PHYSICAL REVIEW VOLUME 100, NUMBER 1 OCTOBER 1, 1955

Structure of Green's Functions in Quantum Field Theory*

YOICHIRO NAMBU†
Institute for Nuclear Studies, University of Chicago, Chicago, Illinois
(Received June 8, 1955)

The Green's functions are vacuum expectation values of time ordered products of field operators in the Heisenberg representation. They give us vital information about the nature of the interacting particles and quanta represented by these field operators. It is shown that the Fourier transforms of the Green's functions (for up to four operators) are expressible as parametric integrals involving invariant energy denominators and real, scalar weight functions which are termed the spectral functions. Relativity, causality, and some other fundamental assumptions of field theory are required to derive the result.

The spectral functions have a simple physical interpretation, and completely specify the structure of the Green's functions. The equations of motion which hold between Green's functions of different order can be translated into the corresponding relations between the spectral functions. Renormalization can be carried out explicitly, bringing the equations for the spectral functions into a manifestly renormalized form. The causality condition also serves as a means of obtaining renormalized quantities without recourse to the usual subtraction procedure. The observed masses and coupling constants occur in the equations for the spectral functions as external parameters fixing the boundary condition. No unobservable bare masses and couplings, nor the (infinite) renormalization constants ever appear in the equations. These quantities, however, are shown to be expressible in terms of the spectral functions.

The Green's functions involving more than four field operators are not considered in this paper.

1. INTRODUCTION

IN the quantum theory of interacting fields the nature of field operators in the Heisenberg representation seems to be of interest and importance, since these operators correspond to physically observable quantities, and there have been various attempts at the study of the Heisenberg operators which are free from the limitations of perturbation theory. Källén,[1] Lehmann,[2] and Gell-Mann and Low[3] have studied the structure of the Green's functions like Δ_F', which is defined for a scalar field by

$$\Delta_F'(x-x')=\langle 0|P(\varphi(x),\varphi(x')|0\rangle, \tag{1.1}$$

where $|0\rangle$ is the actual vacuum state and P the time ordering operator; they have been able to show that Δ_F' can be expressed, quite independently of the details of interaction, in the form

$$\begin{aligned}\Delta_F'(x-x')&=\frac{1}{(2\pi)^4}\int e^{ip(x-x')}\Delta_F'(p)(dp)^4,\\ \Delta_F'(p)&=-i\int_0^\infty \frac{1}{p^2+m^2-i\epsilon}\nu(m^2)dm^2,\end{aligned} \tag{1.2}$$

where the function ν satisfies certain conditions. In deriving this result, two physical requirements are to be made: (1) the theory is Lorentz invariant, and (2) positive energy states form a complete system in the Hilbert space.

A generalization of the above idea was attempted in a previous paper[4] where the Fourier component of a quantity which is related to the scattering matrix for a meson and a particle, was expressed in the form

$$\begin{aligned}M(x,x')&\equiv\langle q|P(j(x),j(x'))|p\rangle\\ &=\frac{1}{(2\pi)^4}\int e^{-ikx'+ilx}\delta(p+k-q-l)\\ &\qquad\times M(k+l,\,p,q)(dk)^4(dl)^4,\\ M(k+l,\,p,q)&\\ &=\int\int\int\frac{\nu_1+(k+l,\,p)\nu_2+(k+l,\,q)\nu_3}{(k+l+\alpha p+\beta q)^2+m^2-i\epsilon}dm^2d\alpha d\beta,\end{aligned} \tag{1.3}$$

$|p\rangle$ or $|q\rangle$ being the one-particle state of momentum p or q which is assumed to have no spin. In achieving this, the causality principle was emphasized in addition to the above two conditions. It is characterized by the requirement that two observables (Heisenberg operators) must commute outside of each other's light cones.

It can be seen, however, that the tools used in deriving Eq. (1.3) are essentially equivalent to those for Eq. (1.2). Indeed, the causality requirement, as in the case of the Kramers-Kronig relation,[5] should generally manifest itself in a relation between real and imaginary components of matrix elements, or between quantities like $\langle m|\{A(x),A(x')\}|n\rangle$ and $\langle m|[A(x),A(x')]|n\rangle\epsilon(x-x')$ which compose the matrix element $\langle m|P(A(x),A(x'))|n\rangle$. For the Green's function Δ_F' the relation followed automatically from the two conditions mentioned above, but in the case of M it had to be recognized as such.

It is a very interesting question to ask whether one can analyze, along a similar line, the structure of a more

* This work was supported by a grant from the U. S. Atomic Energy Commission.
† On leave from Osaka City University, Osaka, Japan.
[1] G. Källén, Helv. Phys. Acta **25**, 417 (1952); Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 12 (1953).
[2] H. Lehmann, Nuovo cimento **11**, 342 (1954).
[3] M. Gell-Mann and F. Low, Phys. Rev. **95**, 1300 (1954).
[4] Y. Nambu, Phys. Rev. **98**, 803 (1955). This will be referred to as NI.
[5] R. de L. Kronig, J. Opt. Soc. Am. **12**, 547 (1926), H. A. Kramers, Atti congr. intern. fisici, Como **2**, 545 (1927).

general variety of quantities, namely matrix elements of time-ordered products of more than two Heisenberg operators. These quantities play an important role in a covariant description of quantum states in field theory, and the renormalization problem can best be treated in this way.

In fact, it has been pointed out in NI that the renormalization not only secures finite results but also satisfies causality requirements, and that conversely by demanding causality in a certain way one could automatically remove divergences.[6] The point is that in the lowest approximation, the imaginary part of a self-energy or a scattering matrix (the damping term) turns out to be finite although the real part without renormalization may be divergent. The causality relation between the two parts will force the latter to be finite and well determined.

In pursuing this end, one finds that the method followed in NI is not quite sufficient for our purposes, and more physical insight is needed. It seems thus worth while, as an orientation to the solution of the problem, to point out here the following observation. Let us consider the relation between the real and imaginary part of a matrix element using the fundamental formula of perturbation theory. Suppose we want to calculate the matrix element $\langle b|S|a\rangle$ of a certain scattering process. This will be given by

$$\langle b|S|a\rangle=\langle b|H|a\rangle+\sum_n \frac{\langle b|H|n\rangle\langle n|H|a\rangle}{E_a-E_n+i\epsilon}+\cdots, \quad (1.4)$$

where H is the interaction Hamiltonian, and the E_n are the free energy eigenvalues. The small imaginary quantity $i\epsilon$ indicates how to handle the poles of S in accordance with the boundary condition. If the particles concerned can be described by real fields (no spin, no charge), we may take the individual elements $\langle n|H|m\rangle$ to be real (for the interactions allowed in field theory). Thus the imaginary part of S will appear if and only if there occur intermediate states with the same energy as the initial state, or in other words, only if competing real processes, which may be the same as the initial or final state, are possible in the intermediate stage.[7]

The above formula does not show immediately the relation between the real and imaginary components since $\langle n|H|m\rangle$ may depend on energy and momentum in general; nor is it of a relativistically covariant form which one would desire. Nevertheless, it is interesting to see that these ideas, which have a definite physical meaning, can be carried over to a relativistic formulation, partly thanks to the "simplicity" of quantum field theory.

In the next section we shall first give the definition of the quantities which we are going to study, and the equations of motion satisfied by these quantities. In Sec. 3 we write down explicit formulas expressing the Green's functions of the first few orders in terms of certain real functions which may be called the spectral functions. Relativity, causality, and some other properties of field theory will be exploited to determine the general character of the spectral functions without going into the details of the interaction. In Sec. 4, we then translate the equations of motion holding for the Green's function into those for the spectral functions. Divergences will arise during this procedure, and it will be shown how the causality consideration again enables one to get the correct manifestly renormalized equations of motion which involve no formal infinities and could in principle be solved without fear of encountering divergences of the usual kind. Formulas giving the relation between the various renormalization constants and the spectral function will be found in the Appendix.

2. GREEN'S FUNCTIONS AND RELATED QUANTITIES

In the present paper, we treat the symmetrical pseudoscalar field coupled to the Dirac nucleon field, although this does not mean any restriction on the theory itself. The Lagrangian density of the meson-nucleon system is

$$\begin{aligned} L=&-\bar\psi\gamma_\mu\partial_\mu\psi-\kappa\bar\psi\psi\\ &-\tfrac{1}{2}(\partial_\mu\varphi_i\partial_\mu\varphi_i+\mu^2\varphi_i\varphi_i)\\ &-gi\bar\psi\gamma_5\tau_i\psi\varphi_i.\end{aligned} \quad (2.1)$$

We have adopted natural units $\hbar=c=1$, and $x_\mu \equiv(x, y, z, it=ix_0)$. ψ stands for the Dirac nucleon field with two isotopic spinor components. τ_i are the usual three vector components of the isotopic spin operator, and φ_i the corresponding ones of the meson field.

The Hamiltonian density of the system easily follows from the Lagrangian (1) by introducing the canonical variables and the commutation relations in the usual way:

$$\begin{aligned} H=&\sum_{k=1}^{3}\bar\psi\gamma_k\partial_k\psi+\kappa\bar\psi\psi\\ &+\tfrac{1}{2}(\pi_i\pi_i+\sum_{k=1}^{3}\partial_k\varphi_i\partial_k\varphi_i+\mu^2\varphi_i\varphi_i)+g_i\bar\psi\gamma_5\tau_i\psi\varphi_i,\end{aligned} \quad (2.2)$$

$$\{\psi_{\alpha s}(\mathbf{x}), \bar\psi_{\beta t}(\mathbf{y}')\}=(\gamma_4)_{\alpha\beta}\delta_{st}\delta(\mathbf{x}-\mathbf{y}'),$$

$$[\varphi_i(\mathbf{x}), \pi_j(\mathbf{y}')]=[\varphi_i(\mathbf{x}), \partial\varphi_j(\mathbf{y}')/\partial x_0']=i\delta_{ij}\delta(\mathbf{x}-\mathbf{y}'),$$

where α, β refer to the spinor components and s, t to the isotopic spin components. The Heisenberg equations

[6] It is true that a convergent theory does not always satisfy causality as in the case of the so-called damping theory, which gives rise to singularities of matrix elements not allowed in a causal theory. But the previous paper suggests the possiblity of finding a convergent and causal theory which will coincide with the orthodox renormalization theory.

[7] Rigorously speaking, there will be, of course, energy shifts of the particles due to interaction (self-energy) so that we have to interpret E_n as the energy of the observed free particles.

of motion for the field operators are

$$\begin{aligned} &\gamma_\mu\partial_\mu\psi+\kappa\psi+gi\gamma_5\tau_i\varphi_i\psi=0,\\ &-\partial_\mu\bar{\psi}\gamma_\mu+\kappa\bar{\psi}+gi\bar{\psi}\gamma_5\tau_i\varphi_i=0,\\ &(\Box-\mu^2)\varphi_i=gi\bar{\psi}\gamma_5\tau_i\psi. \end{aligned} \tag{2.3}$$

The nature of the Heisenberg operators is the main subject of our concern. Let us introduce the following quantities which give us vital information about these operators.[8]

$$\begin{aligned} T(x\alpha s,y\beta t)&\equiv\langle 0|T(\psi_{\alpha s}(x),\bar{\psi}_{\beta t}(y))|0\rangle\\ &=\epsilon(xy)\langle 0|P(\psi_{\alpha s}(x),\bar{\psi}_{\beta t}(y))|0\rangle,\\ T(zi,z'j)&\equiv\langle 0|T(\varphi_i(z),\varphi_j(z'))|0\rangle\\ &=\langle 0|P(\varphi_i(z),\varphi_j(z'))|0\rangle,\\ T(xyz)&\equiv\langle 0|T(\psi(x),\bar{\psi}(y),\varphi_i(z))|0\rangle\\ &=\epsilon(xy)\langle 0|P(\psi(x),\bar{\psi}(y),\varphi_i(z))|0\rangle,\\ T(xyzz')&\equiv\langle 0|T(\psi(x),\bar{\psi}(y),\varphi_i(z),\varphi_j(z'))|0\rangle\\ &=\epsilon(xy)\langle 0|P(\psi(x),\bar{\psi}(y),\varphi_i(z),\varphi_j(z'))|0\rangle,\\ T(x_1x_2y_1y_2)&\equiv\langle 0|T(\psi(x_1),\psi(x_2),\bar{\psi}(y_1),\bar{\psi}(y_2))|0\rangle\\ &=\epsilon(x_1x_2y_1y_2)\langle 0|P(\psi(x_1),\psi(x_2),\bar{\psi}(y_1),\bar{\psi}(y_2))|0\rangle. \end{aligned} \tag{2.4}$$

Here $|0\rangle$ is the vacuum or the lowest energy state of the total interacting meson nucleon field; P means the time ordering of the operators, and $\epsilon(x,y)$, etc., is the sign function which takes the values ± 1 according as the permutation $(x_0,y_0)\to P(x_0,y_0)$ is even or odd; the spin and isotopic spin indices of the operators may be suppressed when not necessary. Alternatively we may write

$$\begin{aligned} T(xy)&=\theta(x-y)\langle 0|\psi(x)\bar{\psi}(y)|0\rangle-\theta(y-x)\langle 0|\bar{\psi}(y)\psi(x)|0\rangle\\ &=\theta(x-y)\textstyle\sum_n\langle 0|\psi(x)|n\rangle\langle n|\bar{\psi}(y)|0\rangle\\ &\quad-\theta(y-x)\textstyle\sum_n\langle 0|\bar{\psi}(y)|n\rangle\langle n|\psi(x)|0\rangle,\\ T(zz')&=\theta(z-z')\langle 0|\varphi(z)\varphi(z')|0\rangle+\theta(z'-z)\langle 0|\varphi(z')\varphi(z)|0\rangle\\ &=\theta(z-z')\textstyle\sum_n\langle 0|\varphi(z)|n\rangle\langle n|\varphi(z')|0\rangle\\ &\quad+\theta(z'-z)\textstyle\sum_n\langle 0|\varphi(z')|n\rangle\langle n|\varphi(z)|0\rangle,\\ T(xyz)&=\theta(x-y)\theta(y-z)\langle 0|\psi(x)\bar{\psi}(y)\varphi(z)|0\rangle\\ &\quad-\theta(y-x)\theta(x-z)\langle 0|\bar{\psi}(y)\psi(x)\varphi(z)|0\rangle\\ &\quad+\theta(x-z)\theta(z-y)\langle 0|\psi(x)\varphi(z)\bar{\psi}(y)|0\rangle+\cdots\\ &=\theta(x-y)\theta(y-z)\textstyle\sum_{n,m}\langle 0|\psi(x)|n\rangle\\ &\quad\times\langle n|\bar{\psi}(y)|m\rangle\langle m|\varphi(z)|0\rangle-\cdots,\\ \theta(x-y)&=\frac{1+\epsilon(x-y)}{2}=\begin{cases}1 & x_0-y_0>0\\ 0 & x_0-y_0<0,\end{cases} \end{aligned} \tag{2.5}$$

where n, m run through a complete set of states of the meson-nucleon field.

The T's give, so to speak, a measure of the correlation of field quantities at different space-time points. In the absence of interaction, $T(x,y)$ and $T(z,z')$ are the Feynman propagation functions $S_F(x,y)$ and $\Delta_F(z,z')$ for the nucleon and meson respectively; in the presence of interaction, they are the quantities $S_F'(x,y)$ and $\Delta_F'(z,z')$ which were originally defined by Dyson[9] in the perturbation theory. The T's of higher orders describe more complicated correlations of more than two field quantities, and are closely related to what Schwinger[8] has generally termed Green's functions.

A higher order T contains a part which derives from lower order correlations, so that it is convenient to introduce further the irreducible correlations ρ[10]:

$$\begin{aligned} &\rho(xy)=T(xy),\quad \rho(zz')=T(zz'),\\ &\rho(xyz)=T(xyz),\\ &\rho(xyzz')=T(xyzz')-T(xy)T(zz'),\\ &\rho(x_1x_2y_1y_2)=T(x_1x_2y_1y_2)+T(x_1y_1)T(x_2y_2)\\ &\qquad\qquad-T(x_1y_2)T(x_2y_1),\ \text{etc.} \end{aligned} \tag{2.6}$$

We shall refer to all the quantities introduced in this section simply as Green's functions when no confusion arises.

If we make use of the interaction representation and Feynman graphs, the meaning of the ρ's is clear. $\rho(x_1\cdots x_ly_1\cdots y_mz_1\cdots z_n)$ corresponds to the totality of graphs in which there are $l+m+n$ external lines starting at x_1, x_2, $\cdots z_n$, and all these lines are connected with each other through the network of internal lines.

The Green's functions of different order are related by equations of motion which are obtained by differentiating them and making use of Eq. (2.3):

$$\begin{aligned} &(\gamma\partial+\kappa)_x\rho(xy)\\ &\quad=-i\delta(xy)-ig\gamma_5\tau_j\int\rho(xyz'j)\delta(x-z')(dz')^4,\\ &(-\gamma^T\partial+\kappa)_y\rho(xy)\\ &\quad=-i\delta(xy)-ig\int\rho(xyz'j)\gamma_5\tau_j\delta(y-z')(dz')^4,\\ &(\Box-\mu^2)_z\rho(zz')\\ &\quad=i\delta(zz')-ig\int\operatorname{tr}(\gamma_5\tau_i\rho(xyz'))\\ &\qquad\qquad\times\delta(x-z)\delta(y-z)(dx)^4(dy)^4,\\ &(\gamma\partial+\kappa)_x\rho(xyz)\\ &\quad=-ig\int\gamma_5\tau_j[\rho(xy)\rho(zz'j)+\rho(xyzz'j)]\delta(x-z')(dz')^4,\\ &(\Box-\mu^2)_z\rho(xyz)\\ &\quad=ig\int\{-\rho(xy)\operatorname{tr}[\gamma_5\tau_i\rho(x'y')]+\rho(xy')\gamma_5\tau_i\rho(x'y)\\ &\qquad+\rho(xx'\alpha s,yy'\beta t)(\gamma_5\tau_i)_{\beta t,\alpha s}\}\\ &\qquad\times\delta(x'-z)\delta(y'-z)(dx')^4(dy')^4. \end{aligned} \tag{2.7}$$

[8] G. C. Wick, Phys. Rev. **80**, 268 (1950); J. Schwinger, Proc. Natl. Acad. Sci. U. S. **27**, 452 and 455 (1951). See also E. Freese, Z. Naturforsch. **8a**, 776 (1953); P. T. Matthews and A. Salam, Proc. Roy. Soc. (London) **A221**, 128 (1953); K. Nishijima, Progr. Theoret. Phys. **10**, 549 (1953); **12**, 279 (1954). The notations do not necessarily agree with those used by these authors.

[9] F. J. Dyson, Phys. Rev. **75**, 486 and 1736 (1949).

[10] E. Freese, Nuovo cimento **2**, 50 (1955).

Under the assumption of adiabatic switching at $t=\pm\infty$, and the boundary condition for the vacuum, they can be converted into integral equations:

$$\rho(xy)=S_F(xy)+g\int S_F(xx')\gamma_5\tau_j\rho(x'yz'j)\times\delta(x'-z')(dx')^4(dz')^4,$$

$$\rho(zi,z'j)=\delta_{ij}\Delta_F(zz')-g\int\Delta_F(zz'')\ \mathrm{tr}[\gamma_5\tau_i\rho(xyz'j)]\times\delta(x-z'')\delta(y-z'')(dx)^4(dy)^4(dz'')^4,$$

$$\begin{aligned}\rho(xyz)&=g\int S_F(xx')\gamma_5\tau_j[\rho(x'y)\rho(zz'j)+\rho(x'yzz'j)]\delta(x'-z')(dx')^4(dz')^4\\&=g\int\Delta_F(zz')\{-\rho(xy)\ \mathrm{tr}[\gamma_5\tau_i\rho(x'y')]+\rho(xy')\gamma_5\tau_i\rho(x'y)+\rho(xx'\alpha s,yy'\beta t)(\gamma_5\tau_i)_{\beta t,\alpha s}\}\\&\quad\times\delta(x'-z')\delta(y'-z')(dx')^4(dy')^4(dz')^4.\end{aligned}\tag{2.8}$$

It is to be noted that the ρ functions of higher order than the first do not have inhomogeneous terms in their equations of motion.

In the following sections we shall analyze the nature of the T's and ρ's up to four external lines. These are of particular interest since it is they that cause infinities in the actual calculation, and also because $\rho(xyzz')$, for example, describes the scattering of a nucleon and a meson which has direct physical significance.[11]

The problem of renormalization, which has not been discussed so far, is usually dealt with by starting from renormalized fields and a renormalized Lagrangian:

$$\begin{aligned}L=&-Z_2[\bar\psi\gamma_\mu\partial_\mu\psi+\kappa\bar\psi\psi]+Z_2\delta_k\bar\psi\psi\\&-\tfrac{1}{2}Z_3[\partial_\mu\varphi_i\partial_\mu\varphi_i+\mu^2\varphi_i\varphi_i]-\tfrac{1}{2}Z_3\delta\mu^2\varphi_i\varphi_i\\&-Z_1gi\bar\psi\gamma_5\tau_i\psi\varphi_i-Z_4\tfrac{1}{4}\lambda\varphi_i\varphi_i\varphi_k\varphi_k.\end{aligned}\tag{2.9}$$

We will not describe this procedure in detail here since we approach the problem in a different fashion. In fact, we shall be treating physically significant renormalized quantities only, and we need not be meticulous about the distinction between renormalized and unrenormalized quantities since the infinite constants which characterize their difference will not occur explicitly in our equations. We shall regard all the Green's functions and operators as renormalized ones, but treat them as if they obeyed the equations of motion for canonical observables, with, however, observed masses and couplings.

[11] The scattering matrix is obtained essentially by the adiabatic limiting process

$$\lim_{\substack{x_0,\,z_0\to\infty\\ y_0,\,z_0'\to-\infty}}\rho(xyzz').$$

See reference 20.

3. REPRESENTATION OF THE GREEN'S FUNCTIONS BY MEANS OF SPECTRAL FUNCTIONS

In accordance with the ideas outlined in the introduction, we start out with the following fundamental assumptions.

(I) Lorentz invariance and other invariances which are inherent in the theory. As was discussed in NI, it is important to note that the Green's functions, which are T-products of operators, should not depend on the choice of the time axis used for the time ordering. This is guaranteed by causality (to be mentioned below) which insures the independence of field quantities (commutability or anticommutability) for spacelike points.

(II) Completeness of the positive energy states. In addition, we will also assume naturally that the one-nucleon or meson state is stable. But we do not necessarily exclude the existence of bound states.

(III) The adiabatic switching of interaction. This has always been necessary if one wants at all to talk about interaction between particles. It must be recognized, however, that its exact meaning is not a uniquely defined one, and one will get different connections between the perturbed and unperturbed field quantities according to different definitions of the adiabatic process. We want to use it in the simple physical sense that the field variables behave as *essentially* free when they are far apart from one another, and these essentially free fields should serve to *define* the physical constants (the "observed" masses and coupling constants). This should be enough for our purpose of finding relations between observable quantities only. On the other hand, the connection between unrenormalized and renormalized quantities is in principle to be obtained by the unrealistic process in which the coupling constant itself is turned on and off instead of separating the particles from each other. It is also to be noted that the bound states do not cause any difficulty since we are not asked to treat them by perturbation once the physical constants are fixed.

(IV) Causality. This will be the main tool in deriving our results. But we use it here in a disguised way which is not immediately recognizable as such. We postulate namely that, apart from spin and isotopic spin matrices, the imaginary part of the Green's functions arises through vanishing denominators which occur if energy-conserving processes are possible in the intermediate stages appearing in Eq. (2.5). The conditions I and II then say that the energy-momentum of such an intermediate state must form a timelike four-vector with positive time component. This suggests that the relativistically invariant form of the energy denominator and the nature of its singularity are determined by the time-ordered character of the Green's functions. Causality is reflected in all these points in an implicit fashion: The timelike energy-momentum vector will guarantee the noninterference of measurements outside the light cone; and the positive energy condition and the time-ordering will fix the analytic behavior of the function with regard to each energy denominator.

The implication of these arguments will become clear as we go over to the actual application below.

A. Modified Propagation Functions $\rho(xy)$ and $\rho(zz')$

We begin with the simplest cases of $\rho(xy)$ and $\rho(zz')$ as an illustration though the result is already known.[1] We observe that, in the first place, the invariance under the inhomogeneous Lorentz transformation, charge conjugation, and isotopic transformation requires the Fourier transforms

$$\rho(pq)=\frac{1}{(2\pi)^8}\int e^{-ipx-iqy}\rho(xy)(dx)^4(dy)^4,$$
$$\rho(kk')=\frac{1}{(2\pi)^8}\int e^{-ikz-ik'z'}\rho(zz')(dz)^4(dz')^4, \tag{3.1}$$

to be of the form

$$\rho(p\alpha s,q\beta t)=i\delta(p+q)\delta_{st}[ip\gamma\rho_1(p^2)+\rho_0(p^2)]_{\alpha\beta},$$
$$\rho(ki,k'j)=-i\delta(k+k')\delta_{ij}\rho(k^2), \tag{3.2}$$

where the spin (α,β) and isotopic spin (s,t,i,j) indices are explicitly written down. The graphs corresponding to these quantities are of course two external nucleon or meson lines of energy-momenta p, $q=-p$ or k, $k'=-k$ which we measure for convenience always in the "out" direction.

Following the assumption (IV) let us next ask the question: under what conditions will they give rise to real processes?

(1) If the incident particle is a real particle, it can go through the same real state without interaction. In this case

$$p^2+\kappa^2=0,\quad k^2+\mu^2=0,$$

where κ and μ are the masses of the real nucleon and meson respectively. (2) If the energy $|p_0|$ or $|q_0|$ of the incident virtual nucleon is large enough, it can split into a real nucleon and at least one real meson[12]; similarly if $|k_0|$ or $|k_0'|$ is large enough, the incident pseudoscalar meson can split into at least three mesons. The condition for this to happen can be written in an invariant way as

$$p^2+(\kappa+\mu)^2\leq 0,$$
$$k^2+(3\mu)^2\leq 0.$$

We thus conclude, according to (IV), that the ρ's must have terms of the form

$$1/(p^2+m^2),\quad m=\kappa\quad\text{or}\quad m\geq\kappa+\mu,$$
$$1/(k^2+n^2),\quad n=\mu\quad\text{or}\quad n\geq 3\mu, \tag{3.3}$$

[12] We excluded here the existence of a bound meson-nucleon system. In case such a bound system exists, $\nu_\lambda(u)$ will have an additional point spectrum between κ^2 and $(\kappa+\mu)^2$.

which become singular in case a real process (1) or (2) is possible. To determine the character of the singularity precisely, we invoke the assumption (II) together with the decomposition (2.5). Since the energy of the intermediate states is positive definite, the Fourier component of a function $\rho(x,y)$ must behave like $\exp(-i|k_0(x_0-y_0)|)$. So we conclude, just as in the case of S_F and Δ_F, that Eq. (3.3) should actually contain $1/(p^2+m^2-i\epsilon)$ and $1/(k^2+n^2-i\epsilon)$. Thus we can write

$$\rho_\lambda(p^2)=\int_0^\infty\frac{\nu_\lambda(u)}{p^2+\bar{u}}du,\quad\lambda=0,1$$
$$\rho(k^2)=\int_0^\infty\frac{\nu(w)}{k^2+\bar{w}}dw, \tag{3.4}$$
$$\bar{u}\equiv u-i\epsilon,\quad\bar{w}\equiv w-i\epsilon,$$

where the real functions ν have the property[12]

$$\nu_\lambda(u)=\nu_\lambda^0\delta(u-\kappa^2)+f\lambda(u)\theta(u-(\kappa+\mu)^2),$$
$$\nu(w)=\nu^0\delta(w-\kappa^2)+g(w)\theta(w-(3\mu)^2). \tag{3.5}$$

We also note that

$$\nu_\lambda(u)=\frac{1}{\pi}\,\text{Im}\rho_\lambda(-u),$$
$$\nu(w)=\frac{1}{\pi}\,\text{Im}\rho(-w). \tag{3.6}$$

On the other hand, from the decomposition (2.5) and (II) and (IV) it is again easy to see that

$$\delta_{st}(ip\gamma_{\alpha\beta}\,\text{Im}\rho_1(p^2)+\delta_{\alpha\beta}\text{Im}\rho_0(p^2))$$
$$=\sum\langle 0|\psi_{\alpha s}|p\rangle\langle p|\bar{\psi}_{\beta t}|0\rangle\theta(p)$$
$$-\sum\langle 0|\bar{\psi}_{\beta t}|-p\rangle\langle -p|\psi_{\alpha s}|0\rangle\theta(-p), \tag{3.7}$$
$$\delta_{ij}\,\text{Im}\rho(k^2)=\sum\langle 0|\varphi_i|k\rangle\langle k|\varphi_j|0\rangle\theta(k)$$
$$+\sum\langle 0|\varphi_j|-k\rangle\langle -k|\varphi_i|0\rangle\theta(-k),$$

where the summation extends over all the spin states with the momentum $\pm p$ or $\pm k$. The second equation indicates[2]

$$\nu(w)\geq 0. \tag{3.8}$$

For ν_1 and ν_0 we observe that $\bar{\psi}=\psi^*\gamma_4$, and

$$(1/\pi)\{\sum|\langle 0|\psi_{\alpha s}|p\rangle|^2\theta(p)-\sum|\langle -p|\psi_{\alpha s}|0\rangle|^2\theta(-p)\}$$
$$=p_0\nu_1(p^2)-(\gamma_4)_{\alpha\alpha}\nu_0(p^2),$$

choosing the rest system for p. Thus both sides must have the same sign as p_0:

$$\epsilon(p)[p_0\nu_1(p^2)-(\gamma_4)_{\alpha\alpha}\nu_0(p^2)]\geq 0,$$

which means[2]

$$\nu_1(u)\geq 0,\quad |p_0|\nu_1(u)=u^{\frac{1}{2}}\nu_1(u)\geq|\nu_0(u)|, \tag{3.9}$$

since the expectation value of γ_4 lies always between -1 and $+1$.

More information on ν is to be obtained by considering the boundary conditions. If we assume, according to (III), that the interaction can be switched off adiabatically at $t=\pm\infty$, then $\rho(xy)$ and $\rho(zz')$ should go in this limit over to the propagation functions for free incoming and outgoing fields:

$$\lim_{\substack{x_0\to\infty\\ y_0\to-\infty}}{}^{*} \langle 0|\psi(x)\bar{\psi}(y)|0\rangle = \lim_{\substack{x_0\to\infty\\ y_0\to-\infty}} \langle 0|\psi_{\text{out}}{}^0(x)\bar{\psi}_{\text{in}}{}^0(y)|0\rangle, \text{ etc.}^{13} \quad (3.10)$$

On the other hand, the initial condition on the behavior of the ρ's for equal times may be checked by the relation

$$\begin{aligned}\epsilon(xy)\rho(xy)|_{x_0=y_0}&=\langle 0|\{\psi(x),\bar{\psi}(y)\}|0\rangle|_{x_0=y_0},\\ \epsilon(zz')\frac{\partial}{\partial z_0}\rho(zz')\bigg|_{z_0=z_0'}&=\left\langle 0\left|\left[\frac{\partial\varphi(z)}{\partial z_0},\ \varphi(z')\right]\right|0\right\rangle\bigg|_{z_0=z_0'},\end{aligned} \quad (3.11)$$

since the right-hand side is the commutation relation for the field variables.

Although we have thus two conditions at hand, we should primarily use the condition at infinity since our derivation of the ν's was based on the over-all behavior of incoming or outgoing waves. Consequently we assume that the asymptotic fields ψ^0, $\bar{\psi}^0$ and φ^0 are normalized in the usual way so that

$$\begin{aligned}\epsilon(xy)\langle 0|T(\psi^0(x),\bar{\psi}^0(y))|0\rangle&=S_F{}^{(\kappa)}(xy),\\ \langle 0|T(\varphi_i(z),\varphi_j(z'))|0\rangle&=\delta_{ij}\Delta_F{}^{(\mu)}(zz'),\end{aligned} \quad (3.12)$$

where the right-hand sides are the ordinary Feynman functions with mass κ and μ, respectively. This leads to

$$\nu_1{}^0=1,\quad \nu_0{}^0=-\kappa,\quad \nu^0=1. \quad (3.13)$$

The field operators being thus normalized, Eq. (3.11) is no more the commutation relation for canonical variables since the adiabatic process is in general not a unitary transformation. Indeed we must conclude from Eqs. (3.11) and (2.9) that

$$\begin{aligned}\langle 0|\{\psi_{\alpha s}(x),\bar{\psi}_{\beta t}(y)\}|0\rangle|_{x_0=y_0}&=Z_2{}^{-1}\delta_{st}(\gamma_4)_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y}),\\ \left\langle 0\left|\left[\frac{\partial\varphi_i(z)}{\partial z_0},\ \varphi_j(z')\right]\right|0\right\rangle\bigg|_{x_0=y_0}&=Z_3{}^{-1}\delta_{ij}\delta(\mathbf{z}-\mathbf{z}'),\end{aligned} \quad (3.14)$$

[13] lim* means the switching off of the interaction at $+\infty$; in other words, $\langle n|\bar{\psi}|0\rangle$, etc., should vanish for all n except for the one-particle states. We do not mean, however, that the particles turn into "bare" particles. The incoming and outgoing waves are equal since they are renormalized waves without the self-energy effect.

where

$$\begin{aligned}Z_2{}^{-1}&=\int\nu_1(u)du=1+\int_{(\kappa+\mu)^2}^{\infty}f_1(u)du\geq 1,\\ Z_3{}^{-1}&=\int\nu(w)dw=1+\int_{2\kappa}^{\infty}g(w)dw\geq 1,\end{aligned} \quad (3.15)$$

which are the renormalization constants introduced by Dyson in relation to the unrenormalized fields ψ_u, $\bar{\psi}_u$, and φ_u obeying the proper commutation relations:

$$\psi_u=Z_2{}^{\frac{1}{2}}\psi,\quad \bar{\psi}_u=Z_2{}^{\frac{1}{2}}\bar{\psi},\quad \varphi_u=Z_3{}^{\frac{1}{2}}\varphi.$$

It is clear from the above result that the boundary condition at infinity is more convenient than the initial condition since we can deal with the renormalized quantities directly and thus the functions ν should be finite. They should also satisfy

$$\int\frac{\nu_i(u)}{u}du<\infty,\qquad \int\frac{\nu(w)}{w}dw<\infty, \quad (3.16)$$

without which Eq. (3.4) would be meaningless.

Mathematically speaking, Eq. (3.4) expresses the fact that $\rho(k^2)\equiv\rho(-\zeta)$, etc., have no poles in the upper half of the complex plane as functions of ζ:

$$\rho(-\zeta)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(-\zeta')}{\zeta'-\zeta-i\epsilon}d\zeta',\quad \zeta\text{ real}, \quad (3.17)$$

since $\text{Im}\rho(-\zeta)=\pi\nu(\zeta)=0$ for real $\zeta<0$ (or ζ_{min}). In case the condition (3.16) is not satisfied, we can still take, for example, a function

$$f(-\zeta)\equiv\rho(-\zeta)/(\zeta-a)^n,\quad n=\text{positive integer},\quad a<\zeta_{\text{min}},$$

such that

$$\int_{-\infty}^{\infty}\frac{\nu(\zeta)}{(\zeta-a)^n}\frac{d\zeta}{\zeta}<\infty, \quad (3.18)$$

and express it as

$$f(-\zeta)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{\zeta'-\zeta-i\epsilon}f(-\zeta')d\zeta',$$

or

$$\rho(k^2)=\int\frac{\nu(w)}{k^2+w}\left(\frac{a+k^2}{a-w}\right)^n dw+\frac{(a+k^2)^n}{(n-1)!}\left(-\frac{\partial}{\partial a}\right)^{n-1}\left[\frac{\rho(a)}{a+k^2}\right]. \quad (3.19)$$

Thus $\rho(k^2)$ is not completely determined by the imaginary part alone; the second term on the right is a polynomial in k^2, and to omit (or separate) this term physically means some sort of renormalization of ρ. In our formalism the boundary condition for ν will serve to fix it. The "renormalized" ρ has a factor $(k^2+a)^n$, and behaves as $O(k^{2n})$ for large k^2. In other words, $\rho(zz')$ has derivatives (of order $\leq 2n$) of the delta function at the origin.

The Green's functions $\rho(xy)$ and $\rho(zz')$ do not have singularities of the above kind. But the quantity

$$M(zz')\equiv\langle 0|T(ig\bar{\psi}\gamma_5\tau_i\psi(z),ig\bar{\psi}\gamma_5\tau_j\psi(z'))|0\rangle, \quad (3.20)$$

which is related to $\rho(zz')$ by

$$(\square-\mu^2)^2\rho(zz')=i(\square-\mu^2)\delta(zz')+M(zz'), \quad (3.21)$$

has the representation

$$M(k^2)=-\delta_{ij}i(k^2+\mu^2)^2\int\frac{1}{k^2+\bar{w}}\frac{\nu^*(w)}{(w-\mu^2)^2}dw,$$
$$\nu^*(w)=(w-\mu^2)^2\nu(w)=\frac{1}{\pi}\operatorname{Im}M(-w). \quad (3.22)$$

This corresponds to the choice $a=\mu^2$, $n=2$ in Eq. (3.19). We shall also meet with this situation later.

B. Three-Body Green's Functions

Let us take the three-body Green's function

$$\rho(xyz)=\langle 0|T(\psi(x),\bar{\psi}(y),\varphi_i(z))|0\rangle, \quad (3.23)$$

and define its Fourier transform

$$\rho(pqz)=\frac{1}{(2\pi)^{12}}\times\int e^{-ipx-iqy-ikz}\rho(xyz)(dx)^4(dy)^4(dz)^4. \quad (3.24)$$

Because of the conservation law

$$p+q+k=0, \quad (3.25)$$

$\rho(pqk)$ is a function of three scalars p^2, q^2, k^2 in addition to the spin functions γ_5, $(p\gamma)\gamma_5$, $\gamma_5(q\gamma)$, and τ_i. γ_5 reflects here the pseudoscalar nature of the field φ_i. We may thus generally write

$$\rho(p\alpha s,q\beta t,k_i)=i\delta(p+q+k)\sum_{\lambda,\lambda'=0,1}[(ip\gamma)^\lambda\gamma_5(-iq\gamma)^{\lambda'}]_{\alpha\beta}\times(\tau_i)_{st}\rho_{\lambda\lambda'}(p^2,q^2,k^2). \quad (3.26)$$

Under the extended charge conjugation $\psi\to C\bar{\psi}$, $\bar{\psi}\to C^{-1}\psi$, $\varphi_i\to-\varphi_i$, $\rho(xyz)$ goes into $-(C^{-1}\rho(yxz)C)^T$; this means for $\rho(pqk)$[14] the invariance under $p\leftrightarrow q$, $\gamma\to-\gamma$ so that $\rho_{\lambda\lambda'}$ must be symmetric:

$$\rho_{\lambda\lambda'}(p^2,q^2,k^2)=\rho_{\lambda'\lambda}(q^2,p^2,k^2). \quad (3.27)$$

[14] The spin matrices are transformed according to $O\to C^{-1}OC$, with $C^{-1}\gamma_\mu C=-\gamma_\mu^T$, $C^{-1}\tau_i C=-\tau_i^T$, etc.

Next we ask, as before, for the condition of producing a real process. In the first place, an incoming external line p, q, or k by itself can give rise to such a process just as in the case A. This means that $\rho_{\lambda\lambda'}$ must have factors of the form

$$\frac{1}{p^2+m_1^2-i\epsilon},\quad\frac{1}{q^2+m_2^2-i\epsilon},\quad\frac{1}{k^2+m_3^2-i\epsilon},$$

where

$$m_1,\,m_2=\kappa\quad\text{or}\quad\geq\kappa+\mu,$$
$$m_3=\mu\quad\text{or}\quad\geq 3\mu \quad (3.28)$$

(again excluding bound states). Next, it is possible that any two of the incoming particles together produce a real process by interaction (scattering). This would mean that there should be factors of the form

$$\frac{1}{(p+q)^2+m_4^2-i\epsilon},\quad\frac{1}{(p+k)^2+m_5^2-i\epsilon},\quad\frac{1}{(q+k)^2+m_6^2-i\epsilon}.$$

But the conservation law (3.25) just reduces these factors to those in Eq. (3.28) so that we get no new ones.

In order to proceed further, it is necessary to consider the equations of motion for $\rho(xyz)$: By differentiating it with respect to x, y, and z successively, we obtain

$$\begin{aligned}(\gamma\partial+\kappa)_x(-\gamma^T\partial+\kappa)_y(\square-\mu^2)_z\rho(xyz)\\ =-ig\gamma_5\tau_i\delta(x-y)\delta(x-z)\\ +ig^2\gamma_5\tau_j\langle 0|T(\psi\varphi_j(x),\bar{\psi}(y))|0\rangle\gamma_5\tau_i\delta(y-z)\\ +ig^2\gamma_5\tau_i\langle 0|T(\psi(x),\bar{\psi}\varphi_k(y))|0\rangle\gamma_5\tau_k\delta(x-z)\\ -ig^2\gamma_5\tau_j\langle 0|T(\varphi_j(x),\bar{\psi}\gamma_5\tau_i\psi(z))|0\rangle\delta(x-y)\\ +ig^3\gamma_5\tau_j\langle 0|T(\psi\varphi_j(x),\bar{\psi}\varphi_k(y),\bar{\psi}\gamma_5\tau_i\psi(z))|0\rangle\gamma_5\tau_k.\end{aligned} \quad (3.29)$$

Now we could apply the same argument as above to the T products appearing on the righthand side, and could conclude that they must contain the same factors as Eq. (3.28). Hence the original $\rho_{\lambda\lambda'}$ must necessarily contain the three factors simultaneously. In other words,

$$\rho_{\lambda\lambda'}(p^2q^2k^2)=\int\frac{\nu_{\lambda\lambda'}(uvw)}{(p^2+\bar{u})(q^2+\bar{v})(k^2+\bar{w})}dudvdw, \quad (3.30)$$

which is the desired spectral decomposition.

According to Eq. (3.28), $\nu_{\lambda\lambda'}$ has a point spectrum at the masses for real particles, and a continuous spectrum extending upward from the thresholds for various reactions. By Eq. (3.27), it satisfies the symmetry condition

$$\nu_{\lambda\lambda'}(uvw)=\nu_{\lambda'\lambda}(vuw). \quad (3.31)$$

To see the implication of the boundary condition at infinity, we again let the fields ψ, $\bar{\psi}$, and φ go separately to $+\infty$ or $-\infty$, and suppose that they turn into almost free fields there. In this case the only interaction which can be measured by these waves will be that of the type $ig\bar{\psi}\gamma_5\tau_i\psi\varphi_i$ where g *defines* the observed coupling constant. To be more precise, we mean by the adiabatic procedure that only the first term of the right-hand side of Eq. (3.29) remains (understanding that contributions of the same type from the rest have been incorporated into it). Thus

$$\lim{}^*\rho(pqk) = -igS_F(p)i\gamma_5\tau_iS_F(-q)\Delta_F(k), \quad (3.32)$$

from which it follows that

$$\nu_{\lambda\lambda'}(uvw) = g\nu_{\lambda\lambda'}{}^0\delta(u-\kappa^2)\delta(v-\kappa^2)\delta(w-\mu^2)+\cdots,$$

with

$$\nu_{\lambda\lambda'}{}^0 = \begin{pmatrix} \kappa^2 & -\kappa \\ -\kappa & 1 \end{pmatrix}. \quad (3.33)$$

To find the relation between renormalized and unrenormalized quantities a systematic study is needed since it is more complicated than in the previous case. Deferring the details to the Appendix, we quote here only the main result which gives the unrenormalized charge g_0:

$$\int \nu_{11}(uvw)dudvdw = gZ_1Z_2^{-2}Z_3^{-1} = g_0Z_2^{-1}Z_3^{-\frac{1}{2}}. \quad (3.34)$$

As is well known, there is in general an arbitrariness in defining the observed coupling constant, so that the above definition of g, Eq. (3.33), is not the only possibility, though it seems to be the most natural one in our scheme. A different quantity could be obtained, for example, in the following way. Suppose we take the equation of motion

$$(\Box-\mu^2)_2\rho(xyz) = \langle 0|T(\psi(x),\bar{\psi}(y),ig\bar{\psi}\gamma_5\tau_i\psi(z))|0\rangle, \quad (3.35)$$

and let $\psi(x)$ and $\bar{\psi}(y)$ recede to $+\infty$ and $-\infty$, respectively. We then get the relation

$$\langle p|ig\bar{\psi}\gamma_5\tau_i\psi|-q\rangle = \frac{k^2+\mu^2}{4\kappa^2}\int \frac{(-\kappa)^\lambda(-\kappa)^{\lambda'}}{k^2+\bar{w}}\nu_{\lambda\lambda'}{}^0(\kappa\kappa w) \times dw\langle p|i\gamma_5\tau_i|-q\rangle, \quad (3.36)$$

where $|p\rangle$ and $|-q\rangle$ represent one-nucleon states and $\nu_{\lambda\lambda'}{}^0(\kappa\kappa w)$ the magnitude of ν at the point spectrum $u=v=\kappa^2$. This is nothing but the matrix element for the "current" $ig\bar{\psi}\gamma_5\tau_i\psi$, and we may define the charge as the coefficient of $i\gamma_5\tau_i$ in the static limit $p=-q$, $k=0$:

$$g' \equiv \int \frac{(-\kappa)^{\lambda+\lambda'}}{4\kappa^2}\frac{\mu^2}{w}\nu_{\lambda\lambda'}{}^0(\kappa\kappa w)dw = g+\int_{(2\kappa)^2}^{\infty} \frac{(-\kappa)^{\lambda+\lambda'}}{4\kappa^2}\frac{\mu^2}{w}\nu_{\lambda\lambda'}{}^0(\kappa\kappa w)dw. \quad (3.37)$$

g' is thus directly related to an ideal measurement whereas g does not allow such an interpretation, since a real nucleon cannot absorb or emit a real meson. The customary definition of renormalization due to Dyson[8] gives g'.

C. Four-Body Green's Functions

There are three four-body functions which do not vanish identically. They correspond to meson-nucleon, nucleon-nucleon, and meson-meson scattering respectively.

As the first example, we take

$$\rho(xyzz') = T(xyzz') - T(xy)T(zz'), \quad (3.38)$$

and its Fourier transform

$$\rho(pqkk') = \frac{1}{(2\pi)^{16}}\int e^{-ipx-iqy-ikz-ik'z'}\rho(xyzz') \times (dx)^4(dy)^4(dz)^4(dz')^4. \quad (3.39)$$

In view of the conservation law[15]

$$p+q+k+k'=0, \quad (3.40)$$

we have six independent scalars of the momenta, for which we adopt

$$p^2,\ q^2,\ k^2,\ k'^2,\quad (p+q)^2=(k+k')^2,$$
$$(p+k)^2=(q+k')^2 \quad\text{or}\quad (p+k')^2=(q+k)^2,$$

as will be found convenient below. In addition, there are three scalars formed with the Dirac matrices, which may be chosen as

$$ip\gamma,\ -iq\gamma,\ i(p+k,\gamma) \quad\text{or}\quad i(p+k',\gamma),$$

and the two isotopic tensors δ_{ij} and $\tau_{ij}=[\tau_i,\tau_j]/2i$. ρ should be invariant under charge conjugation C: $p\leftrightarrow q$, $\gamma\rightarrow-\gamma$, $\tau_{ij}\rightarrow-\tau_{ij}$; obviously it is also invariant against the interchange C': $k\leftrightarrow k'$, $i\leftrightarrow j$, but we can easily see that $CC'=1$. Thus we may write the general form of ρ as

$$\rho(pqkk') = i\delta(p+q+k+k')(1+C') \times \sum_{\lambda=0,1}(ip\gamma)^{\lambda_1}(i\overline{p+k}\gamma)^{\lambda_2}(-iq\gamma)^{\lambda_3}(i\tau)^{\lambda_4} \times\rho_{\lambda_1\lambda_2\lambda_3\lambda_4}(p^2,q^2,k^2,k'^2,(p+q)^2,(p+k)^2). \quad (3.41)$$

We can then apply the argument of real processes to $\rho_{\lambda_1\cdots\lambda_4}$ and conclude that it must have factors

$$\frac{1}{p^2+\bar{u}},\quad \frac{1}{q^2+\bar{v}},\quad \frac{1}{k^2+\bar{w}},\quad \frac{1}{k'^2+\bar{w}'},$$
$$\frac{1}{(p+q)^2+\bar{s}_1},\quad \frac{1}{(p+k)^2+\bar{s}^{2\prime}} \quad\text{or}\quad \frac{1}{(p+k')^2+\bar{s}_2{}'}, \quad (3.42)$$
$$u,\ v=\kappa^2 \ \text{or}\ \geq(\kappa+\mu)^2,\quad w,\ w'=\mu^2\ \text{or}\ \geq(3\mu)^2,$$
$$s_1\geq(2\mu)^2,\quad S_2=\kappa^2\ \text{or}\ \geq(\mu+\kappa)^2.^{16}$$

---

[15] Since ρ represents the irreducible correlation, no subdivision like $p+q=0$, $k+k'=0$ can occur, which is the reason for taking ρ rather than T.

[16] The thresholds for w, w', and s_1 are characteristic of the pseudoscalar meson theory.

The indicated behavior of the denominators at a singularity is universal. To show this for $1/(p+k)^2+\bar{s}_2$, for example, we may choose x_0, $z_0>y_0$, z_0' or x_0, $z_0<y_0$, z_0', so that

$$T(xyzz')=\sum_n \langle 0|T(\psi(x)\varphi_i(z))|n\rangle\langle n|T(\bar{\psi}(y)\varphi_j(z')|0\rangle, \quad x_0,\, z_0>y_0,\, z_0'$$
$$=-\sum_n \langle 0|T(\bar{\psi}(y)\varphi_j(z'))|n\rangle\langle n|T(\psi(x)\varphi_i(z))|0\rangle, \quad x_0,\, z_0<y_0',\, z_0'$$

which gives a factor $\exp[-i|(p_0+k_0)(x_0+z_0-y_0-z_0')|]$ similar to the previous cases.

Since $\rho(xyzz')$ obeys an equation of motion

$$(\gamma\partial+\kappa)_x(-\gamma^T\partial+\kappa)_y(\Box-\mu^2)_z(\Box-\mu^2)_{z'}\rho = g^2\{\delta(x-z)\delta(y-z')\gamma_5\tau_i\rho(xy)\gamma_5\tau_j + (z\leftrightarrow z',\, i\leftrightarrow j)\}+\cdots, \tag{3.43}$$

the first four factors always occur simultaneously, but the last one is missing for the first term on the right of Eq. (3.43). At any rate we may write

$$\rho_{\lambda_1\cdots\lambda_4}=\int \frac{\nu_{\lambda_1\cdots\lambda_4}(uvww's_1s_2)dudvdwdw'ds_1ds_2}{(p^2+\bar{u})(q^2+\bar{v})(k^2+\bar{w})(k'^2+\bar{w}')((p+q)^2+\bar{s}_1)((p+k)^2+\bar{s}_2)}, \tag{3.44}$$

if we understand that $\nu_{\lambda_1\cdots\lambda_4}$ may have delta functions at infinity: $s_1\delta(s_1-\infty)=\delta(1-\infty/s_1)$ and/or $s_2\delta(s_2-\infty)=\delta(1-\infty/s_2)$.

On the other hand, the boundary condition for $\rho(xyzz')$ should enable one to define the direct four-body interaction of the form $f_1\bar{\psi}\psi\varphi_i^2+if_2\bar{\psi}\gamma_\mu\tau_{ij}\psi\varphi_i\partial_\mu\varphi_j+\cdots$ as the value of ν at the point spectrum u, $v=\kappa^2$, w, $w'=\mu^2$, s_1, $s_2=\infty$. Since we exclude the existence of such an interaction, however, ν is zero at this point.[17]

The other four-body functions may be treated in a similar way. $\rho(z_1z_2z_3z_4)$ is simpler since no γ matrices are involved. The result is

$$\rho(k_1k_2k_3k_4)=-i\delta(\sum k_i)\tfrac{1}{8}\sum_P \delta(i_1i_2)\delta(i_3i_4) \times\rho(k_1^2,k_2^2,k_3^2,k_4^2,\,(k_1+k_2)^2,\,(k_1+k_3)^2), \tag{3.45}$$

where $\rho(k_1^2\cdots)$ has the same structure as Eq. (3.44):

$$\rho=\int\frac{\nu(w_1w_2w_3w_4s_1s_2)}{\prod_i(k_i^2+\bar{w}_i)\cdot[(k_1+k_2)^2+\bar{s}_1][(k_1+k_3)^2+\bar{s}_2]}\times\prod_i dw_i\cdot ds_1ds_2, \tag{3.46}$$

and $\sum_P$ runs over the permutations of 1, 2, 3, 4. In this case ν does not have the delta functions $\delta(1-\infty/s_1)$ and $\delta(1-\infty/s_2)$ provided that the meson-meson direct interaction $(\lambda/4)\varphi_i^2\varphi_j^2$ does not really exist (which means that the renormalized λ is zero). In case λ is finite, ν should have a point spectrum

$$\nu=\nu^0(w_i)\delta(1-\infty/s_1)\delta(1-\infty/s_2)+\cdots,\quad \nu^0(w_i)=2\lambda\prod_i\delta(w_i-\mu_i^2)+\cdots, \tag{3.47}$$

whereby λ is exactly defined. On the other hand, the unrenormalized λ_0 is expressed by (see Appendix)

$$\int\nu^0(w_i)\prod_i dw_i=2\lambda Z_4Z_3^{-4}=2\lambda_0Z_3^{-2}. \tag{3.48}$$

As for $\rho(xx'yy')$, we have two sets of γ matrices and three independent momenta which give $2\times3=6$ scalars of the form $(p\gamma)$; in addition, five scalars may be formed out of the γ's as is familiar in the β-decay theory. In the isotopic space, there are two scalars 1 and $\tau\cdot\tau'$. We put accordingly

$$\begin{aligned}\rho(pp'qq')=&\delta(p+p'+q+q')\\ &\times\tfrac{1}{4}(1+C)(1-P)\sum_{\nu=1}^{5}(ip\gamma)^{\lambda_1}(ip'\gamma')^{\lambda_2}O_\nu\\ &\times\{[i(p+p',\gamma+\gamma')]^{\lambda_3},[i(p+q,\gamma-\gamma')]^{\lambda_4}\}O_\nu'\\ &\times(-iq\gamma)^{\lambda_5}(-iq'\gamma')^{\lambda_6}(\tau\cdot\tau')^{\lambda_7}\\ &\times\rho_{\lambda_1\cdots\lambda_7}(p^2,p'^2,q^2,q'^2,\,(p+p')^2,\,(p+q)^2).\end{aligned} \tag{3.49}$$

$O_\nu\cdot O_\nu'$ stands for the five scalars $1\cdot1$, $\gamma_\mu\gamma_\mu'$, etc.; P means the permutation $p\alpha s\leftrightarrow p'\alpha's'$, and C the charge conjugation $p\leftrightarrow q$, $p'\leftrightarrow q'$, $\gamma\rightarrow-\gamma$, $\gamma'\rightarrow-\gamma'$, $O_\nu\leftrightarrow O_\nu'$. $\rho_{\lambda_1\ldots\lambda_7}$ may be written as before

$$\rho_{\lambda_1\cdots\lambda_7}=\int\frac{\nu_{\lambda_1\cdots\lambda_7}(uu'vv's_1s_2)dudu'dvdv'ds_1ds_2}{(p^2+\bar{u})(p'^2+\bar{u})(q^2+\bar{v})(q'^2+\bar{v})[(p+p')^2+\bar{s}_1][(p+q)^2+\bar{s}_2]}, \tag{3.50}$$

with similar point spectra as in the case of $\rho(xyzz')$, namely $\delta(u-\kappa^2)$, $\delta(u'-\kappa^2)$, $\delta(v-\kappa^2)$, $\delta(v'-\kappa^2)$, and $\delta(1-\infty/s_1)$.

4. RELATION BETWEEN THE SPECTRAL FUNCTIONS

So far the structure of the Green's functions has been determined in a way which reflects to a large extent only the fundamental assumptions of field theory. Now we are going to take full account of the equations of motion and see how they are translated into relations

[17] We can also show, on the basis of Eq. (3.43), that $\delta(1-\infty/s_2)$ does not occur at all, and at u, $v=\kappa^2$, w, $w'=\mu^2$, $s_1=\infty$, $s_2=\kappa^2$, ν is exactly given by the lowest perturbation result:

$$\nu_{\lambda_1\cdots\lambda_4}=g^2\nu_{\lambda_1\cdots\lambda_4}{}^0\delta(u-\kappa^2)\delta(v-\kappa^2)\delta(w-\mu^2)\times\delta(w'-\mu^2)\delta(1-\infty/s_1)\delta(s_2-\kappa^2),$$
$$\nu_{\lambda_1\cdots\lambda_4}{}^0=(-\kappa)^{2-\lambda_1-\lambda_3}\kappa^{\lambda_2},$$

where g is the constant defined in (ii). But this will need a detailed analysis of Eq. (3.43) (or the relation between different ν functions to be discussed later), and is not of the nature of a boundary condition for ν that can be imposed at our choice.

between the various spectral functions introduced in the last section.

According to Sec. 2, there are a set of equations which connect the Green's functions of different order. If we substitute the parametric representation of the Green's functions developed in the last section in these equations, we should be able to obtain the corresponding equations of motion for the spectral functions ν. In explicitly carrying this out, we naturally meet with divergences due to the singularities of the Green's functions, making the results meaningless unless they are renormalized. Since in our representation of the Green's functions their dependence on the relative coordinates or momenta is clearly exhibited by the energy denominators, such a renormalization procedure can be carried through explicitly without tampering with the ν functions, and the ensuing relation between the ν functions will be in a manifestly renormalized, divergence-free form. We will discuss this procedure for each equation successively.

(i) Let us begin with one of the lowest order equations:

$$(\gamma_\mu\partial_\mu+\kappa)_x\rho(xy)=-i\delta(xy)-ig\gamma_5\tau_j\int\rho(xyz_j')\delta(x-z')(dz')^4. \quad (4.1)$$

Going to the momentum representation and using the formulas Eqs. (3.2), (3.4), (3.26), and (3.30), we get

$$i\int(ip\gamma+\kappa)\frac{(ip\gamma)^\lambda\nu_\lambda(u)}{p^2+\bar{u}}du$$
$$=-i+\frac{g}{(2\pi)^4}\int\delta(p'+k-p)dp'dk\times\int\frac{\gamma_5\tau_j(ip'\gamma)^\lambda\gamma_5\tau_j(-iq\gamma)^{\lambda'}\nu_{\lambda\lambda'}(uvw)}{(p'^2+\bar{u})(q^2+\bar{v})(k^2+\bar{w})}dudvdw$$
$$=-i+\frac{3g}{(2\pi)^4}\int dk\times\int\frac{[-i(p-k,\gamma)]^\lambda(-iq\gamma)^{\lambda'}\nu_{\lambda\lambda'}(uvw)}{[(p-k)^2+\bar{u}](q^2+\bar{v})(k^2+\bar{w})}dudvdw. \quad (4.2)$$

Assuming that the integrations in k and the other variables can be interchanged in order freely, we may first carry out the integration

$$I_\lambda(p)\equiv\frac{-i}{(2\pi)^4}\int\frac{(-i(p-k,\gamma))^\lambda}{[(p-k)^2+\bar{u}](k^2+\bar{w})}dk,\quad\lambda=0,1, \quad (4.3)$$

which is obviously divergent logarithmically. It is formally equivalent to the lowest order self-energy

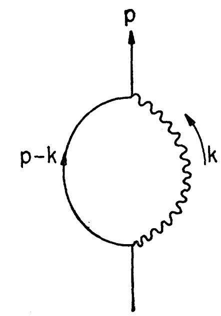

FIG. 1. A self-energy diagram corresponding to Eq. (4.3).

(Fig. 1)

$$\sum(p)=\frac{g^2i}{(2\pi)^4}\int\gamma_5\tau_jS_F(p-k)\gamma_5\tau_j\Delta_F(k)dk$$
$$=-\frac{3g^2i}{(2\pi)^4}\int\frac{i(p-k,\gamma)+\kappa}{[(p-k)^2+\bar{\kappa}^2](k^2+\bar{\mu}^2)}dk,$$

if we put $u=\kappa^2$, $w=\mu^2$, so that the renormalization could be carried out by the standard methods. Instead of doing this, however, we will again make use of the causality requirement in a trick originally due to Lehmann.[2]

$I_\lambda(p)$ is the Fourier component of the function

$$I_\lambda(x)=i[(-\gamma_\mu\partial_\mu)^\lambda\Delta_F^{(u)}(x)]\Delta_F^{(w)}(x), \quad (4.4)$$

where $\Delta_F^{(u)}$, $\Delta_F^{(w)}$ are the Feynman propagation functions with the masses $\sqrt{u}$ and $\sqrt{w}$, and are defined conveniently as

$$(-\gamma_\mu\partial_\mu)^\lambda\Delta_F^{(u)}(x)=\theta(x)(-\gamma_\mu\partial_\mu)^\lambda\Delta^{(u)+}(x)+\theta(-x)(-\gamma_\mu\partial_\mu)^\lambda\Delta^{(u)-}(x), \quad (4.5)$$
$$\Delta_F^{(w)}(x)=\theta(x)\Delta^{(w)+}(x)+\theta(-x)\Delta^{(w)-}(x),$$

$\Delta^{\pm}(x)$ being a function containing only positive or negative frequency part:

$$\Delta^{(u)\pm}(x)=\frac{1}{(2\pi)^3}\int\theta(\pm k)\delta(k^2+u)e^{ikx}dk. \quad (4.6)$$

We now ask for the "imaginary part" of $I_\lambda(p)$ which expresses possible real processes. In the present case, a real process would mean that the incident wave of four-momentum $p(p_0>0)$ splits up into two "particles" of mass $\sqrt{u}$ and $\sqrt{v}$, four-momentum $p-k$ $(p_0-k_0>0)$, and $k(k_0>0)$. Such a process certainly corresponds to the first term of

$$\theta(x)[(-\gamma_\mu\partial_\mu)^\lambda\Delta^{(u)+}(x)]\theta(x)\Delta^{(w)+}(x)$$
$$=\theta(x)[(-\gamma_\mu\partial_\mu)\Delta^{(u)+}(x)]\Delta^{(w)+}(x)$$
$$=\tfrac{1}{2}[(-\gamma_\mu\partial_\mu)\Delta^{(u)+}(x)]\Delta^{(w)+}(x)+\frac{\epsilon(x)}{2}[(-\gamma_\mu\partial_\mu)^\lambda\Delta^{(u)+}(x)]\Delta^{(w)+}(x),$$

which itself is part of $I_\lambda(x)$. If $p_0<0$, we have simply to change Δ^+ into Δ^-. Thus

$$\text{“Im”}I_\lambda(p)=\frac{1}{2}\frac{1}{(2\pi)^2}\int(-i\gamma,\, p-k)^\lambda \times[\theta(p)\theta(p-k)\theta(k)+\theta(-p)\theta(k-p)\theta(-k)] \times\delta((p-k)^2+u)\delta(k^2+v)dk. \quad (4.7)$$

After some calculation (see Appendix 2) it leads to

$$\text{“Im”}I_\lambda(p)=\frac{1}{8\pi}\frac{S(u,w,\,-p^2)}{-p^2}[C(u,w,\,-p^2)(-ip\gamma)]^\lambda$$

$$\begin{aligned}2S(u,w,\,-p^2)&=\{[(\sqrt{u}+\sqrt{w})^2+p^2][(\sqrt{u}-\sqrt{w})^2+p^2]\}^{\frac{1}{2}}, \quad \sqrt{-p^2}\geq\sqrt{u}+\sqrt{w}\\&=0,\quad\text{otherwise}\end{aligned} \quad (4.8)$$

$$C(u,w,\,-p^2)=(-u+w+p^2)/2p^2.$$

$S/2$ is the area of a non-Euclidean triangle formed with the three sides of length $\sqrt{u}$, $\sqrt{w}$, $\sqrt{-p^2}$, and C is the ratio of the projection of the side $\sqrt{u}$ onto $\sqrt{-p^2}$ to $\sqrt{-p^2}$.

Once the imaginary part has been calculated, it is now easy to obtain the whole "renormalized" I_λ by means of the formula (3.19):

$$I_\lambda^{(n)}(p)\equiv\frac{1}{8\pi^2}\int\frac{\bar{u}}{p^2+\bar{u}}\left(\frac{a+p^2}{a-u'}\right)^n\frac{S(uwu')}{u'} \times[C(uwu')(-ip\gamma)]^\lambda du'. \quad (4.9)$$

Since $S/u'\sim1$, $C\sim1$ for large u', n must be ≥1, and $a<\sqrt{u}+\sqrt{w}$. We put the result into the righthand side of Eq. (4.2). Remembering that $p=-q$, we have now a product of two factors

$$\frac{1}{p^2+\bar{u}'}\frac{1}{p^2+\bar{v}}=\frac{1}{v-u'}\left(\frac{1}{p^2+\bar{u}'}-\frac{1}{p^2+\bar{v}}\right).$$

As the last step, we divide Eq. (4.2) by $(ip\gamma+\kappa)$, and again the "imaginary parts" of both sides may be compared for a given p, and thus the relation between the ν's obtained except at the point $u=\kappa^2$. At $u=\kappa^2$, instead, we have to invoke the boundary condition (3.13) to renormalize ν correctly. Thus

$$\nu_1(u)=\delta(u-\kappa^2)+\frac{3g}{8\pi^2}\frac{1}{u-\kappa^2}\int\frac{1}{u-v}\times\left[\frac{S(u'wu)}{u}K_{1\lambda\lambda'}(u'wu)\nu_{\lambda\lambda'}(u'vw)+\left(\frac{u-\kappa^2}{v-\kappa^2}\right)^2\frac{S(u'wv)}{v}K_{1\lambda\lambda'}(u'wv)\times\nu_{\lambda\lambda'}(u'uw)\right]du'dvdw,$$

$$\nu_0(u)=-\kappa\delta(u-\kappa^2)+\frac{3g}{8\pi^2}\frac{1}{u-\kappa^2}\int\frac{1}{u-v}\times\left[\frac{S(u'wu)}{u}K_{2\lambda\lambda}'(u'wu)\nu_{\lambda\lambda'}(u'vw)+\left(\frac{u-\kappa^2}{v-\kappa^2}\right)^2\frac{S(u'wv)}{v}K_{2\lambda\lambda'}(u'wv)\times\nu_{\lambda\lambda'}(u'uw)\right]du'dvdw, \quad (4.10)$$

$$K_{1\lambda\lambda'}(u'wu)=\begin{pmatrix}-1 & \kappa\\ -C(u'wu)\kappa & C(u'wu)u\end{pmatrix},$$

$$K_{2\lambda\lambda'}(u'wu)=\begin{pmatrix}\kappa & -u\\ C(u'wu)u & -C(u'wu)u\kappa\end{pmatrix}.$$

It has been so designed that, with $a=\kappa^2$ and $n=1$ or 2 depending on the individual term, the integrals in Eq. (4.10) are convergent, reduce to zero for $u=\kappa^2$, and that ν_λ satisfies the condition (3.16) if a similar condition holds for $\nu_{\lambda\lambda'}$. The second point is essential for the correct renormalization according to Eq. (3.13). The factor $1/(u-v)$ in the integrands is to be interpreted as the principal value.

The equation of motion for $\rho(zz')$ may be treated in a similar way. We have namely

$$(\Box-\mu^2)_z\rho(zz')=i\delta(zz')-gg\int\mathrm{tr}[\gamma_5\tau_i\rho(xyz')] \times\delta(x-z)\delta(y-z)(dx)^4(dy)^4, \quad (4.11)$$

or substituting the parametric representation,

$$i\int\frac{k^2+\mu^2}{k^2+\bar{w}}\nu(w)dw=i-\frac{ig}{(2\pi)^4}\int\delta(p+q+k)dpdq \times\int\frac{\mathrm{tr}[i\gamma_5\tau_i(ip\gamma)^\lambda\gamma_5\tau_j(-iq\gamma)^{\lambda'}]}{(p^2+\bar{u})(q^2+\bar{v})(k^2+\bar{w})} \times\nu_{\lambda\lambda'}(uvw)dudvdw, \quad (4.12)$$

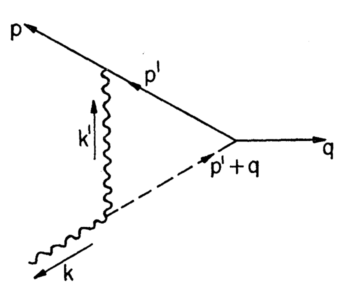

FIG. 2. A vertex part diagram corresponding to Fq. (4.18).

which is of the same nature as Eq. (4.2). We quote the final result only.

$$\nu(w)=\delta(w-\mu^2)+\frac{1}{w-\mu^2}\frac{3g^2}{2\pi^2}\int\frac{1}{w-w'}\times\Bigg[\frac{S(uvw)}{w}H_{\lambda\lambda'}(uvw)\nu_{\lambda\lambda'}(uvw')+\left(\frac{w-\mu^2}{w'-\mu^2}\right)^2\frac{S(uvw')}{w'}H_{\lambda\lambda'}(uvw')\times\nu_{\lambda\lambda'}(uvw)\Bigg]dudvdw', \tag{4.13}$$

$$H_{\lambda\lambda'}(uvw)=\begin{pmatrix}1 & 0\\ 0 & \frac{1}{2}(w-u-v)\end{pmatrix}.$$

(ii) The three-body Green's function $\rho(xyz)$ obeys the equations of motion

$$(\gamma\partial+\kappa)_x\rho(xyzi)=-ig\int\gamma_5\tau_j[\rho(xy)\rho(ziz'j)+\rho(xyziz'j)]\delta(x-z')(dz')^4,$$

$$(\square-\mu^2)_z\rho(xyzi)=ig\int[-\rho(xy)\ \mathrm{tr}[\gamma_5\tau_i\rho(x'y')]+\rho(xy')\gamma_5\tau_i\rho(x'y)+\rho(xx'\alpha s,yy'\beta t)(\gamma_5\tau_i)_{\beta t,\,\alpha s}]\times\delta(x'-z)\delta(y'-z)(dx')^4(dy')^4, \tag{4.14}$$

of which we will consider here only the first equation. Going over to the Fourier component, we get

$$(ip\gamma+\kappa)\rho(pqk)=-ig\gamma_5\tau_j\Bigg[\rho(q)\rho_{ij}(k)\delta(p+q+k)+\frac{1}{(2\pi)^4}\int\rho(p'qkik'j)\delta(p'+k'-p)dp'dk'\Bigg]. \tag{4.15}$$

We may divide ρ into ρ_a and ρ_b, corresponding respectively to the two terms on the right-hand side. Inserting the expression for $\rho(q)$ and $\rho_{ij}(k)$, Eqs. (3.2) and (3.4), we find easily

$$\rho_a(pqk)=-ig\delta(p+q+k)\frac{ip\gamma-\kappa}{p^2+\kappa^2}\gamma_5\tau_i\times\int\frac{(-iq\gamma)^\lambda\nu_\lambda(v)\nu(w)}{(q^2+\bar{v})(k^2+\bar{w})}dvdw, \tag{4.16}$$

or using the representation Eqs. (3.26) and (3.30) for $\rho(pqk)$,

$$\nu_{a\lambda\lambda'}(uvw)=g(-\kappa)^{1-\lambda}\delta(u-\kappa^2)\nu_{\lambda'}(v)\nu(w). \tag{4.17}$$

As for ρ_b, it may further be divided into two corresponding to the two terms of Eq. (3.41):

$$\rho(p'qkk')=\delta(p'+q+k+k')(1+C)(ip'\gamma)^{\lambda_1}\times(i(p'+k,\gamma))^{\lambda_2}(-iq\gamma)^{\lambda_3}(\tau)^{\lambda_4}\rho_{\lambda_1\cdots\lambda_4},$$

which we call ρ_b' and ρ_b''. Let us first consider ρ_b''. We have to carry out the integration[18]

$$I_\lambda(pqk)\equiv\frac{-i}{(2\pi)^4}\times\int\frac{(ip'\gamma)^\lambda\delta(p'+k'-p)}{(p'^2+\bar{u})(k'^2+\bar{w})((p'+q)^2+\bar{s}_1)}dp'dk', \tag{4.18}$$

which is similar to the lowest order vertex part represented by the Feynman diagram Fig. 2, so that we would expect to find I_λ in the form

$$I_\lambda=\int\frac{[i(ap+bq),\gamma]^\lambda}{(p+\bar{u}_1)(q^2+\bar{v}_1)(k^2+\bar{w}_1)}\times f(u_1v_1w_1)du_1dv_1dw_1. \tag{4.19}$$

To obtain directly the renormalized finite expression for I_λ, we again avail ourselves of the method used in connection with (*i*). I_λ is the Fourier component of the quantity

$$I_\lambda(xyz)=-[(\gamma_\mu\partial_\mu)^\lambda\Delta_F^{(u)}(x-y)]\times\Delta_F^{(v)}(x-z)\Delta_F^{(s_1)}(y-z). \tag{4.20}$$

Let us then ask for that part of I_λ where all the three internal lines p', k', p', $+k$ contribute to real processes. Suppose that $p_0>0$, $k_0<0$ (or $p_0<0$, $k_0>0$), and large enough to create real processes. In this case the internal "real" particles must be moving in the direction (or opposite) indicated in Fig. 2. Since p' and $p'+q$ point to opposite directions seen from the external line q, q cannot create a real process under such circumstances. This means that when the two denominators p'^2+u, k'^2+w become zero, the third does not, which amounts to a certain restriction on the range of the values of u, w, and s_1. At any rate, the "imaginary part" of I_λ arising from the above mentioned process comes, using Eq. (2.5), from the ϵ-independent part of

$$-\sum_\pm\theta(\pm(x-y))\theta(\pm(x-z))\theta(\pm(y-z))\times[(\gamma_\mu\partial_\mu)^\lambda\Delta^{(w)\pm}(x-y)]\Delta^{(w)\pm}(x-z)\Delta^{(s_1)\pm}(y-z)$$
$$=-\sum_\pm\frac{1\pm\epsilon(x-y)}{2}\frac{1\pm\epsilon(y-z)}{2}[(\gamma_\mu\partial_\mu)^\lambda\Delta^{(u)\pm}(x-y)]\times\Delta^{(w)\pm}(x-z)\Delta^{(s_1)\pm}(y-z), \tag{4.21}$$

[18] In case $s_1=\infty$, it reduces to the type of Eq. (4.3).

which leads to

$$-\frac{1}{2\pi}\frac{1}{4}\int[\theta(p)\theta(-k)\theta(p')\theta(p'+q)\theta(k')$$
$$+\theta(p)\theta(k)\theta(-p')\theta(-p'-q)\theta(-k')]$$
$$\times(ip'\gamma)^\lambda\delta(p'^2+u)\delta((p'+q)^2+s_1)$$
$$\times\delta(k'^2+w)\delta(p'+k-p)dp'dk'. \quad (4.22)$$

This must be equated to the corresponding "imaginary part" of Eq. (4.19):

$$\text{"Im"}I_\lambda=(i\pi)^2\int\frac{[i(ap+bq,\gamma)]^\lambda}{q^2+v_1}f(u_1v_1w_1)dv_1,$$
$$u_1=-p^2,\quad w_1=-k^2. \quad (4.23)$$

Evaluating Eq. (4.22), we get (see Appendix 2):

$$\text{"Im"}I_\lambda=-\tfrac{1}{16}(\theta(k)\theta(-p)$$
$$+\theta(-k)\theta(p))\frac{[i(a'p+b'q,\gamma)]^\lambda}{S(u_1,w_1,-q^2)}, \quad (4.24)$$
$$4S^2a'=2(u_1+u-w)q^2+(u_1-q^2-w_1)(u-q^2-s_1),$$
$$4S^2b'=2(u-q^2-s_1)u_1+(u_1+u-w)(u_1-q^2-w_1).$$

S is defined in Eq. (4.8), and the arguments u_1, w_1, $-q^2$ subject to the condition

$$\sqrt{u_1}\geq\sqrt{u}+\sqrt{w},\quad \sqrt{w_1}\geq\sqrt{s_1}+\sqrt{w},$$
$$-q^2\leq(\sqrt{s_1}-\sqrt{u})^2. \quad (4.25)$$

Equation (4.23) is still of an integral form, and we do not know the domain of v_1 yet. But if we make an analytic continuation of $-q^2$ to the unknown domain of v_1, Eq. (4.24) should acquire an imaginary part due to the pole at $v_1-i\epsilon$. On the other hand, Eq. (4.24) becomes imaginary only if $-q^2$ is such as to make the factor under the root sign in S [Eq. (4.8)] negative, or

$$(\sqrt{u_1}+\sqrt{w_1})^2\geq-q^2\geq(\sqrt{u_1}-\sqrt{w_1})^2. \quad (4.26)$$

Hence the domain of v_1 should agree with that of $-q^2$ above, and comparing these new "imaginary parts" we finally get

$$f(u_1v_1w_1)=\frac{1}{16\pi^3}\frac{1}{S^*(u_1v_1w_1)},$$
$$S^*(u_1v_1w_1)=[(\sqrt{u_1}+\sqrt{v_1}+\sqrt{w_1})$$
$$\times(\sqrt{u_1}+\sqrt{v_1}-\sqrt{w_1})$$
$$\times(\sqrt{u_1}-\sqrt{v_1}+\sqrt{w_1}) \quad (4.27)$$
$$\times(-\sqrt{u_1}+\sqrt{v_1}+\sqrt{w_1})]^{\frac{1}{2}},$$
$$4S^{*2}a=2(u_1+u-w)v_1-(u_1+v_1-w_1)(u+v_1-s_1),$$
$$4S^{*2}b=-2(u+v_1-s_1)u_1$$
$$-(u_1+u-w)(u_1+v_1-w_1).$$

$S^*/2$ is the area of an Euclidean triangle with the familiar relations

$$\sqrt{u_1}+\sqrt{v_1}\geq\sqrt{w_1}\geq|\sqrt{u_1}-\sqrt{v_1}|,$$
$$\sqrt{v_1}+\sqrt{w_1}\geq\sqrt{u_1}\geq|\sqrt{v_1}-\sqrt{w_1}|, \quad (4.28)$$
$$\sqrt{w_1}+\sqrt{u_1}\geq\sqrt{v_1}\geq|\sqrt{w_1}-\sqrt{u_1}|,$$

which limit the domain of the variables together with the restriction

$$\sqrt{u_1}\geq\sqrt{u}+\sqrt{w},\quad \sqrt{w_1}\geq\sqrt{s_1}+\sqrt{w},$$
$$\sqrt{v_1}\geq\sqrt{u}+\sqrt{s_1}. \quad (4.29)$$

The last condition simply indicates the threshold values of $-p^2$, $-k^2$, and $-q^2$ for creating real processes.

We thus obtain

$$I_\lambda(pqk)=\frac{1}{16\pi^3}$$
$$\times\int\frac{[i(ap+bq,\gamma)]^\lambda}{(p^2+\bar{u}_1)(q^2+\bar{v}_1)(k^2+\bar{w}_1)}\frac{du_1dv_1dw_1}{S^*(u_1v_1w_1)}, \quad (4.30)$$

which is certainly convergent [without use of the modified formula (3.19)] because of the triangular relation (4.28). The θ factors in Eq. (4.24) have dropped out also because of the triangular relation which effectively takes care of them. I_λ being thus determined, we substitute it into the equation for

$$(ip\gamma+\kappa)\rho_b''(pqk)=g\gamma_5\tau_j\frac{1}{16\pi^3}\int\frac{[i(ap+bq,\gamma)]^{\lambda_1}(ip\gamma)^{\lambda_2}(-iq\gamma)^{\lambda_3}(\tau)^{\lambda_4}}{(p^2+\bar{u}_1)(q^2+\bar{v}_1)(k^2+\bar{w}_1)(p^2+\bar{s}_2)(q^2+\bar{v})(k^2+\bar{w}')}$$
$$\times\frac{\nu_{\lambda_1\cdots\lambda_4}(uvww's_1s_2)}{S^*(u_1v_1w_1)}du_1dv_1dw_1dudvdwdw'ds_1ds_2. \quad (4.31)$$

We put the representation (3.30) for $\rho_b''(pqk)$ on the left-hand side, divide the whole thing by $ip\gamma+\kappa$, transform products of factors like $1/(q^2+\bar{v}_1)(q^2+\bar{v})$ on the right into partial fractions, and compare both sides (or their imaginary parts) for a given set of p^2, q^2 and k^2. We shall not write down the ensuing relation between $\nu_{b\lambda\lambda'}''(u_1v_1w_1)$ and $\nu_{\lambda_1\cdots\lambda_4}(uvww's_1s_2)$ since it is too complicated due to the spin matrices.

We are left with ρ_b' that corresponds to the first term of Eq. (3.41). The integral to be considered is now[19]

$$I_{\lambda_1\lambda_2}(pqk)\equiv\frac{-i}{(2\pi)^4}\int\frac{(ip'\gamma)^{\lambda_1}[i(p'+k,\gamma)]^{\lambda_2}\delta(p'+k'-p)}{(p'^2+\bar{u})(k'^2+\bar{w}')[(p'+q)^2+\bar{s}_1][(p'+k)^2+\bar{s}_2]}dp'dk', \tag{4.32}$$

which contains four denominators. A Feynman diagram for this is given in Fig. 3, where four external lines p, k, k, and $-p-2k=q-k$ come in. We assume the representation

$$I_{\lambda_1\lambda_2}''=\int\frac{[i(ap+bq,\gamma)]^{\lambda_1}[i(cp+dq,\gamma)]^{\lambda_2}-\Lambda\delta(2-\lambda_1-\lambda_2)}{(p^2+\bar{u}_1)(k^2+\bar{w}_1)(q^2+\bar{v}_1)}g(u_1v_1w_1)du_1dv_1dw_1. \tag{4.33}$$

To the simultaneous "imaginary part" for the three denominators of Eq. (4.33) with $p_0>0$, $k_0<0$, $(p+k)_0=-q_0>0$, for example, corresponds the following part of Eq. (4.32):

$$\begin{aligned}&\text{“Im”}I_{\lambda_1\lambda_2}\\&=-\frac{2(\pi i)^4}{(2\pi)^4}\int(ip'\gamma)^{\lambda_1}[i(p'+k,\gamma)]^{\lambda_2}\delta(p'^2+\bar{u})\delta(k'^2+w')\\&\quad\times\delta((p'+q)^2+s_1)\delta((p'+k)^2+s_2)\theta(p')\theta(k')\\&\quad\times\{\theta(p'+k)\theta(p'+q)+\theta(-p'-k)\theta(-p''-q)\}\\&\qquad\qquad\times\delta(p'+k'-p)dp'dk'. \qquad (4.34)\end{aligned}$$

By a procedure similar to that for Eq. (4.22) as explained in the Appendix, this yields

$$\begin{aligned}&\text{“Im”}I_{00}=g(u_1v_1w_1)\\&\quad=\frac{1}{64}\frac{1}{|S^*(u_1v_1w_1)|}\\&\qquad\times\delta(u+w'+v_1+w_1-s_1-s_2-u_1)\\&\qquad\times\theta(\sqrt{u_1}-\sqrt{u}-\sqrt{w'})\\&\qquad\times[\theta(\sqrt{w_1}-\sqrt{s_1}-\sqrt{w'})\theta(\sqrt{v_1}-\sqrt{w'}-\sqrt{s_2})\\&\qquad+\theta(\sqrt{w_1}-\sqrt{u}-\sqrt{s_2})\theta(\sqrt{v_1}-\sqrt{s_1}-\sqrt{u})],\\&4S^{*2}a=2(u_1+u-w')v_1\\&\qquad\qquad-(u_1+v_1-w_1)(u+v_1-s_1), \qquad (4.35)\\&4S^{*2}b=-2(u+v_1-s_1)u_1\\&\qquad\qquad-(u_1+u-w)(u_1+v_1-w_1),\\&4S^{*2}c=-2v_1(w_1+w'-v_1-u)\\&\qquad-(u_1+v_1-w_1)(s_1+u_1-2v_1-w_1-u),\\&4S^{*2}d=2u_1(s_1+u_1-2v_1-w_1-u)\\&\qquad\qquad-(u_1+v_1-w_1)(w_1+w'-v_1-u),\\&2\Lambda=-(u+u_1+v_1-w'-s_1-s_2)+2acu_1\\&\qquad\qquad+2bdv_1-(ad+bc)(u_1+v_1-w_1).\end{aligned}$$

[19] In case $s_1=\infty$, the integration reduces to the type of Eq. (4.18).

Finally, we piece together the equations for ν_a, ν_b', and ν_b'' to get one for the total $\nu_{\lambda\lambda'}$. The renormalization has not been completely taken into account since $\nu_{\lambda\lambda'}$ may not yet satisfy the required boundary condition (3.33). If we "correct" it to the value Eq. (3.33) we get the properly renormalized equation. The remaining part of the spectrum is already correct, and will not be affected.

5. DISCUSSION

The significance of what we have achieved here seems to lie in several points. To begin with, the Green's functions which have rather complicated transformation properties have been expressed in terms of the ν functions, or spectral functions, which are real scalar functions depending on scalar variables only. The Green's functions thus obtained satisfy the various requirements of field theory, including relativity and causality. It can easily be seen that the formulas given in NI for some scattering matrices are compatible with the present results. Our new representations are more general in the sense that they show the complete dependence of

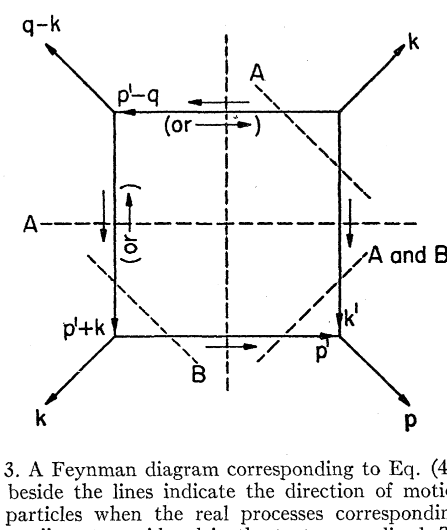

FIG. 3. A Feynman diagram corresponding to Eq. (4.32). The arrows beside the lines indicate the direction of motion of the "free" particles when the real processes corresponding to the "imaginary" part considered in the text are realized. These real processes occur across either set, A or B, of broken lines (intermediate states).

the matrix on all the variables. The results of NI follow from the present ones, but the converse is not true.[20]

As a matter of fact, our derivation of the formulas in Sec. 3 is not to be regarded as a proof; if we start from the fundamental assumptions I to IV, we do not know *a priori* whether we may use Eq. (3.17) rather than Eq. (3.19) to express the general Green's functions in terms of the ν functions. Thus in order to confirm our formulas we would have to show that they are actually consistent with the equations of motion connecting different Green's functions. This was done in our paper up to the four-body functions.

One undesirable character of the old formulas was that the density functions ρ introduced there did not seem to be uniquely defined, and there was no physical reason to prefer one over the other. In the present case, on the other hand, the ν functions have a well-defined meaning since they may be related directly to matrix elements of the "real processes."

As for the problem of renormalization, we have attained two remarkable results. One is that the equations of motion for the ν functions are in a manifestly renormalized form, and involve only renormalized constants. The other is that renormalization can be performed by making use of causality, together with the boundary condition for the field operators at infinity,[21] instead of going through the usual subtraction procedure. To put it more precisely, the real (or dispersive) part of a renormalized Green's function is determined from the imaginary (or absorptive) part via causality. Part of the imaginary part is connected with the corresponding part of different Green's functions via the equations of motion; the rest is theoretically at our disposal, being of the nature of a boundary condition (or inhomogeneous terms in the equation) which is fixed by the given fundamental constants, namely the masses and the couplings, and which finally controls all the quantities. The definition of the coupling constant is more or less a matter of convenience, and there is no absolute reason to prefer one to the other[22]; it would be sufficient if the relation between different coupling constants were known.

The equations of motion for the ν functions are the fundamental relations in our theory, which should provide us with all the necessary information about the system, though we do not know yet whether they have meaningful solutions at all. Neither are we quite sure whether the present renormalization is completely equivalent with the usual one. In the lowest approximation it seems to be the case (except for the slightly different definition of g), but it is well known that a self-energy operator calculated to a certain approximation can give rise to spurious singularities of $\rho(xy)$ or $\rho(zz')$ which may violate causality. It would thus seem that causality is necessary as an additional requirement to perform a consistent calculation if one believes in microscopic causality at all.

Källén[17] has investigated the nature of the renormalization constants, and has shown in particular that we run into contradiction if all the constants are assumed to be finite. In the present paper, not much attention has been focused on this problem since we have been trying to do away with unobservable quantities. The relation of the ν functions to the renormalization constants is discussed briefly in the Appendix.

To make a complete study of all these points mentioned above, it would be necessary to extend our formulation to the Green's functions of all orders which are connected by the equations of motion. This will be done in a separate paper. Though somewhat complicated, there seems to be no essential difficulty. It would also seem possible to give a similar formulation for the relativistic wave functions such as $\langle 0|T(\psi(x),\varphi(z),\cdots)|P\rangle$ which have been studied extensively.[8] Among other things, the bound state or scattering problems may be handled conveniently in our formulation, and a new insight into the boundary condition may be obtained.

The author would like to acknowledge his gratitude to Professor Goldberger and Dr. Freese for their interest and enlightening discussions.

APPENDIX

1. Renormalization Constants

In order to study the various renormalization constants we must start from the renormalized Lagrangian (2.9) which leads to the equations of motion

$$\begin{aligned}(\gamma\partial+\kappa)\psi&=-igZ_2^{-1}Z_1\gamma_5\tau_i\varphi_i\psi-\delta_k\psi,\\ (\square-\mu^2)\varphi_i&=igZ_3^{-1}Z_1\bar{\psi}\gamma_5\tau_i\psi\\ &\quad-\delta\mu^2\varphi_i-\lambda Z_3^{-1}Z_4\varphi_i\varphi_k^2,\end{aligned}\tag{A1}$$

and the commutation relations

$$\begin{aligned}\{\psi_{\alpha s}(r),\bar{\psi}_{\beta t}(r')\}&=Z_2^{-1}(\gamma_4)_{\alpha\beta}\delta_{st}\delta(r-r'),\\ [\varphi_i(r),\partial\varphi_k(r')/\partial t]&=iZ_3^{-1}\delta_{ik}\delta(r-r').\end{aligned}\tag{A2}$$

[20] The quantity

$$M(k,l)\delta(p+k-q-l)=\int\langle p|T(j_i(x)j_j(x'))|q\rangle e^{ikx'-ilx}(dx)^4(dx')^4,$$
$$j_i(x)=ig\bar{\psi}\gamma_5\tau_i\psi(x),$$

introduced in NI Eq. (17) corresponds to

$$\lim_{\substack{ip\gamma+\kappa=0\\ iq\gamma+\kappa=0}}(k^2+\mu^2)(l^2+\mu^2)\bar{v}_p(ip\gamma+\kappa)\rho(p,-q,k,-l)(iq\gamma+\kappa)u_q,$$

where u_q and v_p are the initial and final spinor wave functions. If we take spinless fields as was done in NI, we shall see that the terms ρ_2 and ρ_3 in the numerator of Eq. (1.3) are missing, which could not be derived in general from the earlier considerations.

[21] Ideas related to ours have also been developed by H. Lehmann[2] and W. Zimmermann, Nuovo cimento **11**, 416 (1952), Lehmann, Symanzik, and Zimmermann, Nuovo cimento **1**, 205 (1955). The significance of causality in the problem of renormalization has been noticed by Stueckelberg. See for example, E. C. G. Stueckelberg and T. A. Green, Helv. Phys. Acta **24**, 153 (1951).

[22] G. Källén, Nuovo cimento **12**, 217 (1954).

The equations of motion for the Green's functions become thus

$$(\gamma\partial+\kappa_0)\rho(xy)=-iZ_2^{-1}\delta(x-y)-igZ_2^{-1}Z_1\gamma_5\tau_i \times\langle 0|T(\varphi_i\psi(x),\bar\psi(y))|0\rangle,$$
$$(\square-\mu_0^2)\rho(zz')=iZ_3^{-1}\delta(z-z')+igZ_3^{-1}Z_1 \times\langle 0|T(\bar\psi\gamma_5\tau_i\psi(z),\varphi_j(z'))|0\rangle -\lambda Z_3^{-1}Z_4\langle 0|T(\varphi_i\varphi_k^2(z),\varphi_j(z'))|0\rangle \text{ etc.}, \quad \text{(A3)}$$

where we have put

$$\kappa_0\equiv\kappa+\delta\kappa,\quad \mu_0^2\equiv\mu^2+\delta\mu^2, \quad \text{(A4)}$$

denoting the mechanical masses.

We will treat all the constants Z_i^{-1}, $\delta\kappa$, and $\delta\mu^2$ as if they were finite quantities. Taking the Fourier transforms, we substitute the parametric representation on the left-hand side. But this time we rewrite

$$i(ip\gamma+\kappa)\int\frac{(ip\gamma)\nu_1+\nu_0}{p^2+\bar u}du =-i\int\nu_1du +i\int\frac{(ip\gamma)(\kappa\nu_1+\nu_0)+(u\nu_1+\kappa\nu_0)}{p^2+\bar u}d\bar u, \quad \text{(A5)}$$
$$i(k^2+\mu^2)\int\frac{\nu}{k^2+\bar w}dw=i\int\nu dw+i\int\frac{\mu^2-w^2}{k^2+\bar w}\nu dw,$$

pretending convergence of each integral on the right. Comparing this with the right-hand side of Eq. (A3), we see that the first terms of Eq. (A5) must correspond to the first terms of Eq. (A3) since the other terms would have no singularities of the delta-function type if they were finite. Thus

$$\int\nu_1(u)du=Z_2^{-1},\quad \int\nu(w)dw=Z_3^{-1}. \quad \text{(A6)}$$

Next we differentiate Eq. (A3) once more and obtain

$$(\gamma\partial+\kappa_0)_x(-\gamma^T\partial+\kappa_0)_y\rho(xy) =-iZ_2^{-1}(-\gamma^T\partial+\kappa_0)\delta(x-y) -(gZ_2^{-1}Z_1)^2\langle 0|T(\gamma_5\tau_i\varphi_i\psi(x),\bar\psi\gamma_5\tau_j\varphi_j(y))|0\rangle,$$
$$(\square-\mu_0^2)_z(\square-\mu_0^2)_{z'}\rho(zz') =i(\square-\mu^2)\delta(z-z') -(gZ_3^{-1}Z_1)^2\langle 0|T(\bar\psi\gamma_5\tau_i\psi(z),\bar\psi\gamma_5\tau_j\psi(z'))|0\rangle -i\lambda Z_3^{-2}Z_4\delta_{ij}\delta(z-z')[\langle 0|\varphi_k^2(z)|0\rangle +2\langle 0|\varphi_i\varphi_k(z)|0\rangle] -i\lambda gZ_3^{-2}Z_1Z_4[\langle 0|T(\varphi_i\varphi_k^2(z),\bar\psi\gamma_5\tau_j\psi(z'))|0\rangle +\langle 0|T(\bar\psi\gamma_5\tau_i\psi(z),\varphi_j\varphi_k^2(z'))|0\rangle] +(\lambda Z_3^{-1}Z_4)^2\langle 0|T(\varphi_i\varphi_k^2(z),\varphi_j\varphi_l^2(z'))|0\rangle. \quad \text{(A7)}$$

The equations corresponding to Eq. (A5) become

$$-i(ip\gamma+\kappa_0)^2\int\frac{(ip\gamma)\nu_1+\nu_0}{p^2+\bar u}du=i\int(ip\gamma+\kappa_0)\nu_1du+i\int(\kappa\nu_1+\nu_0)du -i\int\frac{(ip\gamma)\{(\kappa^2+u)\nu_1+2\kappa\nu_0\}+\{2\kappa u\nu_1+(\kappa^2+u)\nu_0\}}{p^2+\bar u}du, \quad \text{(A8)}$$
$$-i(k^2+\mu_0^2)\int\frac{\nu}{k^2+\bar w}dw=-i\int(k^2+\mu_0^2)\nu dw-i\int(\mu_0^2-w)\nu dw-i\int\frac{(\mu_0^2-w)^2}{k^2+\bar w}\nu dw.$$

Comparison with Eq. (A7) leads to

$$\int(\kappa\nu_1+\nu_0)du=0,\quad \text{or}\quad \kappa_0Z_2^{-1}=-\int\nu_0(u)du,$$
$$\int(\mu_0^2-w)\nu dw=5c\lambda Z_3^{-2}Z_4,$$

or

$$\mu_0^2Z_3^{-1}=\int\nu(w)wdw+5c\lambda Z_3^{-2}Z_4, \quad \text{(A9)}$$
$$c\equiv\tfrac{1}{3}\langle 0|\varphi_i(x)\varphi_i(x)|0\rangle=\int\int\frac{\nu(w)}{k^2+\bar w}dwdk.$$

If we take beforehand the Wick product[8]

$$\frac{\lambda}{4}:\varphi_i^2\varphi_k^2:=\frac{\lambda}{4}[\varphi_i^2\varphi_k^2-2\varphi_i^2\langle\varphi_k^2\rangle-4\varphi_i\varphi_k\langle\varphi_i\varphi_k\rangle +\langle\varphi_i^2\rangle\langle\varphi_k^2\rangle+2\langle\varphi_i\varphi_k\rangle^2],$$

in the Lagrangian, the term with c will be absent.

Now let us consider the three-body Green's functions. The equations of motion read

$$(\gamma\partial+\kappa_0)\rho(xyz) =-igZ_2^{-1}Z_1\langle 0|T(\gamma_5\tau_j\varphi_j\psi(x),\bar\psi(y),\varphi_i(z))|0\rangle =-igZ_2^{-1}Z_1[\gamma_5\tau_j\langle 0|T(\varphi_j(x),\varphi_i(z))|0\rangle \times\langle 0|T(\psi(x),\bar\psi(y)|0\rangle+\cdots],$$
$$(\square-\mu_0^2)\rho(zz') =igZ_3^{-1}Z_1\langle 0|T(\psi(x),\bar\psi(y),\bar\psi\gamma_5\tau_i\psi(z))|0\rangle =igZ_3^{-1}Z_1[\langle 0|T(\psi(x),\bar\psi(z))|0\rangle\gamma_5\tau_i \times\langle 0|T(\psi(z),\bar\psi(y))|0\rangle+\cdots],$$
$$(\gamma\partial+\kappa_0)_x(-\gamma^T\partial+\kappa_s)_y\rho(xyz) =(gZ_2^{-1}Z_1)^2\gamma_5\tau_j\langle 0|T(\psi\varphi_j(x),\bar\psi\varphi_k(y),\varphi_i(z))|0\rangle\gamma_5\tau_k +gZ_2^{-2}Z_1\gamma_5\tau_j\delta(x-y) \times\langle 0|T(\varphi_j(x),\varphi_i(x))|0\rangle,\ \text{etc.} \quad \text{(A10)}$$

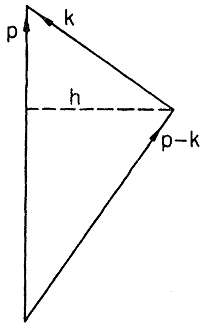

FIG. 4. Vectorial relation for the integral (A15).

$$(\gamma\partial+\kappa_0)_x(-\gamma^T\partial+\kappa_0)_y(\Box-\mu_0^2)_z\rho(xyz)$$
$$=i(gZ_2^{-1}Z_1)^2(gZ_3^{-1}Z_1)\gamma_5\tau_j$$
$$\times\langle 0|T(\psi\varphi_j(x),\bar\psi\varphi_k(y),\bar\psi\gamma_5\tau_i\psi(z))|0\rangle\gamma_5\tau_k$$
$$+i(gZ_2^{-1}Z_1)^2Z_3^{-1}\gamma_5\tau_j$$
$$\times\langle 0|T(\psi\varphi_j(x),\bar\psi(y))|0\rangle\gamma_5\tau_i\delta(y-z)$$
$$+i(gZ_2^{-1}Z_1)^2Z_3^{-1}\gamma_5\tau_i$$
$$\times\langle 0|T(\psi(x),\bar\psi\varphi_k(y))|0\rangle\gamma_5\tau_k\delta(x-z)$$
$$-i(gZ_2^{-1}Z_1)(gZ_3^{-1})\gamma_5\tau_j$$
$$\times\langle 0|T(\varphi_j(x),\bar\psi\gamma_5\tau_i\psi(z))|0\rangle\delta(x-y)$$
$$-igZ_2^{-2}Z_3^{-1}Z_1\gamma_5\delta(x-y)\delta(x-z).$$

Substituting the representation for $\rho(xyz)$, we proceed as before. For example

$$i(ip\gamma+\kappa_0)\int\frac{(ip\gamma)^\lambda\gamma_5\tau_i(-iq\gamma)^{\lambda'}}{(p^2+\bar u')(q^2+\bar v')(k^2+\bar w)}\nu_{\lambda\lambda'}dudvdw$$
$$=-i\int\frac{\gamma_5\tau_i(-iq\gamma)^{\lambda'}}{(q^2+\bar v)(k^2+\bar w)}\nu_{1\lambda'}dudvdw+\cdots. \quad \text{(A11)}$$

This, being dependent only on q^2 and k^2, must correspond to the first term on the right of the first of Eq. (A10). Thus

$$\int\nu_{1\lambda'}(uvw)du=gZ_2^{-1}Z_1\nu_{\lambda'}(v)\nu(w).$$

In a similar way we get the following set of relations

$$\int\nu_{\lambda1}(uvw)dv=gZ_2^{-1}Z_1\nu_\lambda(u)\nu(w),$$
(equivalent to the above)
$$\int\nu_{\lambda\lambda'}(uvw)dw=gZ_3^{-1}Z_1\nu_\lambda(u)\nu_{\lambda'}(v),$$
$$\int\nu_{\lambda1}(uvw)dvdw=gZ_2^{-1}Z_3^{-1}Z_1\nu_\lambda(u), \quad \text{(A12)}$$
$$\int\nu_{11}(uvw)dudv=gZ_2^{-1}Z_1\nu(w),$$
$$\int\nu_{11}(uvw)dudvdw=gZ_2^{-2}Z_3^{-1}Z_1,$$

which are consistent with Eq. (A6).

Finally we take $\rho(z_1\cdots z_4)$ which obeys the equation

$$(\Box-\mu_0^2)\rho(z_1\cdots z_4)$$
$$=\lambda Z_3^{-1}Z_4\langle 0|T(\varphi_1\varphi_i^2,\varphi_2,\varphi_3,\varphi_4)|0\rangle$$
$$+ig\langle 0|T(\bar\psi\gamma_5\tau_i\psi(z_1),\varphi_2,\cdots\varphi_4)|0\rangle+\cdots$$
$$=\lambda Z_3^{-1}Z_4[2\langle 0|T(\varphi_1\varphi_2)|0\rangle\langle 0|T(\varphi_i\varphi_3)|0\rangle$$
$$\times\langle 0|T(\varphi_i\varphi_4)|0\rangle+\cdots]$$
$$+ig\langle 0|T(\bar\psi\gamma_5\tau_i\psi(z_1),\varphi_2,\cdots\varphi_4)|0\rangle+\cdots. \quad \text{(A13)}$$

Substituting the representation Eq. (3.46), we conclude that

$$\int\nu^0(w_i)dw_1=2\lambda Z_4Z_3^{-1}\nu(w_2)\nu(w_3)\nu(w_4), \text{ etc.}$$
$$\int\nu^0(w_i)dw_1dw_2=2\lambda Z_4Z_3^{-2}\nu(w_3)\nu(w_4), \text{ etc.} \quad \text{(A14)}$$
$$\int\nu^0(w_i)dw_1dw_2dw_3=2\lambda Z_4Z_3^{-3}\nu(w_4), \text{ etc.}$$
$$\int\nu^0(w_i)\prod dw_i=2\lambda Z_4Z_3^{-4},$$

ν^0 being defined in Eq. (3.47).

2. Evaluation of Some Integrals

There is an easy way of evaluating the integrals in Sec. 4. First let us take Eq. (4.7). Let

$$\mathcal{I}_\lambda\equiv\sum_\pm\int(ip'\gamma)^\lambda\theta(\pm p)\theta(\pm p')\theta(\pm k)\delta(p'^2+u)$$
$$\times\delta(k^2+w)\delta(p'+k-p)dp'dk. \quad \text{(A15)}$$

The three vectors are related to each other as shown in Fig. 4. The delta functions in the integrand demand that the length of the vectors p' and k be kept fixed as well as the vector p itself, so that the triangle pkp' has a fixed shape and can rotate in the four-dimensional space around the vector p. We may thus write Eq. (A15) as

$$\mathcal{I}_\lambda=\sum_\pm\frac{\theta(\pm p)}{4(uw)^{\frac12}}\int(ip'\gamma)^\lambda\theta(\pm p')\theta(\pm k)$$
$$\times\delta(|p'|-\sqrt{u})\delta(|k'|-\sqrt{w})dP, \quad \text{(A16)}$$

where P is the apex of the triangle opposite the vector p. The integration may be factorized into two, namely that within the plane of the triangle and that resulting from the rotation of the triangle (or the point P) in a two-dimensional (space-like) space orthogonal to it.

For $\lambda=0$, the first integration gives the Jacobian

$$(uw)^{\frac12}/S,$$

which is the ratio of the area of a square with the sides $\sqrt{u}$ and $\sqrt{w}$ to that of the parallogram (denoted by S)

formed with the vectors p' and k. The second integration gives the surface area of a sphere

$$4\pi h^2,$$

where h is the height of the triangle seen from the base p. Thus

$$\mathcal{J}_0=\frac{1}{4(uw)^{\frac{1}{2}}}\frac{(uw)^{\frac{1}{2}}}{S}4\pi h^2=\frac{\pi h^2}{S}=\frac{\pi h^2}{h\sqrt{-p^2}}=\frac{\pi S}{-p^2}, \quad \text{(A17)}$$

and S may be expressed in terms of the three sides of the triangle:

$$S^2=\begin{vmatrix} p'^2 & (p'k) \\ (p'k) & k^2 \end{vmatrix}=p'^2k^2-\frac{(p'^2+k^2-p^2)^2}{4}. \quad \text{(A18)}$$

We have not been careful about the indefinite character of the metric in Minkowski space, but $\mathcal{J}_0$ is obviously a positive definite quantity. From physical consideration, on the other hand, we must have $\sqrt{-p^2}\geq\sqrt{u}+\sqrt{w}$ which is contrary to the usual relation for the three sides of a triangle. S must then be

$$S=\{[(\sqrt{u}+\sqrt{w})^2+p^2][(\sqrt{u}-\sqrt{w})^2+p^2]\}^{\frac{1}{2}}. \quad \text{(A19)}$$

For $\lambda=1$, the vector p' in $ip'\gamma$ gets simply projected onto the axis p as a result of the second integration, so that we get

$$\mathcal{J}_1=c(ip\gamma)\mathcal{J}_0,$$
$$c=(pp')/p^2=\frac{p^2+p'^2-k^2}{-2p^2}=\frac{p^2+w-u}{-2p^2}. \quad \text{(A20)}$$

Next take the integral in Eq. (4.22),

$$\mathcal{J}_\lambda'\equiv\sum_{\pm}\int\theta(\pm p)\theta(\mp k)\theta(\pm p')\theta(\pm p'\pm q)\theta(\pm k')$$
$$\times(ip'\gamma)^\lambda\delta(p'^2+u)\delta((p'+q)^2+s)$$
$$\times\delta(k'^2+w)\delta(p'+k'-p)dp'dk'$$
$$=\sum_{\pm}\frac{1}{8(uws)^{\frac{1}{2}}}\theta(\pm p)\theta(\mp k) \quad \text{(A21)}$$
$$\times\int(ip\gamma)^\lambda\delta(|p'|-\sqrt{u})\delta(|p'+q|-s)$$
$$\times\delta(|k'|-w)\delta(p'+k'-p)dp'dk'.$$

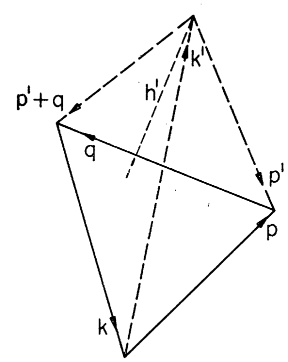

FIG. 5. Vectorial relation for the integral (A21).

The vectorial relations are shown in Fig. 5. This time the three vectors p, q, k and the lengths of the vectors p', k', $p'+q$ are fixed, so that the shape of the tetrahedron in Fig. 5 and its base pqk are fixed, while the apex P is movable with one degree of freedom in the four-dimensional space. The integration may be factorized into that within the space of the tetrahedron and that in the complementary (space-like) space. The first yields the Jacobian

$$(uws)^{\frac{1}{2}}/\Delta$$

where Δ is the volume of the parallelopiped formed with the vectors p', k', $p'+q$. The second gives the perimeter of a circle

$$2\pi h',$$

where h' is the distance between P and the base plane pqk. Thus

$$\mathcal{J}_0'=\frac{\sum}{8(uws)^{\frac{1}{2}}}\frac{(uws)^{\frac{1}{2}}}{\Delta}2\pi h'=\frac{\sum h'}{4\Delta}=\frac{\sum h'}{2Sh'}=\frac{\sum}{2S}, \quad \text{(A22)}$$
$$\sum\equiv\sum_{\pm}\theta(\pm p)\theta(\mp k),$$

S being the area of the base triangle.

The factor $ip'\gamma$ in $\mathcal{J}_1'$ goes, as before, into its projection onto the plane pqk, so that

$$\mathcal{J}_1'=[i(ap+bq,\gamma)]\mathcal{J}_0. \quad \text{(A23)}$$

a and b are determined from

$$(pp')=ap^2+a(pq),\quad (qp')=a(pq)+bq^2,$$

together with

$$(pp')=(p^2+p'^2-k'^2)/2=(p^2+w-u)/2,$$
$$(p'q)=-(p'^2+q^2-(p'+q)^2)/2=(-q^2+u-s)/2, \quad \text{(A24)}$$
$$(pq)=-(p^2+q^2-k^2)/2.$$