leq 20$. Specifically we want to know how much admixture (sum of the

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[1] E. P. Wigner, Phys. Rev. **56**, 519 (1939).
[2] L. E. H. Trainor, Phys. Rev. **85**, 962 (1952).
[3] R. F. Christy, Pittsburgh Conference on Medium Energy Nuclear Physics, 1952 (unpublished).
[4] L. A. Radicati, Phys. Rev. **87**, 521(L) (1952).
[5] M. Gell-Mann and V. L. Telegdi, Phys. Rev. **91**, 169 (1953).
[6] W. M. MacDonald, Phys. Rev. **98**, 60 (1955).

squared amplitudes) of states of different isotopic spin is introduced into a total isotopic spin eigenstate by the Coulomb interaction. We shall call the amount of admixture—"the isotopic spin impurity of the state."

In the shell model, the simplest nuclear state is the ground state of a nucleus consisting of closed shells in neutrons and protons. Although only a few of the nuclear states are of this kind, the ground state and certain low-lying states of a nucleus with one or more particles in open shells can be regarded as the states of a system consisting of a "core" of particles having $J$, $T=0$ and one or more particles in open shells. Since the isotopic spin states of the same spin and parity are separated by a rather large energy $\sim$15 Mev, in lowest approximation the effect of the Coulomb interaction between core nucleons in promoting the isotopic spin impurity of the ground state of the core can be treated separately from the effect upon the isotopic spin state of the Coulomb interaction of the outside nucleons with each other and with the core.[7] (The effect of the outside nucleons on the isotopic spin eigenstate of the core can be taken into account approximately later, if necessary.)

In the present paper we shall determine the core impurity for the ground state of an $N=Z$ even-even nucleus on the Fermi gas model. The results we obtain consequently will be less dependent on the details of nuclear structure than later calculations on the shell model. The second source of isotopic spin impurity, in the wave function for nucleons in open shells, will be treated in a later paper using the $jj$-coupling model with harmonic oscillator wave functions. A recalculation on this model of core impurity for $C^{12}$, which in an $N=Z$ even-even nucleus and also has closed shells in neutrons and protons, will be in agreement with results to be found here.

The conclusion which we shall reach in these calculations on light nuclei ($A\leq 20$) is that the core impurity is much more important than the impurity of the state of the outside nucleons.[8] This result invalidates the basis for some earlier somewhat less rigorous calculations on the mixing of the isotopic spin states by the Coulomb potential.[7]

## II. DECOMPOSITION OF THE COULOMB OPERATOR

The so-called Coulomb perturbation is usually understood to contain not only the Coulomb potential between protons but also the neutron-proton mass difference. In the isotopic spin formalism the perturbation is

$$H_C=\tfrac{1}{8}e^2\sum{}'(1-\tau_{\zeta i})(1-\tau_{\zeta j})r_{ij}^{-1}+(m_n-m_p)c^2T_\zeta+\tfrac{1}{2}A(m_n+m_p)c^2, \quad (1)$$

where $\tau_\zeta$ has the eigenvalue $(+1)$ for a neutron and $(-1)$ for proton. The $\zeta$-component of isotopic spin then has the eigenvalue $T_\zeta=\frac{1}{2}(N-Z)$ for a nucleus with $N$ neutrons and $Z$ protons. The last two terms of Eq. (1) commute with $T^2$ and, in fact, can produce no mixing of states of either the same or different isotopic spin if these states are orthogonal. These two terms merely will produce relative displacements of the levels of different isobars. To determine the effect of the perturbation produced by the first term, which is the Coulomb potential, one must perform a decomposition into irreducible tensors in isotopic spin space. Each of these tensors has different transformation properties under rotations in isotopic spin space, and there will exist different selection rules on the matrix elements of these tensors. The decomposition can easily be made and the selection rules stated which are relevant for our discussion.

$$H_C=S+T^{(10)}+T^{(20)}.$$

Scalar: $S=\frac{1}{8}e^2\sum{}'(1+\frac{1}{3}\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)r_{ij}^{-1},\quad \Delta T=0$

Vector: $T^{(10)}=-\frac{1}{8}e^2\sum{}'(\tau_{\zeta i}+\tau_{\zeta j})r_{ij}^{-1},$
$\Delta T=0,\pm1, 0\nrightarrow 0 \quad (2)$

Tensor: $T^{(20)}=\frac{1}{8}e^2\sum{}'(\tau_{\zeta i}\tau_{\zeta j}-\frac{1}{3}\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)r_{ij}^{-1},$
$\Delta T=0,\pm1,\pm2, 0\nrightarrow 0, 1.$

The quantity $T^{(ij)}$ is the $j$th component of a tensor of rank $i$.[9] The scalar part $S$ commutes with $T^2$ and can be included in the nuclear Hamiltonian without affecting the validity of the isotopic spin. The tensor $T^{(20)}$ can only mix the $T=0$ with the $T=2$ state, and the large energy separation of these two multiplets will enable us to neglect $T^{(20)}$ in computing the impurity of $T=0$ states. One has to consider only $T=0$ states for the normal states of $N=Z$ nuclei for $A\leq 20$, and only the vector component of $H_C$ therefore need be considered.

According to perturbation theory the perturbation $H_p$ introduces into an eigenstate $\Psi_0$ the impurity $p$, defined as the sum of the squared amplitudes of different eigenstates

$$p=\sum\frac{|(\Psi_\nu,H_p\Psi_0)|^2}{(E_0-E_\nu)^2}. \quad (3)$$

By closure, an upper limit on $p$ is

$$p\leq p_M=\frac{(\Psi_0,H_p{}^2\Psi_0)-|(\Psi_0,H_p\Psi_0)|^2}{(E_0-E_1)^2}. \quad (4)$$

Since the perturbation which we shall use,

$$\mathfrak{C}\equiv T^{(10)}=-\tfrac{1}{8}e^2\sum_{i,j}{}'(\tau_{\zeta i}+\tau_{\zeta j})r_{ij}^{-1}, \quad (5)$$

has zero expectation in the $T=T_\zeta=0$ ground state $\Psi_0$, the maximum impurity is

$$p_M=(\Psi_0,\mathfrak{C}^2\Psi_0)/(E_0-E_1)^2. \quad (6)$$

[7] L. A. Radicati, Proc. Phys. Soc. (London) **A66**, 139 (1953); **A67**, 39 (1953).
[8] W. M. MacDonald, Phys. Rev. **98**, 234(A) (1955).
[9] E. P. Wigner, *Gruppentheorie* (Friedrich Viewag and Sohn, Braunschweig, 1931).

The energy difference $E_0-E_1$ which appears in Eq. (6) should be the energy separation of the ground state $T=0$ from the first excited state of the same spin $J$ and parity but having $T=1$. The importance of the distinction between this energy and the separation of the $T=0$ and $T=1$ multiplets is obvious for the odd-odd nuclei where the first state of the $T=1$ multiplet is only a few Mev above the $T=0$ ground state. In most cases, however, the first $T=1$ state of the same spin and parity as the ground state is not known. We are now interested only in the nuclei with $N=Z$ and $A=4n$, however, and for these the first $T=1$ level lies quite high. In this case we can use the energy separation of the $T=0$ and $T=1$ multiplets since the first $T=1$ state with same spin and parity as the ground state cannot lie much higher. The energies $E(T=0)-E(T=1)$ are estimated from the energy separation of the ground state of the $T_\zeta=0$ member of an isobaric triad from the ground states of the $T_\zeta=1$ components. (See Table I.)

### III. SIMPLE ESTIMATE OF ISOTOPIC SPIN IMPURITY

The matrix element of $\mathfrak{C}^2$ can be estimated by a method which is similar to that used for deriving sum rules. The squared sum $\mathfrak{C}^2$ can be written

$$\mathfrak{C}^2=\frac{e^4}{64}\{2\sum_{i,j}{}'(\tau_{\zeta i}+\tau_{\zeta j})^2 r_{ij}^{-2}+2\sum_{i,j,l}{}'(\tau_{\zeta i}+\tau_{\zeta j})(\tau_{\zeta i}+\tau_{\zeta l})\times r_{ij}^{-1}r_{il}^{-1}+\sum_{i,j,k,l}(\tau_{\zeta i}+\tau_{\zeta j})(\tau_{\zeta k}+\tau_{\zeta l})r_{ij}^{-1}r_{kl}^{-1}\},\quad (7)$$

where the primed summations indicate that different indices never assume the same value. The approximation is now made that the expectation values of the reciprocal separations $r_{ij}^{-2}$, $r_{ij}^{-1}r_{il}^{-1}$, and $r_{ij}^{-1}r_{kl}^{-1}$ are the same for every term of each of the three sums appearing in Eq. (7) and are respectively $\langle r_{12}^{-2}\rangle$, $\langle r_{12}^{-1}r_{13}^{-1}\rangle$, and $\langle r_{12}^{-1}r_{34}^{-1}\rangle$. Each of the three sums over the isotopic spin coordinates can be carried out explicitly to yield

$$\langle\mathfrak{C}^2\rangle\approx\tfrac{1}{16}e^4A(A-1)\{\langle r_{12}^{-2}\rangle+(A-4)\langle r_{12}^{-1}r_{13}^{-1}\rangle-(A-3)\langle r_{12}^{-1}r_{34}^{-1}\rangle\}.\quad (8)$$

This expression vanishes for all $r_{ij}$ constant and equal, as it must since then $\mathfrak{C}=-\frac{1}{8}e^2r_{ij}^{-1}T_\zeta=0$.

The averages appearing in Eq. (8) will be calculated by taking the nucleus to be spherical and of uniform density.

$$\langle r_{12}^{-2}\rangle=V^{-2}\int d\mathbf{r}_1d\mathbf{r}_2r_{12}^{-2};$$

$$\langle r_{12}^{-1}r_{13}^{-1}\rangle=V^{-3}\int d\mathbf{r}_1d\mathbf{r}_2d\mathbf{r}_3r_{12}^{-1}r_{13}^{-1};\quad (9)$$

$$\langle r_{12}^{-1}r_{34}^{-1}\rangle=V^{-4}\int d\mathbf{r}_1d\mathbf{r}_2d\mathbf{r}_3d\mathbf{r}_4r_{12}^{-1}r_{34}^{-1},$$

where $V=(4/3)\pi R^3$ is the nuclear volume and the integrations are performed by allowing the points $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, and $\mathbf{r}_4$ to move through the volume of a sphere of radius $R$. These integrations yield

$$\langle r_{12}^{-2}\rangle=(9/4)R^{-2}=2.25R^{-2};$$
$$\langle r_{12}^{-1}r_{13}^{-1}\rangle=(51/35)R^{-2}=1.457R^{-2};$$
$$\langle r_{12}^{-1}r_{34}^{-1}\rangle=(36/25)R^{-2}=1.44R^{-2}.\quad (10)$$

TABLE I. Energy separation in Mev of isotopic multiplets in $T_\zeta=0$ even-even nuclei.

| $A$ | 8 ($Be^8$) | 12 ($C^{12}$) | 16 ($O^{16}$) | 20 ($Ne^{20}$) |
|---|---|---|---|---|
| $E(T=1)-E(T=0)$ | 16.7 | 15.1 | 12.9 | 10.1 |

Inserting these average values in (8), we have finally

$$\langle\mathfrak{C}^2\rangle=\frac{e^4}{16R^2}A(A-1)(0.76+0.017A).\quad (11)$$

This result is very interesting in its dependence on atomic number. From Eq. (7) one might have expected $\langle\mathfrak{C}^2\rangle$ to be proportional to $A^4R^{-2}$ but this result is proportional to $A(A-1)R^{-2}$ for $A$ less than $\sim 100$. If one writes the impurity limit $p_M$ in the form,

$$p_M=\frac{A(A-1)}{2}\frac{(e^2/R)^2}{8(E_0-E_1)^2}(0.76+0.017A),\quad (12)$$

an obvious interpretation is that the core impurity is just the effect of the perturbation of a single-nucleon wave function by another nucleon multiplied by the number of interacting pairs. Stated in this way, no account has been taken of the differences in interaction between nucleons in different orbits. For example, the Coulomb interaction between nucleons in an $s$ orbit is obviously greater than that between nucleons in a $p$ orbit and the perturbation of the single-particle orbits is consequently greater in the first case than in the second. Nevertheless, the qualitative idea suggested by Eq. (12) is correct as will be seen in the shell model calculations.

### IV. THE FERMI GAS MODEL

This simple evaluation is completely independent of the symmetric structure of the nuclear wave function and does not even approximately take account of the fact that the ground states of $A=4n$ nuclei where $A\leq 20$ are of a four-structure with four nucleons in each space state. This feature of Eq. (8) is a consequence of the separation of the space and isotopic spin dependence of the operator $\mathfrak{C}^2$ in Eq. (7) by the introduction of $\langle r_{12}^{-2}\rangle$, $\langle r_{12}^{-1}r_{13}^{-1}\rangle$, and $\langle r_{12}^{-1}r_{34}^{-1}\rangle$. Although the use of such an approximation means that the coefficients of these averages in Eq. (8) are not really correct, one might expect that the value of $\langle\mathfrak{C}^2\rangle$ given by (11) has not been affected much by this error. The reason is that Eq. (11) is mostly a consequence of the necessary condition that $\mathfrak{C}=0$ for all the $r_{ij}$ of Eq. (5) equal to the same constant ($r_{ij}=a$) and of the near equality of $\langle r_{12}^{-2}\rangle$, $\langle r_{12}^{-1}r_{13}^{-1}\rangle$, and $\langle r_{12}^{-1}r_{34}^{-1}\rangle$.

In order to obtain a more correct evaluation of $\langle \mathfrak{C}^2 \rangle$ which includes the four-structure of the ground state wave function, we shall work with the single-particle model and use a formalism similar to that of Rosenfeld.[10] The expression for the density matrix $(i|g|f)$ found there, however, omit terms which are important for this problem. The extension of these methods to the evaluation of three- and four-particle operators has not been necessary previously.

We begin by describing the nuclear ground state by the wave function $\Psi(Q_1,\cdots,Q_A)$ which is antisymmetric in the coordinates $Q_i$, which represent three space coordinates, a spin coordinate, and an isotopic spin coordinate. In the single-particle model, the antisymmetric ground state wave function can be represented by

$$\Psi=\begin{vmatrix}\psi_1(Q_1)\cdots\psi_1(Q_A)\\ \psi_A(Q_1)\cdots\psi_A(Q_A)\end{vmatrix}, \quad (13)$$

where $\psi_\mu(Q_\nu)$ is a single-particle wave function of the form

$$\psi_\mu(Q_\nu)=\phi_\mu(\mathbf{r}_\nu)u_\mu(\boldsymbol{\sigma}_\nu,\boldsymbol{\tau}_\nu). \quad (14)$$

The function $u_\mu(\boldsymbol{\sigma}_\nu,\boldsymbol{\tau}_\nu)$ is a spin and isotopic spin state while $\varphi_\mu(\mathbf{r}_\nu)$ is a space state. The whole development of the expectation values of two-, three-, and four-nucleon operators is quite general up to a point where the "density matrices" are actually evaluated, but from this point we use for the space states plane waves normalized in the nuclear volume and fill all momentum states up to some wave number $k_m$. This choice constitutes the use of the Fermi gas model.[11]

## V. MATRIX ELEMENTS OF MULTIPLE-NUCLEON OPERATORS

### Two-Nucleon Operators

Consider an operator $W_{12}(Q_1,Q_2)$ on the collective coordinates $Q_1$ and $Q_2$. The expectation value of $W_{12}$ will be

$$\langle W_{12}(Q_1,Q_2)\rangle=\int_{Q1}\cdots\int_{QA}\Psi^*(Q_1,\cdots,Q_A)\times W_{12}(Q_1,Q_2)\Psi(Q_1,\cdots,Q_A). \quad (15)$$

This equation can be rewritten by separating the space and spin coordinates (intrinsic spin and isotopic spin) in $Q_1$ and $Q_2$. Let $s$ denote the "total" spin state of both nucleons and $r_1$, $r_2$ denote their space coordinates. Then

$$\langle W(Q_1,Q_2)\rangle=\sum_s\int_{Q3}\cdots\int_{QA}\mathbf{dr}_1\mathbf{dr}_2\Psi^*\times(\mathbf{r}_1,\mathbf{r}_2,s;Q_3,\cdots,Q_A)W_{12}(Q_1,Q_2)\times\Psi(\mathbf{r}_1,\mathbf{r}_2,s;Q_3,\cdots,Q_A). \quad (16)$$

The summation over $s$ is simply the operation of taking a trace over the 16-dimensional spin and isotopic spin space of the two nucleons. We can make a unitary transformation in this space to the representation in which the states are eigenstates of $\tau_{\zeta 1}$ and $\tau_{\zeta 2}$, the $\zeta$ components of the isotopic spin of the nucleons, and of $\sigma_{z1}$ and $\sigma_{z2}$, the intrinsic spin. The Eq. (17) can be written then

$$\langle W(Q_1,Q_2)\rangle=\sum_{s,s'}\int\mathbf{dr}_1\mathbf{dr}_2(s|g(\mathbf{r}_1,\mathbf{r}_2)|s')(s'|W_{12}|s), \quad (17)$$

where $s$ and $s'$ are now quantum numbers for the total spin state of the two nucleons and $(s|g(\mathbf{r}_1,\mathbf{r}_2)|s')$ is the density matrix defined by

$$(s|g(\mathbf{r}_1,\mathbf{r}_2)|s')=\int_{Q3}\cdots\int_{QA}\Psi^*(r_1,r_2,s;Q_3,\cdots,Q_A)\times\Psi(r_1,r_2,s';Q_3,\cdots,Q_A). \quad (18)$$

In the state $\Psi$ given by Eq. (13), we shall take every space state to be completely occupied by four nucleons, two neutrons, and two protons, with each pair of like nucleons having opposite spins. We shall thus be treating the normal states of nuclei with $A=4n$, $T=T_\zeta=0$. In writing the exchange terms which appear in $(s|g(\mathbf{r}_1,\mathbf{r}_2)|s')$, we shall make use of the total spin exchange operator (intrinsic spin and isotopic spin): $P_{\sigma\tau}{}^{(ij)}=\frac{1}{2}(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(1+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)$. The density matrix is then

$$(s|g(\mathbf{r}_1,\mathbf{r}_2)|s')=[A(A-1)]^{-1}\times(s|\sum_{\mu,\nu}^{m}\phi_\mu{}^*(1)\phi_\nu{}^*(2)[\phi_\mu(1)\phi_\nu(2)-P_{\sigma\tau}{}^{12}\phi_\mu(2)\phi_\nu(1)]+\delta(s,s_{12})\sum_{\mu}^{m}|\phi_\mu(1)|^2|\phi_\mu(2)|^2(1-P_{\sigma\tau}{}^{12})|s'), \quad (19)$$

where $s_{12}$ denotes the total spin state (spin and isotopic spin) of nucleons 1 and 2 for which the two nucleons do not both have the same spin and isotopic spin. The $\delta(s,s_{12})$ restricts the sum over spin states $s$ to those states compatible with the requirement that the same single particle state cannot be occupied by both nucleons 1 and 2.

At this point, we introduce for the $\phi_\nu$ of Eq. (19) plane waves normalized to volume $V=(4/3)\pi R^3$,

$$\phi_\mu=V^{-\frac{1}{2}}\exp(i\mathbf{k}_\mu\cdot\mathbf{r}), \quad (20)$$

with the free particle states being taken as dense in $k$-space. We can replace sums over $k_\mu$, therefore, by integrals up to a maximum wave number $k_m$ given by $k_m=(9\pi)^{\frac{1}{3}}/2r_0=1.523/r_0$, where $R=r_0A^{\frac{1}{3}}$. The sums which appear in (19) are then easily evaluated:

$$\sum_{\mu,\nu}^{m}|\phi_\mu(1)|^2|\phi_\nu(2)|^2=A(A-1)/16V^2,$$

$$\sum_{\mu,\nu}^{m}\phi_\mu{}^*(1)\phi_\nu{}^*(2)\phi_\mu(2)\phi_\nu(1)=[A(A-1)/16V^2]G^2(r_{12}),$$

$$G(r)\equiv\frac{3}{k_mr}j_1(k_mr)=3\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{J_{\frac{3}{2}}(k_mr)}{(k_mr)^{\frac{3}{2}}}. \quad (21)$$

[10] L. Rosenfeld, *Nuclear Forces* (Interscience Publishers, Inc., New York, 1948).

[11] H. Bethe and R. Bacher, Revs. Modern Phys. **8**, 82 (1936). See also reference **10**.

The function $G(r_{12})$ is designated as the equivalent nucleon correlation function since it gives the probability of a certain relative separation of two nucleons of the same spin and isotopic spin.

If we insert the sums of Eq. (21) into Eq. (19), we obtain

$$(s|g(\mathbf{r}_1,\mathbf{r}_2)|s')=\left(s\left|\frac{1}{16V^2}[1-P_{\sigma\tau}{}^{12}G^2(r_{12})]\times\left[1+\frac{4}{A-1}\delta(s,s_{12})\right]\right|s'\right). \quad (22)$$

### Three-Nucleon Operator

The derivation of the density matrices for three and four particle operators is very straightforward. Denote the permutations $P$ of the symmetric group by

$$P=\begin{pmatrix}\cdots\alpha_i{}^P\cdots\\ \cdots\beta_i{}^P\cdots\end{pmatrix},$$

where the $\alpha_i{}^P$ designate only the numbers which are not permuted into themselves by $P$. Further, let $\delta_P$ be the sign of the transposition so that $\delta_P=\pm1$ according as $P$ is even or odd. Then, if the expectation value of $W_{123}$ is given by

$$\langle W_{123}\rangle=\sum_{s,s'}\int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \times(s|g(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)|s')(s'|W_{123}|s), \quad (23)$$

the density matrix is just

$$(s|g(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)|s')=\left(s\left|[1+\sum_P \delta_P P_{\sigma\tau}{}^P(\textstyle\prod_i G(r_{\alpha_i{}^P\beta_i{}^P}))]\times\left[1+\frac{4}{A-1}[\delta(s,s_{12})+\delta(s,s_{13})+\delta(s,s_{23})]+\frac{16}{(A-1)(A-2)}[\delta(s,s_{123})+\delta(s,s_{132})]\right]\right|s'\right), \quad (24)$$

where $s_{123}$ is a permissible total spin state for three nucleons occupying the same space state. The permutations $P_{\sigma\tau}{}^P$ on the spin and isotopic spin are all the permutations belonging to the symmetric group on three symbols.

### Four-Nucleon Operator

The matrix element for the four-nucleon operator is just

$$\langle W_{1234}\rangle=\sum_{s,s'}\int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 d\mathbf{r}_4 \times(s|g(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)|s')(s'|W_{1234}|s), \quad (25)$$

where the proper density matrix is

$$(s|g(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)|s')=\Bigg(s\Bigg|[1+\sum_P \delta_P P_{\sigma\tau}{}^P(\textstyle\prod_i G(r_{\alpha_i{}^P\beta_i{}^P}))]\times\Bigg[1+\frac{4}{A-1}[\delta(s,s_{12})+\delta(s,s_{13})+\delta(s,s_{14})+\delta(s,s_{23})+\delta(s,s_{24})+\delta(s,s_{34})]+\frac{16}{(A-1)(A-2)}\times[\delta(s,s_{124})+\delta(s,s_{234})+\delta(s,s_{123})]+\frac{16}{(A-1)(A-2)}\times[\delta(s,s_{12})\delta(s,s_{34})+\delta(s,s_{13})\delta(s,s_{24})+\delta(s,s_{14})\delta(s,s_{23})]+\frac{64}{(A-1)(A-2)(A-3)}\delta(s,s_{1234})\Bigg]\Bigg|s'\Bigg). \quad (26)$$

These expressions are rather lengthy, but the calculation of expectation values with these density matrices is greatly simplified in two ways. First, it will be noticed that all the permutations which are equivalent under the normal subgroup which leaves the operator invariant, contribute equal integrals to the expectation value. Therefore only one of the terms which are equivalent under this group need be considered, multiplied by the number of such terms. Secondly, a large number of terms will be found to contribute nothing to the expectation value when the sum over total spin states is evaluated.

## VI. EVALUATION OF $\langle\mathfrak{C}^2\rangle$ FOR $T_\zeta=0$, $A=4n$ NUCLEI

From Eq. (7), the squared Coulomb operator can be written as

$$\mathfrak{C}^2=\frac{e^2}{16}\{\sum_{i,j}{}'W_{ij}+\sum_{i,j,l}{}'W_{ijl}+\sum_{i,j,k,l}{}'W_{ijkl}\} \quad (27)$$

where

$$W_{ij}=(\tau_{\zeta i}+\tau_{\zeta j})^2 r_{ij}^{-2},\ W_{ijl}=(\tau_{\zeta i}+\tau_{\zeta j})(\tau_{\zeta i}+\tau_{\zeta l})r_{ij}^{-1}r_{il}^{-1},$$
$$W_{ijkl}=(\tau_{\zeta i}+\tau_{\zeta j})(\tau_{\zeta k}+\tau_{\zeta l})r_{ij}^{-1}r_{kl}^{-1}, \quad (28)$$

and primed summations indicate that different indices never assume the same value. From the complete antisymmetry of the nuclear wave function, it follows that

$$\langle\mathfrak{C}^2\rangle=\tfrac{1}{16}e^4\{\tfrac{1}{2}A(A-1)\langle W_{12}\rangle+A(A-1)(A-2)\langle W_{123}\rangle+\tfrac{1}{4}A(A-1)(A-2)(A-3)\langle W_{1234}\rangle\}. \quad (29)$$

The expectation value of each of the multi-nucleon operators is to be found using Eqs. (22), (24), and (26). In finding these expectation values, we shall omit the exchange terms containing integrals over the correlation function $G(r)$ and defer consideration of their magnitude and effect.

To terms of order $1/A$, the expectation value of $W_{12}$ is

$$\langle W_{12}\rangle=\frac{1}{16V^2}\sum_{s,s'}\int \mathbf{dx}_1\mathbf{dx}_2(s|W_{12}|s')(s'|1|s). \quad (30)$$

The sum over the total spin states can be evaluated

$$\mathrm{Tr}W_{12}=32r_{12}^{-2},$$

and the result found:

$$\langle W_{12}\rangle=\langle r_{12}^{-2}\rangle. \quad (31)$$

Neglecting only terms of order $A^{-1}$ in $\langle W_{12}\rangle$ implies that $\langle W_{123}\rangle$ must be correct up to terms of order $A^{-2}$.

$$\langle W_{123}\rangle=\frac{1}{64V^3}\sum_{s,s'}\int \mathbf{dx}_1\mathbf{dx}_2\mathbf{dx}_3(s|W_{123}|s')$$
$$\times\left(s'\left|1+\frac{4}{A-2}[\delta(s,s_{12})+\delta(s,s_{13})+\delta(s,s_{23})]\right|s\right). \quad (32)$$

In evaluating (26), the sums over total spin states give

$$\mathrm{Tr}W_{123}=64r_{12}^{-1}r_{13}^{-1},\quad \mathrm{Tr}[W_{123}\delta(s,s_{12})]=32r_{12}^{-1}r_{13}^{-1},$$
$$\mathrm{Tr}[W_{123}\delta(s,s_{13})]=0.$$

The expectation value is then

$$\langle W_{123}\rangle=\left(\frac{A+4}{A-2}\right)\langle r_{12}^{-1}r_{13}^{-1}\rangle. \quad (33)$$

Finding the expectation value $\langle W_{1234}\rangle$ is slightly more trouble since one must now carry all terms up to order $A^{-3}$. In this calculation some peculiar features arise from the fact that in some isotopic spin states for four nucleons $W_{1234}$ is positive, and in others it is negative. In the case of $W_{12}$ and $W_{123}$, only positive values were possible. A consequence of this fact is that $\mathrm{Tr}W_{1234}=0$. The expression which would give rise to a term in $\langle \mathfrak{C}^2\rangle$ proportional to $A^4$ therefore vanishes here just as it did in the sum rule calculations.

The expectation value of $\langle W_{1234}\rangle$ is found from Eq. (26) to order $A^{-3}$ to be

$$\langle W_{1234}\rangle=\frac{1}{256V^4}\sum_{s,s'}\int \mathbf{dr}_1\mathbf{dr}_2\mathbf{dr}_3\mathbf{dr}_4(s|W_{1234}|s')$$
$$\times\left(s'\left|1+\frac{4}{A-1}[\delta(s,s_{12})+\delta(s,s_{13})+\delta(s,s_{14})\right.\right.$$
$$+\delta(s,s_{23})+\delta(s,s_{24})+\delta(s,s_{34})]+\frac{16}{(A-1)(A-2)}$$
$$\times[\delta(s,s_{124})+\delta(s,s_{134})+\delta(s,s_{234})+\delta(s,s_{123})]$$
$$+\frac{16}{(A-1)(A-2)}[\delta(s,s_{12})\delta(s,s_{34})+\delta(s,s_{13})\delta(s,s_{24})$$
$$\left.\left.+\delta(s,s_{14})\delta(s,s_{23})]\right|s\right). \quad (34)$$

When the traces over spin and isotopic spin have been performed, the result is just

$$\langle W_{1234}\rangle=-\frac{4(A+5)}{(A-2)(A-3)}\langle r_{12}^{-1}r_{34}^{-1}\rangle. \quad (35)$$

Using now Eqs. (29), (31), (33), and (35), the expectation value of $\langle \mathfrak{C}^2\rangle$ is just

$$\langle \mathfrak{C}^2\rangle=\tfrac{1}{16}e^4A(A-1)\{\langle r_{12}^{-2}\rangle+(A+4)\langle r_{12}^{-1}r_{34}^{-1}\rangle$$
$$-(A+5)\langle r_{12}^{-1}r_{34}^{-1}\rangle\}. \quad (36)$$

When the average values from Eq. (10) are inserted, this becomes

$$\langle \mathfrak{C}^2\rangle=\frac{e^4}{16R^2}A(A-1)\{0.878+0.017A\}. \quad (37)$$

As we expected, although the expression for $\langle \mathfrak{C}^2\rangle$ in Eq. (36) appears to be rather different from the result given in Eq. (8), the insertion of numerical values for the averages yields results given by Eqs. (11) and (37) which are nearly the same.

## VII. CORRELATION EFFECTS

The effect of correlations introduced by the Pauli principle on $\langle \mathfrak{C}^2\rangle$ can be seen easily. The averages $\langle r_{12}^{-2}\rangle$, $\langle r_{12}^{-1}r_{13}^{-1}\rangle$, and $\langle r_{12}^{-1}r_{34}^{-1}\rangle$ which we have calculated by assuming a uniform distribution of nuclear matter completely neglects the effects of correlation embodied in the Pauli principle. For example, two nucleons in the same spin and isotopic spin state can never come into coincidence, and for such pairs of nucleons $\langle r_{12}^{-2}\rangle$ should be smaller. The effect of this correlation should be greatest on $\langle r_{12}^{-2}\rangle$ in fact, since contributions from the singularity are sharply reduced. The average $\langle r_{12}^{-1}r_{13}^{-1}\rangle$ is affected somewhat less because a correlation of nucleons 2 and 3 does not reduce the contributions from singularities of $r_{12}^{-1}$ and $r_{13}^{-1}$. Since there is very little reduction of $\langle r_{12}^{-1}r_{34}^{-1}\rangle$ produced by any correlation of nucleons 1, 3 with nucleons 2, 4, this average is affected least of all.

A more physical way of seeing the effect of correlations on the averages proceeds from the physical interpretation of these quantities. The significance of $\langle r_{12}^{-2}\rangle$ lies in providing a measure of the density fluctuations in the nucleus. The quantity $\langle r_{12}^{-1}r_{13}^{-1}\rangle$ on the other hand correlates the "density" in one direction from a point with the "density" in another direction from that point. We may interpret $\langle r_{12}^{-1}r_{13}^{-1}\rangle$ as a measure of the "angular" uniformity about a point. The last quantity $\langle r_{12}^{-1}r_{34}^{-1}\rangle$ correlates the density at one point with that at another point. In this sense $\langle r_{12}^{-1}r_{34}^{-1}\rangle$ measures the uniformity in density of the nucleus. From the physical considerations one sees, therefore, that the effect of correlations will be greatest on $\langle r_{12}^{-2}\rangle$, less on $\langle r_{12}^{-1}r_{13}^{-1}\rangle$, and least of all on $\langle r_{12}^{-1}r_{34}^{-1}\rangle$.

Using this knowledge of the relative effects of correlations on the averages appearing in Eqs. (8) and (36), one sees from these equations that the effect of correlations is to decrease the magnitude of $\langle\mathcal{C}^2\rangle$. This result is reasonable when one realizes that the perturbation $\mathcal{C}$ produces an apparent attraction of neutrons and an apparent repulsion of protons. Since we actually evaluate $\langle\mathcal{C}^2\rangle$, the effect of $\mathcal{C}$ on any pair of neutrons or protons can be regarded as a mutual repulsion. Now if the original wave function contains any correlation of this kind, the perturbation produced by $\mathcal{C}$ will be reduced. The Fermi gas model does provide a correlation between any pair of neutrons or protons which have the same spin. To the extent that this correlation coincides with the mutual repulsion produced by $\mathcal{C}$ will the effect of $\mathcal{C}$ in bringing in higher isotopic spin states be reduced.

The correlation terms in $\langle\mathcal{C}^2\rangle$ will appear as a linear combination $f(A)$ of integrals involving $G(r)$, so that

$$\langle\mathcal{C}^2\rangle=\tfrac{1}{16}e^4\{\langle r_{12}^{-2}\rangle+(A+4)\langle r_{12}^{-1}r_{13}^{-1}\rangle -(A+5)\langle r_{12}^{-1}r_{34}^{-1}\rangle-f(A)\}. \quad (38)$$

Clearly $f(A)$ cannot exceed the value of other terms in curly brackets even if $G(r)$ were unity for $r\leq R$. The short range of $G(r)$ causes the correlation integrals in $f(A)$ to decrease as $k_mR$ increases. The integral which decreases most slowly with increasing $k_mR$ is the correlation integral for $r_{12}^{-2}$, as we expected, and is

$$-V^{-2}\int \mathbf{dr}_1\mathbf{dr}_2 r_{12}^{-2}G(r_{12}).$$

The ratio of $\langle r_{12}^{-2}\rangle$ to this integral is approximately $(k_mR)=(1.523A^{\frac{1}{3}})$, and even for $A\sim 8$ the correlation terms would only reduce $\langle\mathcal{C}^2\rangle$ by about one-third at the very most. Our values for maximum isotopic spin impurity are not presumed to be more accurate than this in any case.

## VIII. IMPURITIES IN THE $T=0$ GROUND STATE OF $T=0$ EVEN-EVEN NUCLEI

The expression for $\langle\mathcal{C}^2\rangle$ which has just been found may be used in Eq. (6) with the energy differences listed in Table I to find $p_M$, the upper limit on the isotopic spin impurity for the ground state of nuclei with $A=4n$ and $T_\zeta=0$. (See Table II.) To be useful these values of $p_M$ should be quite close to the actual value of the isotopic spin impurity for these states found from Eq. (3). We have actually verified that $p\approx p_M$ for the statistical model by explicitly writing the perturbation expansion of Eq. (3), using for the state $\Psi_0$ that given by Eq. (13), and for the excited states $\Psi_\nu$ a Slater determinant of free-particle states with one nucleon excited to the continuum above the Fermi sphere.[12] In addition, the perturbation of a single-particle orbit found in this way agrees with that suggested by Eq. (12).

TABLE II. Statistical model estimate of isotopic spin impurity for ground states of nuclei with $A=4n$ and $T_\zeta=0$.

| Nucleus | $Be^8$ | $C^{12}$ | $O^{16}$ | $Ne^{20}$ |
|---|---|---|---|---|
| $p_M$ | $2.6\times10^{-3}$ | $7.5\times10^{-3}$ | $1.9\times10^{-2}$ | $3.9\times10^{-2}$ |

These values of core impurity, while small, are not negligible and should be detected, for example, in violations of the isotopic spin selection rule for electric dipole transitions. The core impurity is in fact much larger than the isotopic spin impurity produced by Coulomb distortion of the wave function for nucleons outside the core.[13]

We wish to remark that although the neglect of correlations in $\langle\mathcal{C}^2\rangle$ gives a $p_M$ which is an upper limit on isotopic spin impurity, one must not conclude that Eq. (37) represents an upper limit on the predictions of more detailed single-particle models. For even though a nuclear wave function of the form of Eq. (13) contains certain correlations, the individual-particle wave functions are smooth $(|\psi|^2=V^{-1})$ and do not provide a positive internucleon correlation. A wave function, or set of wave functions, which describes a state of pronounced maxima or minima of the nuclear density in certain regions could increase $\langle r_{12}^{-2}\rangle$ and $\langle r_{12}^{-1}r_{13}^{-1}\rangle$ relative to $\langle r_{12}^{-1}r_{34}^{-1}\rangle$ and thus increase $p$. At the same time such a set of wave functions will not provide so large an overlap for all pairs of nucleons so that $p$ can also decrease. This latter result is what we shall actually find when we calculate the isotopic spin impurity of the normal state of $C^{12}$ on the $jj$-coupling model with harmonic oscillator wave functions.

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[12] Wm. M. MacDonald, Princeton dissertation, 1954 (unpublished).

[13] The large value of $2.5\times10^{-3}$ obtained by Radicati (reference 8) for the isotopic spin impurity of the ground state of $N^{14}$ is the result of an unrealistic representation of the Coulomb potential produced by the core nucleons ($C^{12}$). A more rigorous calculation gives $1\times10^{-4}$ with his value of the energy separation.