Connection Between the S-Matrix and the Tensor Force

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The present paper is a study of the radial Schrödinger equations for the case of an interaction between the lth and the $(l+2)$nd angular momentum, produced by the tensor force in the presence of spin-orbit coupling. It contains a number of theorems, known for central potentials, concerning the low-energy behavior of the S-matrix, bound states, and zero-energy resonance. The construction, in two stages, of all potentials belonging to a given S-matrix and given bound states, is described. The step from the spectral function to the potential involves the generalization of the Gel'fand Levitan equation given in a recent paper; that from the S-matrix to the spectral function, a procedure due to Plemelj also outlined in that paper. The latter procedure leads to a restriction on the S-matrix necessary for a short range potential to exist. If there is such a potential, it is uniquely determined by the S-matrix, the binding energies, and as many real, symmetric, positive semidefinite matrices as there are bound states.

1. INTRODUCTION

IN the work of Bargmann,[1,2] Jost and Kohn,[3–5] and Levinson,[6,7] it was shown that the scattering phase shift of one angular momentum, l, given as a function of the energy, if caused by a short-range central potential, together with the L_l bound state energies and a set of L_l positive parameters C_n, uniquely determines that potential. Moreover, the latter can be constructed from the former by solving a Fredholm integral equation first derived by Gel'fand and Levitan.[5,7,8] It is the primary purpose of the present paper to generalize the work of the above authors to such noncentral potentials as that of a combination of spin-orbit coupling and the tensor force.[9,10] As a first step in that direction, the author and Res Jost have recently extended the construction procedure to the (physically fictitious) case of a finite number of coupled radial Schrödinger equations for S-states.[11] The then neglected centrifugal barrier is now fully taken into account.

In addition to the construction of the potential from the S-matrix, the bound states, and as many matrices as there are bound states, a number of generalizations of theorems, known for central potentials, concerning bound states and the low-energy behavior of the S-matrix are proved.

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[10] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York and London, 1952), pp. 94 ff.
[11] R. G. Newton and R. Jost, Nuovo cimento **1**, 590 (1955), hereafter referred to as (I).

Section 2 contains several kinds of solutions of the radial Schrödinger equation with coupling between the lth and the $(l+2)$nd angular momentum, and their properties as functions of the distance and of the energy. The S-matrix is defined and shown to be unitary and symmetric. At the end of the section a theorem is proved concerning the low-energy behavior of the S-matrix, being the generalization of the statement that for a large class of potentials the lth phase shift goes as k^{2l+1} near $k=\sqrt{E}=0$. Section 3 deals with the bound states and the special possibility of a resonance with no bound state at zero energy for $l=0$. While the former are independent of the S-matrix (except for their total number), the latter is not. Subsection 3*a* treats the bound states for negative energy; 3*b* that of zero energy and $l\geqslant 1$; 3*c* the bound state and resonance for zero energy and $l=0$ (coupled with $l=2$); 3*d* the generalization of the theorem at the end of Sec. 2 to the case of bound states or resonance at $E=0$; 3*e* the only connection between bound states and the S-matrix for short range potentials.

Section 4 contains the completeness relation and the definition of the spectral function. The generalization of the Gel'fand Levitan equation derived in (I) connects the latter directly with the potential. In Sec. 5 the construction of the spectral function from the S-matrix, the bound state energies, and a set of real, symmetric, positive semidefinite matrices is described. In contrast to the case of a central potential, this step in the construction is rather involved and, as in (I), leads to certain restrictions on the S-matrix necessary for the existence of a short-range potential.

There are three appendices. The first supplies all the convergence and existence proofs as well as a number of inequalities necessary for the work; Appendix B contains the proof of an ideal-theoretic theorem needed for the existence of an "irregular solution" of the Schrödinger equation as an entire function of the energy. Appendix C proves a special theorem about matrix functions needed for the completeness relation in the case of coupling between S- and D-states.

The general position- and spin-dependent potential between two particles is of the form[10]

$$\mathcal{V}(r)=v_d(r)+v_\sigma(r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+v_t(r)S_{12}. \qquad (1.1)$$

When this is applied to the wave function of two spin $\frac{1}{2}$ particles in a triplet state, whose two relevant radial components, of the lth and the $(l+2)$nd angular momentum, may be combined in the row-matrix or vector

$$U=(u,w),$$

and whose spin-angle components are

$$Y=(Y_{j,l,s}{}^M,Y_{j,l+2,s}{}^M),$$

then, in matrix notation,

$$\mathcal{V}UY^T=UVY^T.$$

Here V is a real, symmetric (2×2)-matrix whose elements are functions of r alone. For $l=0$,

$$V=\begin{pmatrix} v_c & 2\sqrt{2}v_t \\ 2\sqrt{2}v_t & v_c-2v_t\end{pmatrix},$$

where[10] $v_c=v_d+v_\sigma$. Only *two* of the elements in the matrix V are independent.

From the point of view of the S-matrix any restriction on V beyond symmetry and reality is unnatural. No criterion is known for the scattering to arise from a potential V that satisfies the condition necessary to be associated with the $\mathcal{V}$ of (1.1). In order to remove this restriction, a spin-orbit potential of the form $v_o(r)\mathbf{S}\cdot\mathbf{L}$ must be added to (1.1). Such a potential has the property

$$v_o(r)\mathbf{S}\cdot\mathbf{L}UY^T =\tfrac{1}{2}v_o(r)U\begin{pmatrix} J^2-l^2-S^2 & 0 \\ 0 & J^2-(l+2)^2-S^2\end{pmatrix}Y^T.$$

Consequently, the potential

$$\mathcal{V}'=v_d(r)+v_\sigma(r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+v_t(r)S_{12}+v_o(r)\mathbf{S}\cdot\mathbf{L}, \qquad (1.2)$$

when applied to a triplet state wave function, manifests itself as a matrix-multiple of the radial part, UV (the spin-angle matrix that always multiplies from the right being neglected), where V is a real, symmetric (2×2)-matrix function of r and otherwise unrestricted. The matrix V and the set of three potentials v_c, v_t, and v_o stand in one-to-one correspondence to each other. Among the latter it is the tensor force alone that produces coupling between the lth and the $(l+2)$nd angular momentum.

2. SOLUTIONS AND THE S-MATRIX

In view of the statements in the introduction, the Schrödinger equation for the radial part of the wave function containing the components of angular momentum l and $l+2$ can, in the presence of the potential $\mathcal{V}'$ of (1.2), be written as follows[12]:

$$\psi''(E,r)+E\psi(E,r)=\psi(E,r)[r^{-2}l(l+1)(1-P) +r^{-2}(l+2)(l+3)P+V(r)], \qquad (2.1)$$

where

$$P=\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}, \qquad (2.2)$$

and $V(r)$ is a real symmetric (2×2)-matrix function of r; $\psi(E,r)$ is a (2×2)-matrix whose rows are individually vector solutions of (2.1). "A solution of (2.1)" will always mean such a square matrix of two vector solutions.

In the space of (2×2)-matrices it will be useful to define a norm:

$$|M|\equiv 2\max_{\alpha,\beta=1,2}(|M_{\alpha\beta}|), \qquad (2.3)$$

which obeys both the triangle and Schwarz's inequalities. Furthermore, we use the notation $\mathfrak{M}(s)$ for the class of (matrix valued) functions whose sth absolute moment exists[13]:

$$M(r)\epsilon\mathfrak{M}(s) \text{ if } \int_0^\infty dr\, r^s|M(r)|<\infty. \qquad (2.4)$$

The assumptions on the potential will not be entirely fixed. For the main purpose it will be assumed that $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(5)$, although for $l=0$ it will suffice that $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(4+\delta)$, $\delta>0$. For the purpose of one theorem concerning the behavior of the S-matrix at low energies (see the end of this section) we will need $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(2l+6)$. The only other assumption, needed in Sec. 5, will be

$$\int_0^\infty dr\, r|V(r+R)-V(r)|<CR^\delta, \text{ for some } \delta>0. \qquad (2.5)$$

Several different kinds of solutions of (2.1) will be used. The first is $G(k,r)$, $(k^2=E)$, which vanishes at[14]

[12] Differentiation with respect to r will be indicated by a prime; r is the radial distance in units of $\hbar/mc\sqrt{2}$, E the energy in units of mc^2, V the potential in units of mc^2; m is the reduced mass if (2.1) describes the interaction between two particles. The first derivative with respect to k, where $k^2=E$, is denoted by a dot, the second by a dash.

[13] The symbolism customary in mathematics is used: "ϵ" stands for "is a member of;" "$\cap$" stands for the intersection of two classes, i.e., the members of $\mathfrak{M}_1\cap\mathfrak{M}_2$ are all elements which are both in $\mathfrak{M}_1$ and in $\mathfrak{M}_2$.

[14] It is well known that, in contrast to the scalar case (i.e., with no coupling between different angular momenta) in general no "regular" solution can be defined for the tensor force without interference with the logarithmic term. (There is one regular vector solution without a logarithm, but the second row of the matrix solution generally contains log r.) It is correspondingly difficult to define this solution by a boundary condition. The obvious integral equation for this solution diverges at $r=0$ unless the existence of large negative moments of the potential is assumed. If to the inhomogeneity of this integral equation, however, a judicious (V-*dependent*) multiple of the solution of (2.1) with $V=0$ is added, the integral is made to converge. The result is (2.6).

$r=0$. It is defined by the integral equation

$$G(k,r)=G_0(k,r)+(2l+3)\int_1^r dt\, t^{-1}(\mathbf{1}-P)V(t)G_0(k,r)+\int_0^r dt[G(k,t)V(t)\mathcal{G}(k;t,r)-(2l+3)t^{-1}(\mathbf{1}-P)V(t)PG_0(k,r)], \quad (2.6)$$

which, if $V\epsilon\mathfrak{M}(0)$, has a unique solution obtainable by successive approximations.[15] The following functions occur in (2.6): the free solution

$$G_0(k,r)=\begin{pmatrix} k^{-(l+1)}u_l(kr) & 0 \\ 0 & k^{-(l+3)}u_{l+2}(kr)\end{pmatrix}, \quad (2.7)$$

and the Green's function

$$\mathcal{G}(k;t,r)=\begin{pmatrix} g_l(k;t,r) & 0 \\ 0 & g_{l+2}(k;t,r)\end{pmatrix}, \quad t\leqslant r, \quad (2.8)$$

where

$$g_l(k;t,r)=k^{-1}[u_l(kt)v_l(kr)-u_l(kr)v_l(kt)],$$
$$=\tfrac{1}{2}i(-1)^l k^{-1}[w_l(kr)w_l(-kt)-w_l(-kr)w_l(kt)], \quad (2.9)$$

$$u_l(x)=xj_l(x),$$
$$v_l(x)=xn_l(x), \quad (2.10)$$
$$w_l(x)=i(-1)^{l+1}xh_l^{(1)}(-x)=-[v_l(x)+iu_l(x)],$$

and j_l, n_l, $h_l^{(1)}$ are the customary[16] spherical Bessel functions, spherical Neumann functions, and spherical Hankel functions of the first kind, respectively. The solution $G(k,r)$ of (2.6) and (2.1) is[15] an even, entire function[17,18] of k for all r. If $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(1)$, then,[15,19] with $\nu=\mathrm{Im}k$,

$$(\mathbf{1}-P)G(k,r)=(\mathbf{1}-P)[k^{-(l+1)}\sin(kr-\tfrac{1}{2}\pi l)\mathbf{1}+O(k^{-(l+2)}e^{|\nu|r})], \text{ as } |k|\to\infty,$$
$$PG(k,r)=P[-k^{-(l+3)}\sin(kr-\tfrac{1}{2}\pi l)\mathbf{1}+O(k^{-(l+4)}e^{|\nu|r})], \text{ as } |k|\to\infty, \quad (2.11)$$

and

$$(\mathbf{1}-P)G(k,r)=(\mathbf{1}-P)\{[(2l+1)!!]^{-1}r^{l+1}\mathbf{1}+O(r^{l+2})\}, \text{ as } r\to 0,$$
$$PG(k,r)=P\{[(2l+5)!!]^{-1}r^{l+3}\mathbf{1}+O(r^{l+4})\}, \text{ as } r\to 0. \quad (2.12)$$

If we define a Wronskian matrix[20]

$$[\Phi;\Psi]\equiv\Phi(r)\Psi^{T\prime}(r)-\Phi'(r)\Psi^T(r), \quad (2.13)$$

which is independent of r if both Φ and Ψ satisfy (2.1) (due to the symmetry of V), then

$$[G;G]=0. \quad (2.14)$$

An "irregular" solution $I(k,r)$ of (2.1) that for all $r>0$ is an entire function of k and satisfies the equations

$$[I;G]=\mathbf{1}, \quad (2.15)$$

$$[I;I]=0, \quad (2.16)$$

can be defined as follows. Let $\psi_1(k,r)$ and $\psi_2(k,r)$ be two solutions of (2.1) fixed by the boundary conditions

$$\psi_1(k,1)=\psi_2'(k,1)=\mathbf{1}, \quad \psi_1'(k,1)=\psi_2(k,1)=0.$$

They are entire functions of k for all $r>0$. Let

$$G(k,r)=M_1^T(k)\psi_1(k,r)+M_2^T(k)\psi_2(k,r),$$
$$I_1(k,r)=-N_2(k)\psi_1(k,r)+N_1(k)\psi_2(k,r).$$

In order for $I_1(k,r)$ to obey (2.15), N_1 and N_2 must satisfy

$$N_1(k)M_1(k)+N_2(k)M_2(k)=\mathbf{1}. \quad (2.17)$$

Both M_1 and M_2 are entire functions of k. Moreover, (2.12) shows that, for every k_0, the only vector a with the property $M_1(k_0)a=M_2(k_0)a=0$ is $a=0$. It is proved in Appendix B that under these circumstances there exist two matrices N_1 and N_2 which solve (2.17) and which are entire functions of k. Because of (2.14), the solution

$$I(k,r)=[-N_2(k)+A(k)M_1^T(k)]\psi_1(k,r)+[N_1(k)+A(k)M_2^T(k)]\psi_2(k,r),$$

where

$$A(k)=-N_2(k)N_1^T(k)+B(k),$$

and $B(k)$ is symmetric and an entire function of k, is then an entire function of k for all $r>0$ and it satisfies both (2.15) and (2.16).

A further solution of (2.1) is defined by the boundary condition

$$\lim_{r\to\infty}\exp[i(kr-\tfrac{1}{2}\pi l)]\mathcal{E}F(k,r)=\mathbf{1}, \quad (2.18)$$

where

$$\mathcal{E}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \quad (2.19)$$

If $V\epsilon\mathfrak{M}(3)$, $F(k,r)$ is[15] an analytic function of k in $\mathrm{Im}k<0$, continuous for $\mathrm{Im}k\leqslant 0$, except at $k=0$; at the latter point,

$$\mathcal{K}_l(k)F(k,r)\equiv F_e(k,r), \quad (2.20)$$

with

$$\mathcal{K}_l(k)=k^l(\mathbf{1}-P)+k^{l+2}P, \quad (2.21)$$

[15] See Appendix A for the proof.

[16] See, for example L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), pp. 77 ff. The functions u_l and v_l are identical with those used by Walter Kohn, Revs. Modern Phys. **26**, 292 (1954) and are called j_l and n_l in reference 6.

[17] The notions analyticity, continuity, etc., when applied to a matrix valued function, are always meant to hold for each matrix element.

[18] Such a regular solution can presumably be defined under the weaker assumption $V\epsilon\mathfrak{M}(1)$ for which (2.6) in general, diverges. It will, however, in the following be convenient to have an equation for G whose series of successive approximations converges. Therefore it was assumed that $V\epsilon\mathfrak{M}(0)$.

[19] Throughout this paper, the notations O and o will be used to mean a *matrix* all of whose elements are $O(\cdots)$ or $o(\cdots)$.

[20] A superscript T will stand for transposition, an asterisk for complex conjugation.

is continuous. The asymptotic behavior in $\mathrm{Im}k \leqslant 0$ is the following[15]:

$$F(k,r)=\exp[-i(kr-\tfrac{1}{2}\pi l)]\mathcal{E}+O(r^{-1}e^{-|\nu|r}), \quad \text{as } r\to\infty, \quad k\neq 0, \quad (2.22)$$

$$F_e(0,r)=(2l-1)!!(\mathbf{1}-P)[r^{-l}\mathbf{1}+o(r^{-l-2})] + (2l+3)!!P[r^{-l-2}\mathbf{1}+o(r^{-l-4})], \quad r\to\infty, \quad (2.23)$$

$$F(k,r)=\exp[-i(kr-\tfrac{1}{2}\pi l)]\mathcal{E}+O(|k|^{-1}e^{-|\nu|r}), \quad \text{as } |k|\to\infty, \quad r>0. \quad (2.24)$$

If[15] $V\epsilon\mathfrak{M}(2)$, then[12] $P\dot{F}_e(0,r)$ exists (for $r>0$); if $V\epsilon\mathfrak{M}(4)$, so does $(\mathbf{1}-P)\dot{F}_e(0,r)$. If $V\epsilon\mathfrak{M}(3)$, then $P\dot{F}_e(0,r)\equiv 0$ and[12] $P\bar{F}_e(0,r)$ exists; if $V\epsilon\mathfrak{M}(5)$, then, for $l\geqslant 1$, $(\mathbf{1}-P)\dot{F}_e(0,r)\equiv 0$ and $(\mathbf{1}-P)\bar{F}_e(0,r)$ exists. The same is true for the first derivative of F with respect to r.

The solution $F(k,r)$ satisfies, for $\mathrm{Im}k\leqslant 0$,

$$[F(k,r);F(k,r)]=0, \quad (2.25)$$

and for $\mathrm{Im}k=0$,

$$[F(k,r);F(-k,r)]=2i(-1)^l k\mathbf{1}; \quad (2.26)$$

for real k it also has the property

$$F(-k,r)=(-1)^l F^*(k,r). \quad (2.27)$$

By means of the definitions[21]

$$F(k)\equiv[F(k,r);G(k,r)]\mathcal{K}_l(k), \quad (2.28)$$

$$F'(k)\equiv-[F(k,r);I(k,r)]\mathcal{K}_l(k^{-1}), \quad (2.29)$$

$G(k,r)$ can readily be expressed in terms of $F(k,r)$ and $F(-k,\, r)$, for $\mathrm{Im}k=0$:

$$G(k,r)=(2ik)^{-1}\mathcal{K}_l(k^{-1})[(-1)^l F^T(k)F(-k,\,r) - F^T(-k)F(k,r)]. \quad (2.30)$$

Similarly, $F(k,r)$ can be expressed in terms of G and I, for $\mathrm{Im}k\leqslant 0$:

$$F(k,r)=F'(k)\mathcal{K}_l(k)G(k,r)+F(k)\mathcal{K}_l(k^{-1})I(k,r). \quad (2.31)$$

The function $F(k)$ is of fundamental importance in the following. It is analytic in $\mathrm{Im}k<0$, continuous for $\mathrm{Im}k\leqslant 0$, except possibly at $k=0$, where

$$F_e(k)\equiv\mathcal{K}_l(k)F(k)\mathcal{K}_l(k^{-1}), \quad (2.32)$$

however, is continuous. The statements following (2.24) apply to $F_e(0)$: If $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(2)$, then $P\dot{F}_e(0)$ exists; if $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(4)$, so does $(\mathbf{1}-P)F_e(0)$; if $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(3)$, then $P\dot{F}_e(0)=0$ and $P\bar{F}_e(0)$ exists; if $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(5)$, then, for $l\geqslant 1$, $(\mathbf{1}-P)\dot{F}_e(0)=0$ and $(\mathbf{1}-P)\times\bar{F}_e(0)$ exists. Furthermore, if $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(1)$, then

$$F(k)=\mathbf{1}+O(k^{-1}), \text{ as } |k|\to\infty \text{ in } \mathrm{Im}k\leqslant 0. \quad (2.33)$$

In contrast to (2.27), $F(k)$ has the property that for real k

$$F(-k)=F^*(k). \quad (2.34)$$

[21] The functions $F(k)$ and $F'(k)$ of (I) although *defined* differently, satisfy (2.28) and (2.29) *mutatis mutandis*.

Finally, insertion of (2.30) and (2.26) in (2.14) yields the relation

$$F^T(k)F(-k)=F^T(-k)F(k). \quad (2.35)$$

The S-matrix[22] is obtained from (2.30) and (2.35) as

$$S(k)=F(k)F^{-1}(-k). \quad (2.36)$$

Owing to (2.35) it is symmetric, and due to (2.34),

$$S(-k)=S^*(k)=S^{-1}(k), \quad (2.37)$$

which, together with its symmetry, demonstrates its unitarity.

It is well known that for central potentials sufficiently small at infinity, the asymptotic phase ($\tfrac{1}{2}\log \mathrm{Im}S$) of the lth angular momentum tends to zero at $k=0$ as k^{2l+1}, provided there is no bound state (of angular momentum l) with zero binding energy.[23] The generalization of this theorem to the present case is given in the remainder of this section.

It is proved in Appendix A that, if $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(2l+6)$, then

$$\mathrm{Im}F_e(k)=(\mathbf{1}-P)O(k^{2l+1})+P\,O(k^{2l+5}), \text{ as } k\to 0. \quad (2.38)$$

Suppose now that $\det F_e(0)\neq 0$. Then, owing to (2.36) and (2.34)

$$S(k)=\mathcal{K}_0(k^{-1})[\mathbf{1}+i\mathrm{Im}F_e(k)(\mathrm{Re}F_e(k))^{-1} \times[\mathbf{1}-i\mathrm{Im}F_e(k)(\mathrm{Re}F_e(k))^{-1}]^{-1}\mathcal{K}_0(k) = \mathbf{1}+\begin{pmatrix} O(k^{2l+1}) & O(k^{2l+3}) \\ O(k^{2l+3}) & O(k^{2l+5}) \end{pmatrix}, \quad \text{as } k\to 0. \quad (2.39)$$

The modification of this statement in case $\det F_e(0)=0$ will be given in part (d) of the next section.

3. BOUND STATES AND RESONANCE

a. $k\neq 0$.—Suppose that for $k=k_0$, $\mathrm{Im}k<0$, $\det F(k)=0$, so that there exists a vector $a\neq 0$ for which (2.31) yields

$$aF(k_0,r)=aF'(k_0)\mathcal{K}_l(k_0)G(k_0,r). \quad (3.1)$$

The vector solution $aF(k_0,r)$ of (2.1) thus vanishes both at $r=0$ and $r=\infty$; at the latter point exponentially. This means that $-k_0^2$ in an eigenvalue of (2.1), or the energy of a bound state. One can show from (2.15), via (2.12) and similar estimates for G', that

$$I(k,r)=(\mathbf{1}-P)[(2l-1)!!r^{-l}\mathbf{1}+o(r^{-l})] +P[(2l+3)!!r^{-(l+2)}\mathbf{1}+o(r^{-(l+2)})], \quad \text{as } r\to 0; \quad (3.2)$$

consequently,

$$\lim_{r\to 0} bI(k,r)=0 \text{ implies } b=0. \quad (3.3)$$

[22] See C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **22**, No. 19, 19 (1946).

[23] This theorem was proved rigorously for central potentials in $\mathfrak{M}(2l+2)$, by David S. Carter in a Ph.D. thesis at Princeton University, 1952 (unpublished).

It follows, therefore, from the fact that $\det\mathcal{K}_l(k_0^{-1})\neq 0$, by (2.31), that if $-k_0^2$, $\mathrm{Im}k_0<0$, is a bound state energy, then there must exist a vector $a\neq 0$ so that $aF(k_0)=0$, and hence $\det F(k_0)=0$. Thus the nonzero eigenvalues of (2.1) correspond exactly to the points in the lower half plane where $F(k)$ is a singular matrix. It is proved in the standard fashion that these eigenvalues are all real. The functions $F(k)$ and $F^{-1}(k)$ are therefore both analytic for $\mathrm{Im}k<0$, except at a finite number[24] of points on the negative imaginary axis, $k=-iK_i$, $K_i>0$, where $F^{-1}(k)$ has poles. As in (I), we shall now prove that at $k=-iK_i$, $F^{-1}(k)$ has exactly a *simple* pole.

Since[12] $[\dot{G};G]$ vanishes at $r=0$, one obtains, as in (I), (3.1),

$$[\dot{G}(k,r);G(k,r)]=2k\int_0^r dtG(k,t)G^T(k,t). \quad (3.4)$$

On the other hand, (2.14), (2.16), (2.28), and (2.31) lead to

$$\begin{aligned}[\dot{F}(k,r);G(k,r)]=\dot{F}(k)\mathcal{K}_l^{-1}-F(k)\dot{\mathcal{K}}_l\mathcal{K}_l^{-2}\\ +F'(k)\mathcal{K}_l[\dot{G}(k,r);G(k,r)]\\ +F(k)\mathcal{K}_l^{-1}[\dot{I}(k,r);G(k,r)]. \quad (3.5)\end{aligned}$$

Now suppose that the vectors a and b solve

$$aF(-iK)=0, \quad (3.6a)$$

$$a\dot{F}(-iK)+bF(-iK)=0. \quad (3.6b)$$

Substitution of (2.31) in (2.25) yields, for $\mathrm{Im}k\leqslant 0$:

$$F(k)F'^T(k)-F'(k)F^T(k)=0. \quad (3.7)$$

Equations (3.6a) and (3.6b) therefore imply

$$\begin{aligned}a\dot{F}(-iK)F'^T(-iK)a^T\\ =-bF(-iK)F'^T(-iK)a^T\\ =bF'(-iK)F^T(-iK)a^T=0. \quad (3.8)\end{aligned}$$

Multiplication of (3.5), taken at $k=-iK$, by a on the left and by $\mathcal{K}_l(-iK)F'^T(-iK)a^T$ on the right then leads, via (3.1) and (3.4), to

$$a[\dot{F}(-iK,r);F(-iK,r)]a^T=-2iK\int_0^r dtaF(-iK,t)F^T(-iK,t)a^T. \quad (3.9)$$

The function $F(-iK,r)$ being real, the absolute value of the right-hand side increases monotonely, while that of the left vanishes at $r=\infty$. It follows that $a=0$ and, according to (I), Appendix A, therefore that $F^{-1}(k)$ has a simple pole at $k=-iK$.

We can now reduce the case of bound states to that without bound states. For that purpose we need for every bound state a real, symmetric projection P_n which satisfies[25]

$$\mathrm{range}P_n\equiv\mathrm{kernel}(\mathbf{1}-P_n)\equiv\mathrm{kernel}F(-iK_n). \quad (3.10)$$

It describes the mixture of angular momenta for which K_n^2 is a bound state energy, in the sense that if the ratio of the two components of the bound state wave function is asymptotically equal to c, then

$$P_n=(1+c^2)^{-1}\begin{pmatrix}1 & -c\\ -c & c^2\end{pmatrix}. \quad (3.10')$$

This is not directly related to the probability ratio of the two states in the sense in which the ground state of the deuteron is a 4% D-state. The connection between the two strictly speaking, involves the potential.

If $\{-K_n^2\}$, $n=1,\cdots,L$, $0<K_1<\cdots<K_L$, are the bound state energies and P_n the corresponding projections fulfilling (3.10), then one defines a matrix $R\equiv R_L(k)$ as follows:

$$R_0=\mathbf{1},$$

$$R_n(k)=\left(\mathbf{1}-P_n'\frac{2iK_n}{k+iK_n}\right)R_{n-1}(k),\ n=1,\cdots,L, \quad (3.11)$$

where P_n' is again a real, symmetric projection, related to P_n by

$$\begin{aligned}\mathrm{range}P_n'\equiv\mathrm{kernel}(\mathbf{1}-P_n')\\ \equiv\mathrm{range}[R_{n-1}(-iK_n)P_nR_{n-1}^{-1}(-iK_n)]. \quad (3.12)\end{aligned}$$

One then forms

$$F_N(k)\equiv R(k)F(k), \quad (3.13)$$

and

$$S_N(k)\equiv R(k)S(k)R^{-1}(-k)=F_N(k)F_N^{-1}(-k). \quad (3.14)$$

It was proved in (I), Sec. 3, that F_N and S_N have all the relevant properties of F and S: $F_N(k)$ is analytic for $\mathrm{Im}k<0$, continuous for $\mathrm{Im}k\leqslant 0$, except at $k=0$, $\lim F_N(k)=\mathbf{1}$ as $|k|\to\infty$; S_N is unitary and symmetric and satisfies (2.37). In addition, however, $F_N^{-1}(k)$ is also analytic for $\mathrm{Im}k<0$.

b. $k=0$, $l\geqslant 1$.—It is evident from (2.23) that for $l\geqslant 1$, $F_e(0,r)$ is always square integrable at infinity. It is therefore again clear from (2.31) that a necessary and sufficient condition for $k^2=0$ to be the energy of a bound state, is that $\det F_e(0)=0$; i.e., that there exist a vector $a\neq 0$ so that $aF_e(0)=0$ and therefore

$$aF_e(0,r)=aF_e'(0)G(0,r). \quad (3.15)$$

It is now, however, not true that $kF_e^{-1}(k)$ is continuous at $k=0$. We shall prove that $k^2F_e^{-1}(k)$ exists at $k=0$ and, if and only if $k^2=0$ is an eigenvalue, it does not vanish there.[26]

[24] See (I), reference 19.

[25] As in (I), we mean by the range of a matrix M the set of all vectors x for which there exist vectors y so that $x=yM$; the kernel of M is the set of all x so that $xM=0$. By a projection we mean merely an idempotent matrix, $P^2=P$.

[26] We assume in the following that $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(5)$.

Consider the system

$$aF_e(0)=0, \quad a\ddot{F}_e(0)+bF_e(0)=0. \tag{3.16}$$

By (3.15) both $aF_e(0,r)$ and $aF_e'(0,r)$ vanish at $r=0$. Moreover, since $\dot{F}_e(0)=0$, by (2.31),

$$bF_e(0,r)=bF_e'(0)G(0,r)+bF_e(0)I(0,r), \tag{3.17}$$

$$a\ddot{F}_e(0,r)=a\frac{\partial^2}{\partial k^2}[F_e'(k)G(k,r)]_{k=0}+a\ddot{F}_e(0)I_e(0,r). \tag{3.18}$$

The second equation in (3.16) then shows that $a\ddot{F}_e(0,r)+bF_e(0,r)$ vanishes at $r=0$ and so does its derivative. Consequently

$$f(r)\equiv[aF_e(0,r);\, a\ddot{F}_e(0,r)+bF_e(0,r)]\to 0, \quad r\to 0. \tag{3.19}$$

The differential equation (2.1), on the other hand, yields[27]

$$f'(r)=-2aF_e(0,r)F_e^T(0,r)a^T, \tag{3.20}$$

which is certainly nonpositive (a and $F_e(0,r)$ being real). Now, as $r\to\infty$

$$[aF_e(0,r);\, a\ddot{F}_e(0,r)]\to[aF_{e0}(0,r);\, a\ddot{F}_{e0}(0,r)]\to 0,$$

and similarly,

$$[aF_e(0,r);\, b\dot{F}_e(0,r)]\to 0.$$

Therefore $f(0)=f(\infty)=0$. Since (3.20) states that $f(r)$ is monotonic, it follows that $f'(r)\equiv 0$ and hence by (3.20),

$$aF_e(0,r)=0.$$

As $r\to\infty$, (2.23) shows that

$$\det F_e(0,r)\to(2l-1)!!(2l+3)!!r^{-l-2},$$

and therefore $a=0$. The theorem of (I), Appendix A, then states that the inverse of

$$M(k)\equiv F_e(0)+\tfrac{1}{2}k^2\ddot{F}_e(0)$$

has exactly a double pole at $k=0$. Since $\dot{F}_e(0)=0$, and

$$F_e^{-1}(k)=[\mathbf{1}+M^{-1}(k)\mathcal{R}(k)]^{-1}M^{-1}(k), \tag{3.21}$$

where

$$\mathcal{R}(k)=o(k^2) \quad \text{as} \quad k\to 0,$$

it follows that

$$Q_0\equiv\lim_{k\to 0}k^2F_e^{-1}(k) \tag{3.22}$$

always exists and

$$E=0 \text{ is an eigenvalue if and only if } Q_0\neq 0. \tag{3.23}$$

(The sufficiency follows clearly from the fact that $\dot{F}_e(0)=0$.)

The bound state at $k=0$ can now be removed as in the previous procedure. Let P_0' be a projection that satisfies (3.10) with respect to $F_e(0)$. Then

$$R_{00}(k)=(\mathbf{1}-P_0'ik^{-1})^2 \tag{3.24}$$

will be used in the same way as the other R_n's. We form

$$F_{e1}(k)=R_{00}(k)F_e(k) \tag{3.25}$$

and find that $F_{e1}(k)$ and $F_{e1}^{-1}(k)$ both exist at $k=0$. One can then write

$$F(k)=R_0^{-1}(k)F_1(k), \tag{3.26}$$

where

$$R_0(k)=\mathcal{K}_0(k^{-1})R_{00}(k)\mathcal{K}_0(k), \tag{3.27}$$

$$F_1(k)=\mathcal{K}_0(k^{-1})F_{e1}(k)\mathcal{K}_0(k). \tag{3.28}$$

Instead of choosing P_0' symmetric, we make it such that

$$(\mathbf{1}-P)P_0'P=0, \tag{3.29}$$

which is always possible.[28] With that choice of P_0'

$$R_0(\infty)=\mathbf{1}, \tag{3.30}$$

and consequently $F_1(\infty)=\mathbf{1}$. Equation (3.27) with (3.24) now replaces the first line in (3.11). The matrix S_N of (3.14) will, if $k^2=0$ is a bound state energy, in general not be unitary or symmetric. It will be noticed that in contrast to the discussion in (I), Sec. 3, P_0' is independent of the S-matrix. For $l\geqslant 1$ one always has

$$S(0)=\mathbf{1}. \tag{3.31}$$

c. $k=0$, $l=0$.—Contrary to the case of no coupling between angular momenta, where $E=0$ never is a bound state energy for $l=0$ [if $V\epsilon\mathfrak{M}(2)$],[29] and for $l\geqslant 1$ it is a bound state if and only if $f(0)=0$, in the present case $E=0$ may or may not be an eigenvalue if $F_e(0)$ is a singular matrix. Equation (2.23) shows that, while $PF_e(0,r)$ is always square integrable, $(\mathbf{1}-P)F_e(0,r)$ is not. The criterion for a bound state at $E=0$ is therefore that $PF_e(0,0)=0$. As before, it follows from (2.31) that a necessary and sufficient condition for the existence of a bound state of zero binding energy is that $F_e(0)=0$.

If $\det F_e(0)=0$ we distinguish three cases:

(1) The case of a bound state[30] and no resonance; by this we mean $(\mathbf{1}-P)F_e(0)\neq 0$, $PF_e(0)=0$;
(2) The case of a bound state and a resonance at the same time; then $F_e(0)=0$;
(3) The case of a resonance and no bound state, i.e., $PF_e(0)\neq 0$, $\det F_e(0)=0$.

In the following we assume that $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(4)$, in which case $\dot{F}_e(0)$ exists, $P\dot{F}_e(0)=0$, and $P\ddot{F}_e(0)$ exists. We now wish to prove that Q_0, (3.22), exists.

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[27] As (I), (3.1).

[28] If the kernel of $F_e(0)$ is represented by $(1,C)$, then

$$P_0'=\begin{pmatrix}0 & 0\\ C^{-1} & 1\end{pmatrix};$$

if it is $(1,0)$, then $P_0'=\mathbf{1}-P$; if it is the whole space, then $P_0'=\mathbf{1}$.

[29] See V. Bargmann, Proc. Natl. Acad. Sci. U. S. 38, 961 (1952).

[30] This occurs in the lower row of F_e, if $l=0$. It must be remembered, though, that the first *column* refers to S-components and the second *column* to D-components.

Consider the system

$$aF_e(0)=0,\quad a\dot{F}_e(0)+bF_e(0)=0, \tag{3.32a}$$

$$a\ddot{F}_e(0)+b\dot{F}_e(0)+cF_e(0)=0. \tag{3.32b}$$

Insertion of (2.31) in (2.26) yields

$$F(k)F'^T(-k)-F'(k)F^T(-k)=2ik\mathbf{1}. \tag{3.33}$$

If this is multiplied by $\mathcal{K}_l(k)$ on the left, $\mathcal{K}_l(-k)$ on the right, and differentiated, it becomes for $l=0$, $k=0$,

$$\dot{F}_e(0)F_e'^T(0)-F_e(0)\dot{F}_e'^T(0)-\dot{F}_e'(0)F_e^T(0)+F_e'(0)\dot{F}_e^T(0)=2i(\mathbf{1}-P). \tag{3.34}$$

Multiplication by a on the left and by a^T on the right leads, by (3.32a) and (3.7), to

$$2ia(\mathbf{1}-P)a^T=-2bF_e(0)F_e'^T(0)a^T=-2bF_e'(0)F_e^T(0)a^T=0.$$

This means that

$$\text{Eqs. (3.32a) imply } a(\mathbf{1}-P)=0. \tag{3.35}$$

Consequently $a\ddot{F}_e(0)$ exists. Now consider separately the three cases distinguished above.

(1) $PF_e(0)=0$, $(\mathbf{1}-P)F_e(0)\neq 0$. Then $b(\mathbf{1}-P)=0$ and (3.32) becomes

$$aF_e(0)=0,\quad a\ddot{F}_e(0)+cF_e(0)=0. \tag{3.32$'$}$$

The arguments following (3.16) apply equally here and show that (3.32) implies $a=0$.

(2) $F_e(0)=0$. In that case (3.32b) reduces to

$$a\ddot{F}_e(0)+b\dot{F}_e(0)=0. \tag{3.32b$''$}$$

Similarly as (3.18) we obtain

$$b\dot{F}_e(0,r)=b\frac{\partial}{\partial k}[F_e'(k)G(k,r)]_{k=0}+b\dot{F}_e(0)I(0,r), \tag{3.36}$$

while (3.18) still holds. One now uses (3.36) in place of (3.17) to prove, quite as before, that $a=0$.

(3) $\det F_e(0)=0$, $PF_e(0)\neq 0$. In that case the first line of (3.32a) and (3.35) imply $a=0$.

This completes the proof that the system (3.32) implies $a=0$. That statement, however, is a necessary and sufficient condition[31] for the inverse of

$$M(k)=F_e(0)+k(\mathbf{1}-P)\dot{F}_e(0)+\tfrac{1}{2}k^2P\ddot{F}_e(0) \tag{3.37}$$

to have at most a double pole at $k=0$. We now write

$$F_e(k)=M(k)+\mathcal{R}(k),\quad \mathcal{R}(k)=(\mathbf{1}-P)o(k)+P\,o(k^2),$$

and

$$\mathcal{R}_1(k)\equiv F_e^{-1}(k)-M^{-1}(k)=-[M^{-1}\mathcal{R}(\mathbf{1}+M^{-1}\mathcal{R})^{-1}]M^{-1}. \tag{3.38}$$

It is proved in Appendix C that, since $M(k)$ has at most a double pole at $k=0$, (3.35) is a necessary and sufficient condition for $M^{-1}(k)(\mathbf{1}-P)$ to have at most a *simple* pole at $k=0$. Therefore $M^{-1}\mathcal{R}=o(1)$ as $k\to 0$ and we can write

$$F_e^{-1}(k)=[\mathbf{1}+o(1)]M^{-1}(k),\quad \text{as}\quad k\to 0; \tag{3.39}$$

hence Q_0 of (3.22) exists.

Moreover, neither in case (1) nor in case (2) can $M^{-1}(k)$ have a *simple* pole at $k=0$, because then (3.32a) would, by (I), Appendix A, imply $a=0$; in both cases, however, the second component of a is quite undetermined by (3.32a). Conversely, if $M^{-1}(k)$ has a simple pole at $k=0$, then (3.32a) must force $a=0$. But since $P\dot{F}_e(0)=0$, the lower row of $F_e(0)$ can then not be zero and $PF_e(0)\neq 0$; we are then in case (3) and $E=0$ is not an eigenvalue. Consequently, (3.23) holds also for $l=0$. In addition, in view of the statement following (3.38),

$$\lim_{k\to 0} kF_e^{-1}(k)(\mathbf{1}-P) \text{ exists for } l=0. \tag{3.40}$$

We are now in a position to remove the singularity of F_e^{-1} at $k=0$ as before. Again one forms $F_{e1}(k)$ by (3.25), but now

$$R_{00}(k)=\begin{cases}(\mathbf{1}-Pik^{-1})^2, \text{ in case (1)},\\ (1-ik^{-1})(\mathbf{1}-Pik^{-1}), \text{ in case (2)},\\ (\mathbf{1}-P_0'ik^{-1}), \text{ in case (3)},\end{cases} \tag{3.41}$$

where

$$\text{range}P_0'\equiv\text{kernel}(\mathbf{1}-P_0')\equiv\text{kernel}F_e(0), \tag{3.42}$$

and

$$(\mathbf{1}-P)P_0'=0. \tag{3.43}$$

The proof that $F_{e1}(0)$ and $F_{e1}^{-1}(0)$ both exist is straightforward and analogous to the previous proof in (I), Sec. 3. R_0 of (3.27) with R_{00} of (3.41) again satisfies (3.30).

In case (3), there is a connection between the kernel of $F_e(0)$ and the S-matrix. One can write

$$F_e(k)=F_e(0)+k(\mathbf{1}-P)\dot{F}_e(0)+\tfrac{1}{2}k^2P\ddot{F}_e(0)+(\mathbf{1}-P)o(k)+P\,o(k^2),$$

$$F_e^{-1}(k)=N_{-1}k^{-1}+\mathfrak{R}(k),\quad \mathfrak{R}(k)=o(k^{-1}),$$

where N_{-1} and $\mathfrak{R}$ must satisfy (even if $\mathfrak{R}(0)$ does not exist),

$$N_{-1}F_e(0)=F_e(0)N_{-1}=0,$$

$$\mathfrak{R}(0)F_e(0)+N_{-1}(\mathbf{1}-P)\dot{F}_e(0)=F_e(0)\mathfrak{R}(0)+(\mathbf{1}-P)\dot{F}_e(0)N_{-1}=\mathbf{1},$$

where $\mathfrak{R}(0)F_e(0)=\lim\mathfrak{R}(k)F_e(0)$ as $k\to 0$. For a modified S-matrix one then obtains

$$S_e(0)=F_e(0)\mathfrak{R}(0)-(\mathbf{1}-P)\dot{F}_e(0)N_{-1}=\mathbf{1}-2(\mathbf{1}-P)\dot{F}_e(0)N_{-1}\equiv\mathbf{1}-2P_0'', \tag{3.44}$$

where

$$S_e(k)=\mathcal{K}_0(k)S(k)\mathcal{K}_0(k^{-1}), \tag{3.45}$$

and P_0'' is a projection which satisfies (3.42) and, in addition

$$PP_0''=0.$$

[31] The proof of this is entirely parallel to that of (I), Appendix A.

The equality of the ranges of P_0 and P_0'', together with (3.43), determines the former uniquely. For the S-matrix itself, therefore

$$\begin{aligned} S(k) &= \mathcal{K}_0(k^{-1})[1-2P_0''+o(1)]\mathcal{K}_0(k) \\ &= 1-2(1-P)P_0''(1-P)+(1-P)o(1)(1-P) \\ &\quad +P\,o(1)P+(1-P)o(k^2)P+P\,o(k^{-2})(1-P), \end{aligned}$$

and thus, because it is symmetric,[32]

$$S(0)=-\mathcal{E}. \tag{3.46}$$

d. Generalization of (2.39).—In case $\det F_e(0)=0$ we now write, for $l\geqslant 1$ and case (1) of $l=0$:

$$\begin{aligned} S(k) &= \mathcal{K}_0^{-1}(1-k^{-2}P_0')^{-1}[1+iR_{00}^*\,\mathrm{Im}F_e(\mathrm{Re}F_{e1})^{-1}] \\ &\quad \times[1-iR_{00}\,\mathrm{Im}F_e(\mathrm{Re}F_{e1})^{-1}]^{-1}(1-k^{-2}P_0')\mathcal{K}_0 \\ &= \mathcal{K}_0^{-1}[1+2i\,\mathrm{Im}F_e(\mathrm{Re}F_{e1})^{-1}(1-k^{-2}P_0') \\ &\quad +\cdots]\mathcal{K}_0, \end{aligned} \tag{3.47}$$

where $P_0'=P$ for $l=0$. This leads, for $l\geqslant 1$, to

$$S(k)=1+\begin{pmatrix} O(k^{2l-1}) & O(k^{2l+1}) \\ O(k^{2l+1}) & O(k^{2l+3}) \end{pmatrix}, \quad \text{as} \quad k\to 0, \tag{3.48}$$

and for $l=0$, since it is symmetric, to

$$S(k)=1+\begin{pmatrix} O(k) & O(k^3) \\ O(k^3) & O(k^3) \end{pmatrix}. \tag{3.48'}$$

In case (2) of $l=0$, one writes

$$R_{00}(k)=(1-k^2P)(1-ik^{-1}(1-P))[1-2i(k/k^2-1)P]$$

and proves similarly as in (3.47), if $V\epsilon\mathfrak{M}(4+\delta)$, $0<\delta\leqslant 1$, that

$$S(k)=-\mathcal{E}+\begin{pmatrix} O(k) & O(k^2) \\ O(k^2) & O(k^3) \end{pmatrix}; \tag{3.48''}$$

in case (3), $l=0$, finally

$$S(k)=-\mathcal{E}+\begin{pmatrix} O(k^\delta) & O(k^{2+\delta}) \\ O(k^{2+\delta}) & O(k^4) \end{pmatrix}. \tag{3.48'''}$$

e. The Determinant.—Consideration of the determinants leads to a connection between the S-matrix and the number of bound states. If one defines an asymptotic phase continuously between $k=\infty$ and $k=0$, as

$$\tfrac{1}{2}\log\det S(k)\equiv i\eta(k), \tag{3.49}$$

then it is shown as in (I), end of Sec. 3, that if m is the number of bound states (counted twice if more than one mixture has a bound state at the same energy), then

$$\eta(0+)-\eta(\infty) = \begin{cases} \pi(m+\tfrac{1}{2}), & \text{for } l=0, \text{ if } k=0 \text{ is a resonance,} \\ \pi m, & \text{otherwise.} \end{cases} \tag{3.50}$$

[32] $\mathcal{E}$ is defined in (2.19).

4. SPECTRAL FUNCTION AND POTENTIAL

The completeness relation is derived as in (I), Sec. 4. We shall give here only the necessary changes for the present case of the admixture of higher angular momenta. In the integral on the left hand side of (I), (4.2), the contribution due to the small semicircle in (I), Fig. 1, will now not vanish. Equation (I), (4.13) will therefore be modified to read

$$\begin{aligned} \mathfrak{F}(r) &= 2\pi^{-1}\mathcal{P}\int_{-\infty}^{\infty}k^2dk\int_{r-\epsilon}^{r}dt\mathfrak{F}(t)G^T(k,t) \\ &\quad \times[F_e^T(k)\mathcal{K}_l^{-2}(k)F_e(-k)]^{-1}G(k,r) \\ &\quad -2\int_{r-\epsilon}^{r}dt\mathfrak{F}(t)G^T(0,t)Q_0F_e(0,r) \\ &\quad +2\sum_{n=1}^{L}\int_{r-\epsilon}^{r}dt\mathfrak{F}(t)G_n^T(t)C_nG_n(r), \end{aligned} \tag{4.1}$$

where $\mathcal{P}$ denotes the principal value of the integral, $G_n(r)\equiv G(-iK_n,r)$, and

$$\begin{aligned} C_n\equiv -Q_nF_e'(-iK_n) &= \int_0^{\infty}drC_nG_n(r)G_n^T(r)C_n \\ &= \int_0^{\infty}drQ_nF_{en}(r)F_{en}^T(r)Q_n^T, \end{aligned} \tag{4.2}$$

with

$$Q_n\equiv \lim_{k\to -iK_n}(k^2+K_n^2)F_e^{-1}(k), \tag{4.3}$$

$$F_e'(k)\equiv\mathcal{K}_l(k)F'(k)\mathcal{K}_l(k).$$

The matrix C_n is real, symmetric and positive semidefinite. Equation (4.2) also holds for $K_0=0$ and defines C_0. From (2.31) one obtains

$$Q_0F_e(0,r)=-C_0G(0,r),$$

so that the last two terms in (4.1) can be combined into one sum of the form of the last term, n running from 0 to L. In view of (3.23), C_0 has the same connection with a bound state at $k=0$ as C_n has with one at $k=-iK_n$.

In the first term of (4.1),

$$\begin{aligned} k^2[F_e^T(k)\mathcal{K}_l^{-2}(k)F_e(-k)] \\ =[kF_e^{-1}(-k)\mathcal{K}_l(k)][kF_e^{-1}(k)\mathcal{K}_l(k)]^T. \end{aligned}$$

For $l\geqslant 1$ this obviously exists at $k=0$; owing to (3.40), it also does for $l=0$. The principal value sign can therefore be dropped in (4.1). As a result the completeness relation becomes

$$\int G^T(\sqrt{E},t)d\mathrm{P}(E)G(\sqrt{E},r)=\delta(t-r), \tag{4.4}$$

or

$$\int_0^\infty dr\mathfrak{F}(r)\mathfrak{F}^T(r)=\int\left[\int_0^\infty dr\mathfrak{F}(r)G^T(\surd E,r)\right]$$
$$\times dP(E)\left[\int_0^\infty dr\mathfrak{F}(r)G^T(\surd E,r)\right]^T,$$

where the spectral function $P(E)$ is defined by

$$P(-\infty)=0,$$

$$dP(E)/dE=\begin{cases}\pi^{-1}E^{l+\frac{1}{2}}\mathcal{K}_0(\surd E)[F^T(\surd E)\\ \quad\times F(-\surd E)]^{-1}\mathcal{K}_0(\surd E), & E>0,\\ \sum_{n=0}^{L}C_n\delta(E-E_n),\quad E_0=0, & E\leqslant 0.\end{cases}\tag{4.5}$$

It is a real, symmetric, positive semidefinite matrix function of E.

It was shown in (I), Sec. 4, that one can write, for $n>0$,

$$C_n=Q_n\mathcal{K}_l(-iK_n)A_n\mathcal{K}_l(-iK_n)Q_n^T,\tag{4.6}$$

where

$$A_n=\int_0^\infty P_nF_n(r)F_n^T(r)P_ndr\tag{4.7}$$

is a real, symmetric, positive-semidefinite matrix with the property that

$$\text{kernel}A_n\equiv\text{kernel}P_n.\tag{4.8}$$

In fact, with the notation (3.10′) we can write (in the case of no degeneracy; if $F(-iK_n)=0$, then $P_n=\mathbf{1}$ and *three* parameters are needed to fix A_n).

$$A_n=(1+c^2)^{-1}dP_n,\tag{4.9}$$

where[33]

$$d=\int_0^\infty dr\{[F_{n11}(r)+cF_{n21}(r)]^2$$
$$+[F_{n12}(r)+cF_{n22}(r)]^2\}.\tag{4.10}$$

For $n=0$, the projection P_0 to be used in (4.7) must be a real, symmetric projection, with

$$\text{range}P_0\equiv\text{range}P_0',\tag{4.11}$$

i.e., the projection of (3.10′) rather than that of reference 28.

The connection between the spectral function and the potential via the generalized Gel'fand Levitan equation is precisely the same as in (I), Sec. 4. If the subscript 1 denotes quantities belonging to a suitable comparison potential V_1, and

$$\mathfrak{G}(s,r)=-\int G_1^T(\surd E,s)$$
$$\times d[P_1(E)-P(E)]G_1(\surd E,r),\tag{4.12}$$

then there exists a matrix function $\mathfrak{K}(s,r)$ which is the unique solution of the integral equation

$$\mathfrak{K}(s,r)+\mathfrak{G}(s,r)+\int_0^r dt\mathfrak{G}(s,t)\mathfrak{K}(t,r)=0,\quad s\leqslant r,\tag{4.13}$$

and satisfies

$$2(d/dr)\mathcal{K}(r,r)=V(r)-V_1(r),\tag{4.14}$$

5. CONSTRUCTION OF V FROM S

The potential $V(r)$ can now be constructed from a given S-matrix, L bound state energies, K_n^2, and L real, symmetric, positive-semidefinite matrices A_n. Except for (3.50), all these quantities are independent of each other. From A_n one obtains, via (4.8) or (4.9), P_n; by means of the procedures of Sec. 3 one then removes from S the bound states and, for $l=0$, the resonance at $k=0$ if $S(0)=-\mathcal{E}$, forming S_N. We then define

$$F_{N0}(k)=\mathcal{K}_0\left(\frac{-ik}{k-i}\right)F_N(k)\mathcal{K}_0^{-1}\left(\frac{-ik}{k-i}\right),\tag{5.1}$$

and

$$S_{N0}(k)=\mathcal{K}_0\left(\frac{-ik}{k-i}\right)S_N(k)\mathcal{K}_0^{-1}\left(\frac{-ik}{k+i}\right).\tag{5.2}$$

The functions $F_{N0}(k)$ and $F_{N0}^{-1}(k)$ are analytic for $\text{Im}k<0$ and continuous for $\text{Im}k\leqslant 0$. Moreover,

$$F_{N0}(\infty)=\mathbf{1}.\tag{5.3}$$

The new S-matrix can be written

$$S_{N0}(k)=F_{N0}(k)\mathcal{K}_0\left(\frac{k+i}{k-i}\right)F_{N0}^{-1}(-k),\tag{5.4}$$

and, although in general not unitary or symmetric, it satisfies

$$S_{N0}(k)S_{N0}(-k)=\mathbf{1},\tag{5.5}$$

and

$$S_{N0}(\infty)=\mathbf{1},\tag{5.6}$$

$$S_{N0}(-k)=S_{N0}^*(k).\tag{5.7}$$

After the transformation

$$z=(k+i)/(k-i),\tag{5.8}$$

which takes the lower half plane into the interior of the unit circle, (5.4) becomes, in the notation of (I), Sec. 5,

$$M(t)=\Phi_+(t)N(t)\Phi_+^{-1}(t^{-1}),\tag{5.9}$$

where, with $t=z$ for $|z|=1$,

$$M(1)=\mathbf{1},\quad M(t)M(t^{-1})=\mathbf{1},\quad M(t^{-1})=M^*(t),\tag{5.10}$$

[33] The relative probability for the $(l+2)$-state is

$$p=d^{-1}\int_0^\infty dr(F_{n12}+cF_{n22})^2,$$

and depends both on c and d. It can be obtained from the latter two variables only via $F(k,r)$.

and

$$N(z)=(1-P)+Pz^2. \quad (5.11)$$

It is proved in Appendix C that if $V\epsilon\mathfrak{M}(4+\delta)$ and (2.5) is satisfied, then M fulfills a Hölder condition. The Plemelj theorem,[34,35] quoted at the end of (I), Sec. 5, is therefore applicable. Accordingly, Eq. (5.9) can always be satisfied with a matrix $N(z)$,

$$N_{ij}(z)=\delta_{ij}z^{\mu_i}, \quad (5.12)$$

the integers μ_i, or "indices," being uniquely (up to permutations) determined by $M(t)$. The solution $\Phi_+(t)$ as well as the indices are obtainable by solving a Fredholm equation. Since the present situation is in general not the "normal case" of (I), Sec. 5, the reader is referred to references 34 and 35 for the procedure of solving (5.9) and obtaining the indices. If the latter have the values 0 and 2, (5.9) and (5.11) together have a solution and one obtains $F_{N0}(k)$, $F_N(k)$, and $F(k)$.

Once $F(k)$ is obtained, the spectral function is formed via (4.5), (4.6), and (4.3). Equations (4.12) to (4.14) then lead to the potential. The latter will automatically be symmetric, since $P(E)$ is so.[36] If the potential so constructed is in $\mathfrak{M}(0)\cap\mathfrak{M}(5)$ (for $l=0$, $\mathfrak{M}(0)\cap\mathfrak{M}(4+\delta)$ is sufficient), then it will have $S(k)$ as an S-matrix and K_n^2, P_n as bound state energies and projections.

If the indices do not have the values 0 and 2, then there exists no "short range" potential (i.e., $\epsilon\mathfrak{M}(5)$) as either $F_{N0}(k)$ or $F_{N0}^{-1}(k)$, or both, will be forced to have a singularity at $k=0$.[37] So long as $F_{N0}^{-1}(k)$ is continuous at $k=0$, the Gel'fand Levitan equation can be solved and a potential found. If, however, $F_{N0}^{-1}(k)$ has a singularity at $k=0$, then one has to shift the latter, if necessary, to several different points on the negative imaginary k-axis (so that at each point $F_{N0}^{-1}(k)$ has exactly a simple pole), where they produce bound states; at the same time $F_{N0}(k)$ will become infinite at $k=0$.

One can, in such a manner, always find an $F(k)$ for which the Gel'fand Levitan equation (4.13) can be solved and a potential constructed. Unless the S-matrix is such that the indices are 0 and 2, that potential will not be of short range and, in addition, it may produce more bound states than were originally contemplated.[38]

The author takes great pleasure in thanking Dr. Res Jost for many fruitful and stimulating conversations. Appendix B, indeed, is entirely his work.

[34] J. Plemelj, Monatshefte Math. und Physik, **19**, 211 (1908).
[35] N. I. Muskelishvili, *Singular Integral Equations* (Croningen, Holland, 1953), pp. 381 ff.
[36] See (I) for the proof.
[37] One writes

$$z=\frac{z}{z+1}\,\frac{z^{-1}+1}{z^{-1}},$$

and thus distributes the extra powers of z onto $\Phi_+(t)$ and $\Phi_+^{-1}(t^{-1})$.
[38] If $F_e(k)$ is $O(k^{-\mu})$ at $k=0$, then more bound states will not alter (3.50).

APPENDIX A

The purpose of this appendix is to supply a number of existence proofs and inequalities that are essential to the support of the body of the paper. All of the estimates below will rest on inequalities satisfied by the solutions of (2.1) with $V=0$. We shall list the essential properties of these functions first.

The following are well known[16]:

$$\left.\begin{aligned} u_l(x)&=[(2l+1)!!]^{-1}x^{l+1}+O(x^{l+3}),\\ v_l(x)&=-(2l-1)!!x^{-l}+O(x^{-l+2}),\\ w_l(x)&=(2l-1)!!x^{-l}+O(x^{-l+2}),\quad l>0;\end{aligned}\right\}\text{as } x\to 0, \quad (A.1)$$

$$\left.\begin{aligned} u_l(x)&=\sin(x-\tfrac{1}{2}\pi l)+O(x^{-2}e^{|\mathrm{Im}x|}),\\ v_l(x)&=-\cos(x-\tfrac{1}{2}\pi l)+O(x^{-2}e^{|\mathrm{Im}x|}),\\ w_l(x)&=i^le^{-ix}+O(x^{-1}e^{\mathrm{Im}x}).\end{aligned}\right\}\text{as } |x|\to\infty. \quad (A.2)$$

If $\nu=\mathrm{Im}k$, one then easily obtains the following inequalities[6]:

$$|u_l(kr)|\leqslant Ce^{|\nu|r}\left(\frac{|k|r}{1+|k|r}\right)^{l+1}, \quad (A.3)$$

$$|v_l(kr)|\leqslant Ce^{|\nu|r}\left(\frac{1+|k|r}{|k|r}\right)^{l}, \quad (A.4)$$

$$|w_l(kr)|\leqslant Ce^{\nu r}\left(\frac{1+|k|r}{|k|r}\right)^{l}. \quad (A.5)$$

The Green's function obeys the following[6]:

$$|g_l(k;t,r)|\leqslant Ce^{|\nu|(r-t)}\frac{r}{1+|k|r}\left(\frac{r}{1+|k|r}\,\frac{1+|k|t}{t}\right)^{l},\quad \text{for } t\leqslant r. \quad (A.6)$$

In addition we require an estimate for $\dot{g}_l$. By considering separately the cases where $|k|t\leqslant 1$ and $|k|t\geqslant 1$, one obtains in a straightforward manner, for $\mathrm{Im}k=0$,

$$|\dot{g}_l(k;t,r)|\leqslant C\frac{|k|r^3}{(1+|k|r)^2}\left(\frac{r}{1+|k|r}\,\frac{1+|k|t}{t}\right)^{l},\quad t\leqslant r. \quad (A.7)$$

For the purpose of proving the convergence of the successive approximations [Liouville-Neumann series, or Born expansion] to (2.6) and estimates on G, we rewrite (2.6):

$$\Delta G(k,r)=\phi(k,r)+\int_0^r dt\Delta G(k,t)V(t)\mathcal{G}(k;t,r), \quad (A.8)$$

where

$$\Delta G(k,r)=G(k,r)-G_0(k,r),$$

and

$$\begin{aligned}\phi(k,r)=\int_0^r dt[&G_0(k,t)V(t)\mathcal{G}(k;t,r)\\ &-(2l+3)^{-1}(1-P)V(t)PG_0(k,r)]\\ &+(2l+3)\int_1^r dt\,t^{-1}(1-P)V(t)PG_0(k,r).\end{aligned} \quad (A.9)$$

The only part of the estimate on ϕ that requires detailed consideration is the (12)-term, containing

$$\mathfrak{f}(k;t,r)=kg_{l+2}(k;t,r)u_l(kl)-(2l+3)(kt)^{-1}u_{l+2}(kr),\quad t\leqslant r.\quad (A.10)$$

For $|k|t\leqslant 1$ one may use both terms in g separately, its second part together with the second term on the right of (A.10); for $|k|t\geqslant 1$ one estimates both terms on the right of (A.10) separately. In either case one readily obtains

$$|\mathfrak{f}(k,t,r)|\leqslant Ce^{|\nu|r}\left(\frac{|k|r}{1+|k|r}\right)^{l+1}\left(\frac{|k|t}{1+|k|t}\right).$$

Accordingly, the (12)-element of the first integral in (A.9) is, in absolute value,

$$\leqslant Ce^{|\nu|r}\left(\frac{r}{1+|k|r}\right)^{l+2}\int_0^r dt|V_{12}(t)|.$$

The (12)-element of the second integral in (A.9), its only nonzero one, is in absolute value

$$\leqslant Ce^{|\nu|r}\left(\frac{r}{1+|k|r}\right)^{l+2}\left(\frac{1+r}{1+|k|r}\right)\int_0^\infty dt|V_{12}(t)|,$$

and one easily obtains

$$|\phi_{11}|\leqslant Ce^{|\nu|r}\left(\frac{r}{1+|k|r}\right)^{l+2}\int_0^r dt|V_{11}(t)|.$$

Under the assumption, then, that $V\epsilon\mathfrak{M}(0)$,

$$|(1-P)\phi(k,r)|\leqslant Ce^{|\nu|r}\left(\frac{r}{1+|k|r}\right)^{l+2}\left(1+\frac{r}{1+|k|r}\right).\quad (A.11a)$$

Similarly, one obtains,

$$|P\phi(k,r)|\leqslant Ce^{|\nu|r}\left(\frac{r}{1+|k|r}\right)^{l+3}\int_0^r dt\frac{t}{1+|k|t}|V(t)|.\quad (A.11b)$$

If we now write

$$\Delta G(k,r)=\sum_{n=0}^{\infty}G^{(n)}(k,r),$$

where

$$G^{(0)}(k,r)=\phi(k,r),$$

$$G^{(n)}(k,r)=\int_0^r dtG^{(n-1)}(k,t)V(t)\mathcal{G}(k;t,r),\quad n\geqslant 1,$$

and define

$$\Xi^{(n)}(k,r)\equiv e^{-|\nu|r}\left(\frac{1+|k|r}{r}\right)^{l+3}\left[\left(1+\frac{r}{1+|k|r}\right)^{-1}(1-P)+\left(\frac{1+|k|r}{r}\right)^{2}P\right]G^{(n)}(k,r),$$

then (A.6) and (A.11) show that the series $\sum\Xi^{(n)}$ is dominated by $\sum h_n$, where

$$h_0(r)=C(|k|+r^{-1}),$$

$$h_n(r)=C\int_0^r dth_{n-1}(t)|V(t)|t(1+|k|t)^{-1},\quad n\geqslant 1,$$

which, in turn, is dominated by the series $\sum\xi^{(n)}$, where

$$\xi^{(0)}=C(|k|+r^{-1}),$$

$$\xi^{(n)}=C\left[\int_0^r dt(1+t)|V(t)|\right]^n\Big/ n!,\quad n\geqslant 1.$$

Then

$$\sum\xi^{(n)}=C(|k|+r^{-1}-1)+C\exp\left[C\int_0^r dt(1+t)|V(t)|\right].$$

It follows that if $V\epsilon\mathfrak{M}(0)$, then $\sum\Xi^{(n)}$ and hence the Born series of (A.8) converges for all k and r. Furthermore, if $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(1)$, then ΔG satisfies a set of inequalities, which, when resubstituted in (A.8) yield, for all k and all r,

$$|(1-P)\Delta G(k,r)|\leqslant Ce^{|\nu|r}\left(\frac{r}{1+|k|r}\right)^{l+2}\left(1+\frac{r}{1+|k|r}\right),$$

$$|P\Delta G(k,r)|\leqslant Ce^{|\nu|r}\left(\frac{r}{1+|k|r}\right)^{l+3}\int_0^r dt\frac{t}{1+|k|t}\times\left(1+\frac{t}{1+|k|t}\right)^2|V(t)|.\quad (A.12)$$

Since G is now defined by a series of entire functions of k which converges for all finite k, it is itself an entire function of k. Moreover, (A.2) and (A.12) show that for every $r<\infty$, as $|k|\to\infty$,

$$(1-P)G(k,r)=(1-P)[k^{-(l+1)}\sin(kr-\tfrac{1}{2}\pi l)\mathbf{1}+O(k^{-(l+2)}e^{|\nu|r})],$$
$$PG(k,r)=P[-k^{-(l+3)}\sin(kr-\tfrac{1}{2}\pi l)\mathbf{1}+O(k^{-(l+4)}e^{|\nu|r})].\quad (A.13)$$

Also, for every k, as $r\to 0$,

$$(1-P)G(k,r)=(1-P)[((2l+1)!!)^{-1}r^{l+1}\mathbf{1}+O(r^{l+2})],\quad (A.14)$$
$$PG(k,r)=P[((2l+5)!!)^{-1}r^{l+3}\mathbf{1}+o(r^{l+4})].$$

One now readily verifies that G satisfies the differential equation (2.1). Equation (A.8) also shows that, as $r\to 0$, $\lim[G'(k,r)-G_0'(k,r)]=0$, and therefore, with

(A.14),

$$[G;G]=0. \quad (A.15)$$

For later purposes we need an estimate on $\dot{G}(k,r)$, where k is real. Differentiation of (A.8) with respect to yields

$$\Delta\dot{G}(k,r)=\Psi(k,r)+\int_0^r dt\Delta\dot{G}(k,t)V(t)\mathcal{G}(k;t,r), \quad (A.16)$$

where

$$\Psi(k,r)=\dot{\phi}(k,r)+\int_0^r dt\Delta G(k,t)V(t)\mathcal{G}^{\cdot}(k;t,r). \quad (A.17)$$

One obtains from (A.10) that

$$|\mathfrak{f}^{\cdot}(k,t,r)|\leqslant Cr\left(\frac{|k|t}{1+|k|t}\right)\left(\frac{|k|r}{1+|k|r}\right).$$

As a result of this and similar estimates on the other integrals in (A.9), it is easily found that, for $\mathrm{Im}k=0$,

$$\begin{aligned}|(1-P)\dot{\phi}(k,r)| &\leqslant C|k|^{-1}r\left(\frac{r}{1+|k|r}\right)^{l+1}\left(1+\frac{r}{1+|k|r}\right),\\ |P\dot{\phi}(k,r)| &\leqslant C|k|^{-1}r\left(\frac{r}{1+|k|r}\right)^{l+3}.\end{aligned} \quad (A.18)$$

The second term on the right of (A.17) is estimated by means of (A.7) and (A.12). If $V\epsilon\mathfrak{M}(0)$, then

$$\begin{aligned}\left|\int_0^r dt(1-P)\Delta G(k,t)V(t)\mathcal{G}^{\cdot}(k;t,r)\right| &\leqslant C|k|^{-1}r\left(\frac{r}{1+|k|r}\right)^{l+2}\left(1+\frac{r}{1+|k|r}\right)^2,\\ \left|\int_0^r dtP\Delta G(k,t)V(t)\mathcal{G}^{\cdot}(k;t,r)\right| &\leqslant C|k|^{-1}r\left(\frac{r}{1+|k|r}\right)^{l+4}\left(1+\frac{r}{1+|k|r}\right).\end{aligned} \quad (A.19)$$

The integral equation (A.16) leads from (A.18) and (A.19) to (A.20) in the same fashion as (A.8) does from (A.11) to (A.12).

$$\begin{aligned}|(1-P)\Delta\dot{G}(k,r)| &\leqslant C|k|^{-1}r\left(\frac{r}{1+|k|r}\right)^{l+1}\left(1+\frac{r}{1+|k|r}\right)^4,\\ |P\Delta\dot{G}(k,r)| &\leqslant C|k|^{-1}r\left(\frac{r}{1+|k|r}\right)^{l+3}\left(1+\frac{r}{1+|k|r}\right)^3.\end{aligned} \quad (A.20)$$

This completes the inequalities needed from $G(k,r)$.

For the purpose of estimating $F(k,r)$, we write

$$F_e(k,r)=F_{e0}(k,r)+\int_r^\infty dtF_e(k,t)V(t)\mathcal{G}(k;r,t), \quad (A.21)$$

or

$$\Delta F_e(k,r)=\varphi(k,r)+\int_r^\infty dt\Delta F_e(k,t)V(t)\mathcal{G}(k;r,t), \quad (A.21')$$

where

$$\Delta F_e(k,r)=F_e(k,r)-F_{e0}(k,r),$$

$$\varphi(k,r)=\int_r^\infty dtF_{e0}(k,t)V(t)\mathcal{G}(k;r,t),$$

$$F_{e0}(k,r)=k^l[(1-P)w_l(kr)+Pk^2w_{l+2}(kr)].$$

According to (A.5) and (A.6), for $\nu=\mathrm{Im}k\leqslant 0$,

$$\begin{aligned}|(1-P)\varphi(k,r)| &\leqslant Ce^{-|\nu|r}\left(\frac{1+|k|r}{r}\right)^{l+2}\times\int_r^\infty dt\left(\frac{t}{1+|k|t}\right)^3|V(t)|,\\ |P\varphi(k,r)| &\leqslant Ce^{-|\nu|r}\left(\frac{1+|k|r}{r}\right)^{l+2}\times\int_r^\infty dt\left(\frac{t}{1+|k|t}\right)|V(t)|.\end{aligned} \quad (A.22)$$

If we now write

$$\Delta F_e(k,r)=\sum_{n=0}^\infty F^{(n)}(k,r),$$

where

$$F^{(0)}(k,r)=\varphi(k,r),$$

$$F^{(n)}(k,r)=\int_r^\infty dtF^{(n-1)}(k,t)V(t)\mathcal{G}(k;r,t), \quad n\geqslant 1,$$

and

$$\Upsilon^{(n)}(k,r)=e^{|\nu|r}\left(\frac{r}{1+|k|r}\right)^{l+2}\times\left\{\left[\int_r^\infty dt\left(\frac{t}{1+|k|t}\right)^3|V(t)|\right]^{-1}(1-P)+\left[\int_r^\infty dt\left(\frac{t}{1+|k|t}\right)|V(t)|\right]^{-1}P\right\}F^{(n)}(k,r),$$

then $\sum\Upsilon^{(n)}$ is dominated by the series $\sum\xi_n$, where

$$\xi_0(r)=C,\quad \xi_n(r)=C\int_r^\infty dt|V(t)|t\xi_{n-1}(t),\quad n\geqslant 1.$$

Then $\xi_n(r)=C[C\int_r^\infty dt\, t|V(t)|]^n/n!$ and $\sum\xi_n\leqslant C'$, if $V\epsilon\mathfrak{M}(1)$. Consequently the Born series of (A.21) converges for all $r>0$ and all k for which $\mathrm{Im}k\leqslant 0$. (For

$k=0$, it is required that $V\epsilon\mathfrak{M}(3)$.) Thus, $F_e(k,r)$ is, for all $r>0$, an analytic function of k for $\mathrm{Im}k<0$, continuous for $\mathrm{Im}k\leqslant 0$. Moreover, it satisfies the following inequalities:

$$\begin{aligned}|(\mathbf{1}-P)\Delta F_e(k,r)| &\leqslant Ce^{-|\nu|r}\left(\frac{1+|k|r}{r}\right)^{l+2} \times\int_r^\infty dt\left(\frac{t}{1+|k|t}\right)^3|V(t)|,\\ |P\Delta F_e(k,r)| &\leqslant Ce^{-|\nu|r}\left(\frac{1+|k|r}{r}\right)^{l+2} \times\int_r^\infty dt\left(\frac{t}{1+|k|t}\right)|V(t)|.\end{aligned}\qquad (A.23)$$

For $\mathrm{Im}k\leqslant 0$, therefore, as $r\to\infty$,

$$\begin{aligned}F_e(k,r)&=F_{e0}(k,r)+o(r^{-3}e^{-|\nu|r})\\ &=(ik)^l e^{-ikr}\mathcal{K}_0(k^2)\mathcal{E}+O(r^{-1}e^{-|\nu|r}),\quad k\neq 0,\\ F_e(0,r)&=(2l-1)!!\{(\mathbf{1}-P)[r^{-l}\mathbf{1}+o(r^{-l-2})]\\ &\quad+(2l+3)(2l+1)P[r^{-l-2}\mathbf{1}+o(r^{-l-4})]\},\end{aligned}\qquad (A.24)$$

where $\mathcal{E}$ is defined by (2.19).

For every $r>0$, as $|k|\to\infty$, $\mathrm{Im}k\leqslant 0$,

$$F(k,r)=i^l e^{-ikr}\mathcal{E}+O(|k|^{-1}e^{-|\nu|r}).\qquad (A.25)$$

We now wish to prove the existence of $\dot F_e(k,r)$ and estimate it, for real k. Accordingly we differentiate (A.21) with respect to k:

$$\dot F_e(k,r)=\psi(k,r)+\int_r^\infty dt\dot F_e(k,t)V(t)\mathcal{G}(k;r,t)\qquad (A.26)$$

$$\psi(k,r)\equiv\dot F_{e0}(k,r)+\int_r^\infty dtF_e(k,t)V(t)\dot{\mathcal{G}}(k;r,t).\qquad (A.27)$$

Now

$$\left.\begin{aligned}|(\mathbf{1}-P)\dot F_{e0}(k,r)| &\leqslant\begin{cases}Cr, & l=0,\\ Cr|k|(|k|+r^{-1})^{l-1} & l\geqslant 1,\end{cases}\\ |P\dot F_{e0}(k,r)| &\leqslant Cr|k|(|k|+r^{-1})^{l+1}.\end{aligned}\right\}\qquad (A.28)$$

The inequalities (A.7) and (A.23) yield for $r\geqslant r_0>0$, $|k|\leqslant k_0<\infty$,

$$\begin{aligned}&\left|\int_r^\infty dt(\mathbf{1}-P)F_e(k,t)V(t)\dot{\mathcal{G}}(k;r,t)\right|\\ &\quad\leqslant C\left(\frac{1+|k|r}{r}\right)^{l+2}\int_r^\infty dt\,t^4|V(t)|\left(\frac{|k|t}{1+|k|t}\right)\\ &\left|\int_r^\infty dtPF_e(kt,)V(t)\dot{\mathcal{G}}(k;r,)t\right|\\ &\quad\leqslant C\left(\frac{1+|k|r}{r}\right)^{l+2}\int_r^\infty dt\,t^2|V(t)|\left(\frac{|k|t}{1+|k|t}\right).\end{aligned}\qquad (A.29)$$

Therefore, for $r\geqslant r_0>0$, $|k|\leqslant k_0<\infty$,

$$\left.\begin{aligned}&|(\mathbf{1}-P)\psi(k,r)|\\ &\quad\leqslant\begin{cases}Cr,\ l=0, & \text{if } V\epsilon\mathfrak{M}(4)\text{ and if }V\epsilon\mathfrak{M}(5),\\ Cr(|k|+r^{-1})^{l-1}, & \text{if } V\epsilon\mathfrak{M}(4),\\ Cr(|k|+r^{-1})^{l-1}|k|, & \text{if } V\epsilon\mathfrak{M}(5),\end{cases}\Big\}\,l\geqslant 1,\\ &|P\psi(k,r)|\\ &\quad\leqslant\begin{cases}Cr(|k|+r^{-1})^{l+1}, & \text{if } V\epsilon\mathfrak{M}(2),\\ Cr(|k|+r^{-1})^{l+1}|k|, & \text{if } V\epsilon\mathfrak{M}(3).\end{cases}\end{aligned}\right\}\qquad (A.30)$$

The inequalities (A.30) lead to estimates on $\dot F_e(k,r)$, via (A.26) in the same way as (A.22) leads to (A.23). For the existence of $P\dot F_e(0,r)$ it is sufficient that $V\epsilon\mathfrak{M}(2)$, while for that of $(\mathbf{1}-P)\dot F_e(0,r)$ we require $V\epsilon\mathfrak{M}(4)$. If $V\epsilon\mathfrak{M}(3)$, then $P\dot F_e(0,r)\equiv 0$ and $P\ddot F_e(0,r)$ exists; if $V\epsilon\mathfrak{M}(5)$, then, for $l\geqslant 1$, $(\mathbf{1}-P)\dot F_e(0,r)\equiv 0$ and $(\mathbf{1}-P)\ddot F_e(0,r)$ exists. The same is true for $F_e(k,r)$ and therefore for $F_e(k)$, if in addition $V\epsilon\mathfrak{M}(0)$.

For the purpose of further estimates it is convenient to introduce an integral representation for $F(k)$. We write

$$F_e(k)=[F_{e0}(k,r);G(k,r)]+[\Delta F_e(k,r);G(k,r)].$$

The last term vanishes as $r\to\infty$. Since, moreover,

$$[F_{e0}(k,r);\mathcal{G}(k;t,r)]=F_{e0}(k,t),$$

and

$$[F_0(k,r);Pu_{l+2}(kr)]=kP,$$

one obtains from the integral equation for G, (2.6),

$$\begin{aligned}F_e(k)=\mathbf{1}+\int_0^\infty dt[F_{e0}(k,t)V(t)G^T(k,t)\\ -(2l+3)t^{-1}PV(t)(\mathbf{1}-P)]\\ +(2l+3)\int_1^\infty dt\,t^{-1}PV(t)(\mathbf{1}-P),\end{aligned}\qquad (A.31)$$

or

$$\begin{aligned}F(k)-\mathbf{1}=\int_0^\infty dt[F_0(k,t)V(t)G_0^T(k,t)\mathcal{K}_l(k)\\ -(2l+3)k^{-2}t^{-1}PV(t)(\mathbf{1}-P)]\\ +(2l+3)k^{-2}\int_1^\infty dt\,t^{-1}PV(t)(\mathbf{1}-P)\\ +\int_0^\infty dtF_0(k,t)V(t)\Delta G^T(k,t)\mathcal{K}_l(k).\end{aligned}\qquad (A.31')$$

(For the convergence of the last integral we require $V\epsilon\mathfrak{M}(0)$.) If $V\epsilon\mathfrak{M}(1)$, then (A.12) shows that the last term in (A.31′) is $O(k^{-2})$ as $|k|\to\infty$. The first term is $O(k^{-1})$ as $|k|\to\infty$, if $V\epsilon\mathfrak{M}(0)$. Therefore, if $V\epsilon\mathfrak{M}(0)$

$\cap\mathfrak{M}(1)$, then

$$F(k)=1+O(k^{-1}),\ \text{as}\ |k|\to\infty\ \text{in Im}k\leqslant 0. \quad (A.32)$$

We further wish to estimate the derivative of $F(k)$. The first term in (A.31′) contains expressions such as $k^{-1}\varphi_{22}(k)$, where

$$\varphi_{22}(k)=\int_0^\infty dt V_{22}(t)w_{l+2}(kt)u_{l+2}(kt).$$

The boundedness of φ_{22} is readily proved. For the derivative we write

$$u_l(x)=\begin{cases}(-1)^{l/2}\left(\dfrac{x}{1+x}\right)^l \sin x+\alpha_l(x), & l \text{ even}\\ (-1)^{(l+1)/2}\left(\dfrac{x}{1+x}\right)^{l+1}\cos x+\alpha_l(x), & l \text{ odd},\end{cases}$$

$$w_l(x)=e^{-ix}\beta_l(x),$$

and easily prove the following inequalities (for real x) from (A.2) and (A.5):

$$|\alpha_l(x)|\leqslant C\frac{x^{l+1}}{(1+x)^{l+2}},\quad |\beta_l(x)|\leqslant C\left(\frac{1+x}{x}\right)^l,$$

$$|\alpha_l'(x)|\leqslant C\frac{x^l}{(1+x)^{l+1}},\quad |\beta_0'(x)|=0,$$

$$|\beta_l'(x)|\leqslant C\frac{(1+x)^{l-1}}{x^{l+1}},\quad l\geqslant 1.$$

We now have

$$\dot\varphi_{22}(k)=\int_0^\infty dt\, tV_{22}(t)\left\{\frac{\partial}{\partial kt}[\alpha_{l+2}(kt)w_{l+2}(kt)]\right.$$
$$-(-1)^{l/2}e^{-2ikt}\left(\frac{kt}{1+kt}\right)^{l+2}\beta_{l+2}(kt)$$
$$\left.-(-1)^{l/2}e^{-ikt}\sin kt\frac{\partial}{\partial kt}\left[\left(\frac{kt}{1+kt}\right)^{l+2}\beta_{l+2}(kt)\right]\right\}$$

for even l, and a similar expression for odd l. The first and third terms are $O(k^{-1})$ as $|k|\to\infty$ if $V\epsilon\mathfrak{M}(0)$ $\times(O(k^{-\delta})$ if $V\epsilon\mathfrak{M}(1-\delta))$. Consider the second;

$$\chi_l(x)\equiv\beta_l(x)[x/(1+x)]^l$$

is bounded, and

$$|\chi_l'(x)|\leqslant c/(1+|x|)^2.$$

Therefore, with $\pi|k|^{-1}=\xi$,

$$|\chi_l(k(t+\xi))-\chi_l(kt)|$$
$$=\pi|\chi_l'(kt')|\leqslant c/(1+|k|t)^2\leqslant c/(t|k|).$$

One then proves precisely as in (I). (C.5) to (C.7), from (2.5) that the second term in $\dot\varphi_{22}$ is $O(k^{-\delta})$ as $|k|\to\infty$. Therefore

$$\dot\varphi_{22}(k)=O(k^{-\delta})\ \text{as}\ |k|\to\infty.$$

One proceeds similarly for

$$\varphi_{00}(k)=\int_0^\infty dt V_{11}(t)w_l(kt)u_l(kt),$$

$$\varphi_{02}(k)=\int_0^\infty dt V_{12}(t)w_l(kt)u_{l+2}(kt),$$

$$\varphi_{20}(k)=\int_0^\infty dt V_{21}(t)[w_{l+2}(kt)u_l(kt)-(2l+3)(kt)^{-1}].$$

φ_{20} requires a little extra care; there one sets

$$u_l(x)=[(2l+1)!!]^{-1}\left(\frac{x^{l+1}}{1+x^{l+2}}\right)+\gamma_l(x),$$

$$w_l(x)=(2l-1)!!x^{-l}+e^{-ix}\zeta_l(x),$$

and obtains

$$w_{l+2}(x)u_l(x)-(2l+3)x^{-1}=-(2l+3)\frac{x^{l+1}}{1+x^{l+2}}$$
$$+(2l+3)!!x^{-l}\gamma_l(x)+\zeta_{l+2}(x)e^{-ix}u_l(x),$$

whose integrals exist separately. The first two terms contribute $O(k^{-1})$ to the k-derivative, and the last $O(k^{-\delta})$.

Consequently, the first integral in (A.31) yields, when differentiated with respect to k, $O(k^{-\delta})$ as $|k|\to\infty$. The derivative of the second is clearly $O(k^{-3})$; that of the last consists of two parts:

For $|k|\geqslant k_0>0$, by (A.5) and (A.12),

$$\left|\int_0^\infty dt\dot F_0(kt)V(t)\Delta G^T(k,t)\right|$$
$$\leqslant C\int_0^\infty dt|V(t)|(t+|k|^{-1})|k|^{-2}\leqslant C'|k|^{-2}$$

if $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(1)$;

$$\left|\int_0^\infty dt F_0(kt)V(t)\Delta\dot G^T(k,t)\right|\leqslant C|k|^{-2},$$

by the use of (A.20). Therefore it follows that

$$\dot F(k)=O(k^{-1-\delta}),\ \text{as}\ |k|\to\infty\ \text{in Im}k\leqslant 0. \quad (A.33)$$

The function $F_{N0}(k)$ of (5.1) combines the features of $F_e(k)$ at the origin with those of $F(k)$ at infinity. It is continuous and has a continuous derivative for all real k, and it satisfies (A.32) and (A.33). It was proved in (I), Appendix C, that therefore $F_{N0}(k)$, when transformed to the unit circle, satisfies a Hölder condition, (I), (C.15).

Finally, the behavior of $\dot{F}_e(k,r)$ near $k=0$ must be examined. From (A.21) we obtain

$$\dot{F}_e(k,r)-\dot{F}_e(0,r)=\dot{F}_{e0}(k,r)-\dot{F}_{e0}(0,r)$$
$$+\int_r^\infty dtF_e(0,t)V(t)[\mathcal{G}^\cdot(k;r,t)-\mathcal{G}^\cdot(0;r,t)]$$
$$+\int_r^\infty dt[F_e(k,t)-F_e(0,t)]V(t)\mathcal{G}^\cdot(k;r,t)$$
$$+\int_r^\infty dt\dot{F}_e(0,t)V(t)[\mathcal{G}(k;r,t)-\mathcal{G}(0;r,t)]$$
$$+\int_r^\infty dt[\dot{F}_e(k,t)-\dot{F}_e(0,t)]V(t)\mathcal{G}(k;r,t). \quad (A.34)$$

We use this equation for $l=0$ only, and multiplied on the left by $(1-P)$. Regarding (A.34) as an integral equation for $\dot{F}_e(k,r)-\dot{F}_e(0,r)$, we estimate first the inhomogeneity. For $r\geqslant r_0>0$, $|k|\leqslant k_0<\infty$, if $V\epsilon\mathfrak{M}(4+\delta)$,

$$|(1-P)[\dot{F}_{e0}(k,r)-\dot{F}_{e0}(0,r)]|\leqslant Cr|\sin kr|\leqslant Cr^{1+\delta}|k|^\delta,$$

$$\left|\int_r^\infty dt(1-P)F_e(0,t)V(t)[\mathcal{G}^\cdot(k;r,t)-\mathcal{G}^\cdot(0;r,t)]\right|$$
$$\leqslant C\int_r^\infty dt|V(t)|\frac{|k|t^5}{(1+|k|t)^4}\leqslant C'|k|^\delta,$$

$$\left|\int_r^\infty dt(1-P)[F_e(k,t)-F_e(0,t)]V(t)\mathcal{G}^\cdot(k;r,t)\right|$$
$$\leqslant C|k|\int_r^\infty dt|V(t)|\frac{|k|t^5}{(1+|k|t)^4}\leqslant C'|k|^\delta,$$

$$\left|\int_r^\infty dt(1-P)\dot{F}_e(0,t)V(t)[\mathcal{G}(k;r,t)-\mathcal{G}(0;r,t)]\right|$$
$$\leqslant C\int_r^{1/|k|}dt|V(t)|t\frac{|k||k'|t^5}{(1+|k'|t)^4}+C\int_{1/|k|}^\infty dt|V(t)|t\left(\frac{t}{1+|k|t}\right)^3$$
$$\leqslant C\int_r^{1/|k|}dt\,t^{4+\delta}|V(t)||k|^\delta+C\int_{1/|k|}^\infty dt\,t^{4+\delta}|V(t)||k|^\delta\leqslant C'|k|^\delta,$$

where we have used the mean value theorem and (A.6), (A.7), and (A.30). In the same manner as previously, one then obtains from (A.34) by successive approximations, that, if $V\epsilon\mathfrak{M}(4+\delta)$, then

$$|(1-P)[\dot{F}_e(k,r)-\dot{F}_e(0,r)]|\leqslant C|k|^\delta, \quad (A.35a)$$

for $l=0$. It follows from (A.30) and (A.26) that $V\epsilon\mathfrak{M}(4+\delta)$ also implies (A.35a) for $l\geqslant 1$, and

$$|P[\dot{F}_e(k,r)-\dot{F}_e(0,r)]|\leqslant C|k|^\delta \quad (A.35b)$$

for all l. As a consequence,

$$S_{N0}(k)=1+O(k^\delta),\text{ as } k\rightarrow 0$$

and then, as in (I) Appendix C, after the transformation (5.8) to the unit circle, S_{N0} satisfies a Hölder condition:

$$|M(t_1)-M(t_2)|\leqslant A|t_1-t_2|^\mu \quad (A.36)$$

provided that $V\epsilon\mathfrak{M}(4+\delta)$ and V obeys (2.5).

A further theorem is obtained by taking the imaginary part of (A.31):

$$\text{Im}F_e(k)=-k^{2l+1}\mathcal{K}_0{}^2(k)\int_0^\infty dtG_0(k,t)V(t)G^T(k,t). \quad (A.37)$$

If $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(2l+4)$, then it follows that

$$(1-P)\ \text{Im}F_e(k)=O(k^{2l+1}),\text{ as } k\rightarrow 0, \quad (A.38a)$$

and if $V\epsilon\mathfrak{M}(0)\cap\mathfrak{M}(2l+6)$, then

$$P\ \text{Im}F_e(k)=O(k^{2l+5}),\text{ as } k\rightarrow 0. \quad (A.38b)$$

APPENDIX B

In this Appendix, we shall prove an ideal-theoretic theorem which is needed for the purpose[39] of defining an "irregular" solution of (2.1) as an entire function of E.

All matrices in the following are understood to be matrices over the ring of entire functions of a complex variable z (i.e., all elements are entire functions). The subsequent three theorems were proved by O. Helmer.[40]

Theorem 1.—Every finitely generated ideal (in the ring of entire functions) is a principal ideal[41]:

$$[\![\alpha_1,\alpha_2,\cdots\alpha_r]\!]=[\![\mu]\!].$$

The function μ is the "largest common divisor" of the set $\{\alpha_1,\cdots,\alpha_r\}$ and we also use the notation $[\![\alpha_1,\cdots,\alpha_r]\!]=\mu$. The common zeros of $\{\alpha_1,\cdots,\alpha_r\}$ and their multiplicities determine μ.

Theorem 2.—If $[\![\alpha_1,\cdots,\alpha_n]\!]=\mu$, then there exists an $(n\times n)$-matrix with the vector $(\alpha_1,\cdots,\alpha_n)$ as its first row and whose determinant is μ.

Corollary.—Every matrix $A=\|\alpha_{ik}\|$ can be made triangular by means of a left multiplication by an

[39] See the paragraph after (2.14).

[40] O. Helmer, Duke Math. J. **6**, 351 (1940); and Bull. Am. Math. Soc. **49**, 225 (1943).

[41] Here is a reminder of the definitions: An ideal is a set with the property that if it contains α, and β is any member of the ring, then it also contains $\beta\alpha$; it is generated by the set $\{\alpha_1,\cdots,\alpha_r\}$ if every member of the ideal is a "linear combination" of $\alpha_1,\cdots,\alpha_r$ with coefficients in the ring. A principal ideal is an ideal generated by a single element. We use the notation $[\![\alpha_1,\cdots,\alpha_r]\!]$ for the ideal generated by the functions $\{\alpha_1,\cdots,\alpha_r\}$. The ideal generated by the unity is, of course, the whole ring.

invertible[42] matrix T:

$$B=TA=\begin{bmatrix}\beta_{11} & \beta_{12} & \cdots & \beta_{1n}\\ 0 & \beta_{22} & \cdots & \beta_{2n}\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & \beta_{nn}\end{bmatrix}.$$

Proof.—We prove this by induction. For $n=1$ the statement is trivial. Let $[\![\alpha_{11},\alpha_{21},\cdots,\alpha_{n1}]\!]=\mu$ and $\sum\alpha_k\alpha_{k1}=\mu$, with $[\![\alpha_1,\cdots,\alpha_n]\!]=1$, which is always possible; let T_1 have the vector $(\alpha_1,\cdots,\alpha_n)$ as its first row and let T_1 be invertible. The matrix T_1A has then in its left upper corner a divisor of all elements of its first column. By means of a subtraction of suitable multiples of the first row from the other rows (which can be accomplished by left multiplication by an invertible matrix), a matrix of the following kind can be obtained:

$$\begin{bmatrix}\mu & ** & \cdots & *\\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{bmatrix}.$$

One then applies the induction hypothesis to A_1.

Remark.—If the first column of A vanishes, then there exist invertible matrices T and T' so that

$$B=TA=\begin{bmatrix}0 & \beta_{12} & \beta_{13} & \cdots & \beta_{1n}\\ 0 & 0 & \beta_{23} & \cdots & \beta_{2n}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix},$$

and

$$B'=T'A=\begin{bmatrix}0 & 0 & 0 & \cdots & 0\\ 0 & \beta_{22}' & \beta_{23}' & \cdots & \beta_{2n}'\\ 0 & 0 & \beta_{33}' & \cdots & \beta_{3n}'\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & & \beta_{nn}'\end{bmatrix}.$$

Theorem 3 (*Elementary divisor theorem*).—To every matrix A there exist two invertible matrices T_1 and T_2 so that

$$T_1AT_2=N=\begin{bmatrix}\epsilon_1 & & & & & 0\\ & \epsilon_2 & & & & \\ & & \ddots & & & \\ 0 & & & \epsilon_k & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & 0\end{bmatrix},$$

where ϵ_l is a divisor of ϵ_{l+1}.

We can now prove the following

Theorem.—Let M_1 and M_2 be two $(n\times n)$-matrices with the property that for every z_0 it follows from

$$M_1(z_0)a=0 \quad\text{and}\quad M_2(z_0)a=0$$

that $a=0$. Then there exist two matrices N_1 and N_2 so that

$$N_1M_1+N_2M_2=\mathbf{1}.$$

In other words: the left ideal $\mathfrak{M}=[\![M_1,M_2]\!]$ generated by M_1 and M_2 is the entire ring.

Proof.—Again we proceed by induction. For $n=1$, the theorem is equivalent to Theorem 1.

Clearly, $[\![T_1M_1S,T_2M_2S]\!]=\mathfrak{M}S$, if T_1 and T_2 are invertible. We assume the same about S. Due to Theorem 3 and the corollary to Theorem 2 we may, then, assume that the generators of $\mathfrak{M}S$ are of the form

$$M_1'=\begin{bmatrix}\epsilon_1 & & & & & \\ & \epsilon_2 & & 0 & & \\ & & \ddots & & & \\ & & & \epsilon_k & & \\ 0 & & & & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & 0\end{bmatrix},\quad M_2'=\begin{bmatrix}\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n}\\ 0 & \alpha_{22} & \cdots & \alpha_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \alpha_{nn}\end{bmatrix}.$$

The hypothesis of our theorem evidently still holds for M_1 and M_2. If it is applied to a column vector whose only nonzero component is the first, it follows that $[\![\epsilon_1,\alpha_{11}]\!]=1$, i.e., there exist α and β so that $\alpha\epsilon_1+\beta\alpha_{11}=1$. With these one constructs the matrices

$$A_{11}=\begin{bmatrix}\alpha & 0 & \cdots & 0\\ 0 & & & \\ \vdots & & \mathbf{1} & \\ 0 & & & \end{bmatrix},\quad A_{12}=\begin{bmatrix}\beta & 0 & \cdots & 0\\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix},$$

$$A_{21}=\begin{bmatrix}-\alpha_{11} & 0 & \cdots & 0\\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix},\quad A_{22}=\begin{bmatrix}\epsilon_1 & 0 & \cdots & 0\\ 0 & & & \\ \vdots & & \mathbf{1} & \\ 0 & & & \end{bmatrix}.$$

The matrices

$$M_1''=A_{11}M_1'+A_{12}M_2',\quad M_2''=A_{21}M_1'+A_{22}M_2',$$

for which

$$M_1'=A_{22}M_1'-A_{12}M_2'',\quad M_2=-A_{21}M_1''+A_{11}M_2'',$$

generate the same left ideal as M_1' and M_2'. Both are triangular:

$$M_1''=\begin{bmatrix}1 & \beta\alpha_{12} & \beta\alpha_{13} & \cdots & \beta\alpha_{1n}\\ 0 & \epsilon_2 & 0 & \cdots & 0\\ 0 & 0 & \epsilon_3 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix},$$

$$M_2''=\begin{bmatrix}0 & \alpha_{12}'' & \alpha_{13}'' & \cdots & \alpha_{1n}''\\ 0 & \alpha_{22}'' & \alpha_{23}'' & \cdots & \alpha_{2n}''\\ 0 & 0 & \alpha_{33}'' & \cdots & \alpha_{3n}''\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \alpha_{nn}''\end{bmatrix}.$$

[42] We call a matrix T invertible if T^{-1} exists and is a matrix over the ring of entire functions.

In both matrices one now subtracts the first column multiplied by $\beta\alpha_{1k}$ from the kth column, which corresponds to an allowed right multiplication. This diagonalizes M_1'', does not change M_2'', and transforms the ideal $\mathfrak{M}S$ into $\mathfrak{M}S_1$. Finally one transforms M_2'', according to the remark after the corollary to Theorem 2, in such a way that $\mathfrak{M}S_1$ is generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & M_1^{(0)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & M_2^{(0)} \end{pmatrix}.$$

The $[(n-1)\times(n-1)]$-matrices $M_1^{(0)}$ and $M_2^{(0)}$ again satisfy the hypothesis of the theorem and the latter thus follows by induction.

APPENDIX C

Theorem.—Let $M(z)$ be an $(n\times n)$-matrix valued function, analytic in a neighborhood of the origin and $\det M(z)\neq 0$ for all z in that neighborhood, except possibly at $z=0$; furthermore, let $M^{-1}(z)$ have at most a double pole at $z=0$; and let T be a constant $(n\times n)$-matrix. Then a necessary and sufficient condition for $M^{-1}(z)T$ to have at most a simple pole at $z=0$ is that the system

$$aM_0=0, \quad aM_1+bM_0=0, \tag{C.1}$$

where $M_0=M(0)$, $M_1=(\partial M/\partial z)_{z=0}$, implies $aT=0$.

Proof.—If we expand

$$M(z)=M_0+M_1z+\cdots,$$
$$M^{-1}(z)=N_{-2}z^{-2}+N_{-1}z^{-1}+N_0+\cdots,$$

then the matrices M_i and N_i must satisfy

$$\begin{aligned} M_0N_{-2}=0, \quad M_0N_{-1}+M_1N_{-2}=0, \\ M_0N_0+M_1N_{-1}+M_2N_{-2}=\mathbf{1}, \end{aligned} \tag{C.2}$$

and the same equations with the order of M and N reversed.

Suppose that $M^{-1}T$ has a simple pole at $z=0$. Then $N_{-2}T=0$. Multiplication of the last equation in (C.2) by a on the left and T on the right then yields, by (C.1) and the second and first equation in (C.2),

$$aT=aM_1N_{-1}T=-bM_0N_{-1}T=bM_1N_{-2}T=0.$$

This proves the necessity. The sufficiency is obtained immediately from the first two equations in (C.2) with the order of M and N reversed.