Quantum Calculation of Coulomb Excitation. I*

L. C. BIEDENHARN, *The Rice Institute, Houston, Texas*

AND

J. L. McHALE AND R. M. THALER, *University of California, Los Alamos Scientific Laboratory, Los Alamos, New Mexico*

(Received May 4, 1955)

A complete quantum mechanical treatment for Coulomb excitation is presented. Angular correlations as well as total cross section are developed. The reduction to a form suitable for computation of the radial Coulomb integrals which are needed to evaluate the transition matrix elements is discussed in detail. The relation of the quantum mechanical treatment to the classical treatment is established, and it is shown that the results may be presented as the classical result plus a quantum mechanical correction.

INTRODUCTION

COULOMB excitation of nuclear levels is an old concept in nuclear physics which was first considered in the pioneer work of Cockcroft and Walton. The study of this process has advanced considerably both experimentally[1] and theoretically[2] since that time. Interest in Coulomb excitation was further stimulated by the work of Bohr and Mottelson[3] who indicated the importance of the role played by electric quadrupole excitation. Numerous experiments have since verified this conclusion.

The increase in the number and accuracy of the experiments has brought about an increased need for accurate theoretical discussion of Coulomb excitation. Ter-Martirosyan[2] has treated the problem from the classical point of view. In this work the charged particles are considered as traveling in definite Kepler orbits with the Coulomb field causing the nuclear transition. To be valid this approximation requires that[4]

$$\eta \equiv z_1 z_2 e^2/\hbar v \gg 1.$$

For many experiments, however, $\eta \sim 3$. This approximation further requires that the energy absorbed by the nucleus be small compared to the energy of the incident particles. The second requirement results from the fact that $\hbar \rightarrow 0$ implies $\hbar\omega = E \rightarrow 0$, so that no energy loss is considered in the classical treatment. For both large and small bombarding energies, therefore, the classical approximation fails.[5]

The present experimental results on the total cross section have, nevertheless, been in agreement with the classical predictions. Not only is the shape of the excitation function reproduced, but in several cases the absolute cross section has been measured and found to be in excellent agreement.[6] It should be noted, however, that the agreement exceeds the accuracy ($\sim$10%) of the experiments.

In addition to measurements on the total cross section, one may also measure angular distributions to obtain further information concerning the Coulomb excitation process. A particularly desirable measurement considers the directional correlation with incident beam of the gamma rays emitted by the excited nuclei, averaging over emergent particle directions. Calculations on this process have been made using Ter-Martirosyan's methods.[7] Although there are still only a few experiments of this type, the results seem to be clear cut. It is found that the measured results[8] deviate quite significantly ($\sim$20%) from the classical predictions, in a region where the total cross section shows good agreement. For some cases the anisotropy measured is *larger* than predicted. Since external influences always tend to diminish the anisotropy, this may indicate that the classical approximation is at fault.

Several quantum calculations[9,10] have been made to test the classical approximation. However, these calculations have neglected energy loss. Breit and collaborators[9] numerically calculated the total cross section for $\eta = 8.156$ and found excellent agreement between the quantum and classical result. Biedenharn

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] Lark-Horowitz, Risser, and Smith, Phys. Rev. **55**, 878 (1939); C. McClelland and C. Goodman, Phys. Rev. **91**, 760 (1953); T. Huus and C. Zupančič, Kgl. Danske Videnskab. Selskab, Mat-fys. Medd. **28**, No. 1 (1953); G. M. Temmer and N. P. Heydenburg, Phys. Rev. **93**, 351, 906 (1954), among others more recently. See also a forthcoming review article by A. Bohr and B. R. Mottelson.

[2] L. Landau, Physik Z. U.S.S.R. **1**, 88 (1932); V. Weisskopf, Phys. Rev. **53**, 1018 (1938); E. Guth, Phys. Rev. **68**, 280 (1945); R. Huby and H. C. Newns, Proc. Phys. Soc. (London) **A64**, 619 (1951); C. Mullin and E. Guth, Phys. Rev. **82**, 141 (1951); K. A. Ter-Martirosyan, J. Exptl. Theoret. Phys. (U.S.S.R.) **22**, 284 (1952).

[3] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat-fys. Medd. **27**, No. 16 (1953).

[4] N. Bohr, Kgl. Danske Videnskab. Selskab, Mat-fys. Medd. **18**, No. 8 (1948).

[5] K. A. Ter-Martirosyan [J. Exptl. Theoret. Phys. (U.S.S.R.) **22**, 284 (1952)], has also given classical calculations that apply for the case where $\Delta E \sim E$, but numerical results are not available. The statement is meant to apply only to calculations based upon no energy loss as given in extensive tabulations of the report by K. Alder and A. Winther, CERN/T/KA-AW-1, October, 1954 (unpublished).

[6] F. K. McGowan and P. H. Stelson, Oak Ridge National Laboratory Report, ORNL 1798, December, 1954 (unpublished); Phys. Rev. **99**, 112 (1955).

[7] K. Alder and A. Winther, Phys. Rev. **91**, 1578 (1953).

[8] P. H. Stelson and F. K. McGowan, Phys. Rev. **98**, 249(A) (1955); F. K. McGowan and P. H. Stelson, Phys. Rev. **99**, 127 (1955).

[9] G. Breit and P. B. Daitch, Phys. Rev. **96**, 1447 (1954); Daitch, Lazarus, Hull, Benedict, and Breit, Phys. Rev. **96**, 1449 (1954); K. Alder and A. Winther, Phys. Rev. **96**, 237 (1954).

[10] L. C. Biedenharn and C. M. Class, Phys. Rev. **98**, 691 (1955).

and Class[10] employed the exact Coulomb integrals for no energy loss, given in Sec. II, and found that the classical approximation was in excellent agreement with the exact results for $\eta > 3$. For the directional correlations, however, this calculation predicted larger anisotropies than the classical treatment.

Since these considerations indicate that the classical approximation is inadequate, it appeared reasonable to attempt to take the Coulomb field rigorously into account in a quantum mechanical treatment.[11] In this connection it is of interest to note that for dipole transitions, the problem was effectively solved in parabolic coordinates by Sommerfeld[12] many years ago. The cases of greatest interest are, however, the more complicated quadrupole transitions. For multipoles higher than the dipole it appears futile to attempt the integrations in parabolic coordinates, and recourse to the spherical representation is indicated.

In the following, therefore, the problem is formulated in the spherical representation and expressions for both the total cross section and angular correlations are obtained. The formulation of the angular correlation problem is considerably simplified by recognition of the formal analogy to the correlation problem in the internal conversion process.[13] For internal conversion, transitions between a bound state and a continuum state must be considered, whereas here the transitions are between two continuum states. Moreover, for the final state, there is no coherent mixing of different angular momenta because it is assumed that this state is not observed experimentally.

The second part of this paper is concerned with the problem of evaluating the radial Coulomb integrals. Since the excitation and correlation involve infinite sums over angular momenta, of the radial Coulomb matrix elements, it is necessary to estimate the number of L's which enter in a significant way. To a sufficient approximation this can be done as follows. For the quadrupole case the radial Coulomb matrix element is of the form $[\int_0^\infty drr^{-3}F_L(\eta_1,k_1r)F_L(\eta_2,k_2r)]^2$. For large values of L, the magnitude of the Coulomb function $F_L(\eta,kr)$ at the classical turning point, $kr\sim L$, varies as $L^{1/6}$.[14] Because of this slow variation the crude estimate is made that the amplitude of F_L is unity for all L beyond the turning point. Inside the turning point, the functions F_L behave roughly as $(r/L)^{L+1}$. Thus, the contribution to the integral from $r<L$ is of the order of L^{-3}. Outside the turning point, the Coulomb functions F_L oscillate. If one neglects the phase difference between the two F_L's and takes their product to be of the order of unity everywhere, one obtains an overestimate of the outside contribution to be of the order of L^{-2}. Thus one gets as a rough estimate that the integrals are, in general, of the order of L^{-2}. Using this estimate in the summation for the total cross section, it is clear that the sum will involve, in the limit of large L, a summation over terms of the order of $L[\int_0^\infty dr \times r^{-3}F_L(\eta_1,k_1r)F_L(\eta_2,k_2r)]^2\sim L^{-3}$, and thus this part of the sum will be of the order of L^{-2}. For good accuracy one, therefore, needs about one hundred terms and it is clearly essential to simplify the work as far as possible. Direct numerical integration of the radial integrals does not appear feasible, since the Coulomb functions oscillate and many oscillations must be integrated before asymptotic formulas are applicable. It would thus seem most desirable to use the properties of the Coulomb functions to cast the required integrals into a more tractable form.

This is, however, not a very straightforward task. The range of physical interest has the parameter $\rho\equiv k_i/k_f$ of the order of unity. Typical experimental values are $1<\rho<1.1$. But for $\rho\sim 1$ the integrals do not have simple properties. A simple example of such behavior is afforded by the related spherical Bessel function integrals, to which the desired integrals reduce in the limit of zero nuclear charge. These are cases of the Sonine-Schafheitlin discontinuous integral.[15] It is possible, nonetheless, to get useful rapidly converging series in the region of $\rho\sim 1$. The necessary development for such results are presented in this paper. In addition, various recursion relations are also presented. The use of the recursion relations makes it unnecessary to compute the integrals for all values of L. Rather, the integrals may be computed at appropriate intervals in L and the recursion relations used to bridge the gaps. However, repeated application of these relations causes great loss in numerical accuracy, so that they can only be used over a very limited range of L.

The classical orbit calculation results from the present quantum treatment under the simultaneous limiting process $\eta\to\infty$; $\rho\to 1$; $\eta(1-\rho)\to\xi=$finite. From the series results for the Coulomb integrals one gets in this limit confluent forms, which are the series form of the classical integrals. It is interesting also to obtain this result in a more elementary fashion without the necessity of using series, and this is done in the concluding section. Sommerfeld[12] has discussed a closely related limiting form for the electric dipole case to obtain Kramer's approximate bremsstrahlung formula.[16]

The radial Coulomb matrix elements should be valuable in problems other than Coulomb excitation and it is hoped that the present discussion will be of more general interest.

Numerical calculations of electric quadrupole excitation have been completed and submitted for publication.

[11] A preliminary account was given by L. C. Biedenharn and M. E. Rose, Oak Ridge National Laboratory Report, ORNL 1789, September 1954 (unpublished).

[12] A. Sommerfeld, *Wellenmechanik* (F. Ungar Publishing Company, New York, 1947).

[13] Rose, Biedenharn, and Arfken, Phys. Rev. **85**, 5 (1952).

[14] Biedenharn, Gluckstern, Hull, and Breit, Phys. Rev. **97**, 542 (1955).

[15] W. Magnus and F. Oberhettinger, *Special Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1949), p. 35.

[16] H. A. Kramers, Phil. Mag. **46**, 836 (1923).

I. FORMAL DEVELOPMENT

The Coulomb excitation problem can be schematically described as the following process. A (Coulomb-distorted) plane wave of spinless particles, of wave vector $\mathbf{k}_1$ and charge z_1 is incident upon a nucleus of charge z_2. By means of the electromagnetic coupling between the nucleus and incident particles, the nucleus undergoes a a transition from a state $J_i\pi_i$ to a state $J_f\pi_f$, and the incident particles emerge at infinity as a (Coulomb distorted) plane wave of wave vector $\mathbf{k}_2$. Here J, π represent the nuclear angular momentum and parity. We seek to find first: the total cross section for this process, averaged over the directions of $\mathbf{k}_2$, and second: the directional correlation with respect to the incident direction $\mathbf{k}_1$ of a subsequent nuclear radiation, again averaging over the directions of the emergent particles, $\mathbf{k}_2$. The center of mass of the system is considered to be at the nucleus.

Total Cross Section

The electromagnetic coupling can be expressed in terms of the interaction energy density between the two transition charge and current systems (nuclear and bombarding particles), that is:

$$H_{\text{int}}=(\mathbf{j}_N\cdot\mathbf{G}\cdot\mathbf{j}_P+\boldsymbol{\varrho}_N\cdot\boldsymbol{\varrho}_P g). \qquad (1)$$

Here $\mathbf{j}_N$ and $\mathbf{j}_P$ are the current operators for the nuclear and particle systems, respectively, and similarly the $\boldsymbol{\varrho}$'s are charge density operators. The $\mathbf{G}$ and g are dyadic and scalar Green's functions for the infinite domain with outgoing waves at infinity. Introducing a multipole expansion for the Green's functions,[17] and assuming that the bombarding particles always remain exterior to the nucleus, one may write for the Lth magnetic multipole the interaction energy density,

$$\begin{aligned}H_L^{(m)}=4\pi\textstyle\sum_M(-ik)\mathbf{j}_N\cdot[L(L+1)]^{-\frac{1}{2}}\\ \times j_L(kr_N)[\mathbf{L}Y_L^{M*}(\theta_N,\phi_N)]\\ \times\mathbf{j}_P\cdot[L(L+1)]^{-\frac{1}{2}}h_L^{(1)}(kr)\mathbf{L}Y_L^M(\theta,\phi)\end{aligned}$$

and for the Lth electric multipole,

$$\begin{aligned}H_L^{(e)}=4\pi\textstyle\sum_M i[L(L+1)]^{-\frac{1}{2}}j_L(kr_N)\mathbf{j}_N\cdot\boldsymbol{\nabla}\\ \times\mathbf{L}Y_L^{M*}(\theta_N,\phi_N)k^{-1}[L(L+1)]^{-\frac{1}{2}}\\ \times h_L^{(1)}(kr)\mathbf{j}_P\cdot\boldsymbol{\nabla}\times\mathbf{L}Y_L^M(\theta,\phi). \qquad (2)\end{aligned}$$

Here the $j_L(kr)$ are spherical Bessel functions, the $h_L^{(1)}(kr)$ are spherical Hankel functions, and k is the wave number for the virtual gamma quantum absorbed by the nucleus. The operator $\mathbf{L}$ is the rotation operator, $\mathbf{L}=-i\mathbf{r}\times\boldsymbol{\nabla}$. The coordinates r_N, θ_N, ϕ_N refer to the nucleus and r, θ, ϕ are the coordinates of the bombarding particle.

The usual perturbation treatment then leads immediately to the result for the total cross section for excitation by electric (e) or magnetic (m) multipole of order L.

$$\begin{aligned}\sigma_L^{(e,m)}=4\pi\frac{k_2}{k_1}\left(\frac{2m}{\hbar^2}\right)^2\mathfrak{S}\int d\mathfrak{k}_2|\langle\psi_2(\mathbf{k}_2,\mathbf{r})\Psi_f|\\ \times H_L^{(e,m)}|\Psi_i\psi_1(\mathbf{k}_1,\mathbf{r})\rangle|^2. \qquad (3)\end{aligned}$$

In Eq. (3), $\mathfrak{S}$ denotes the appropriate summation over final states, and averaging over initial states for the nucleus (wave functions Ψ_f and Ψ_i, respectively), and $\mathfrak{k}$ is a unit vector in the $\mathbf{k}$ direction. The (Coulomb-distorted) plane waves for the bombarding particles are, for the incident state:

$$\psi_1(\mathbf{k}_1,\mathbf{r})=\left(\frac{2\pi\eta_1}{e^{2\pi\eta_1}-1}\right)^{\frac{1}{2}}e^{i\mathbf{k}_1\cdot\mathbf{r}}\,{}_1F_1(-i\eta_1,1;i(k_1r-\mathbf{k}_1\cdot\mathbf{r})), \qquad (4)$$

with $\eta_1\equiv z_1z_2e^2/\hbar v_1$, and for the emergent state:

$$\psi_2(\mathbf{k}_2,\mathbf{r})=\left(\frac{2\pi\eta_2}{e^{2\pi\eta_2}-1}\right)^{\frac{1}{2}}e^{i\mathbf{k}_2\cdot\mathbf{r}}\,{}_1F_1(i\eta_2,1;-i(k_2r+\mathbf{k}_2\cdot\mathbf{r})), \qquad (5)$$

with η_2 defined analogously.

These latter wave functions are exact wave functions for a particle of charge z_1 in the field of a fixed charge z_2 as discussed in detail by Sommerfeld.[12] They reduce to distorted plane waves at infinity, with the appropriate outgoing and incoming spherical parts respectively. The wave function ψ_2 will be observed to be the time and space-reversed emergent wave, so as to have an incoming scattered spherical component.[12]

The integral over the nucleus is evaluated formally in terms of reduced matrix elements.[18] That is, for magnetic multipoles,

$$\begin{aligned}\langle\Psi_f|[L(L+1)]^{-\frac{1}{2}}j_L(kr_N)\mathbf{j}_N\cdot\mathbf{L}Y_L^{M*}(\theta_N,\phi_N)|\Psi_i\rangle\\ \equiv C(J_iLJ_f;M_iMM_f)(f\|mL\|i), \qquad (6)\end{aligned}$$

and for electric multipoles,

$$\begin{aligned}\langle\Psi_f|k^{-1}[L(L+1)]^{-\frac{1}{2}}j_L(kr_N)\mathbf{j}_N\cdot\boldsymbol{\nabla}\times\mathbf{L}Y_L^{M*}(\theta_N,\phi_N)|\Psi_i\rangle\\ \equiv C(J_iLJ_f;M_iMM_f)(f\|eL\|i). \qquad (7)\end{aligned}$$

The projectile wave functions are expanded in the spherical representation, with the result[12]:

$$\begin{aligned}\psi_1(\mathbf{k}_1,\mathbf{r})=\sum_{l,m}[4\pi(2l+1)]^{\frac{1}{2}}\frac{e^{i(\sigma_l-\sigma_0)}}{k_1r}\\ \times F_l(\eta_1,k_1r)D_{m0}{}^l(\mathfrak{k}_1)i^lY_l{}^m(\theta,\phi) \qquad (8)\end{aligned}$$

and

$$\begin{aligned}\psi_2(\mathbf{k}_2,\mathbf{r})=\sum_{l',m'}[4\pi(2l'+1)]^{\frac{1}{2}}\frac{e^{-i(\sigma_{l'}-\sigma_0)}}{k_2r}\\ \times F_{l'}(\eta_2,k_2r)D_{m'0}{}^{l'}(\mathfrak{k}_2)^*i^{l'}Y_{l'}{}^{m'}(\theta,\phi). \qquad (9)\end{aligned}$$

Here $D_{mn}{}^l$ are rotation matrices.[19]

It is useful now to introduce the reduced matrix elements for the transition multipole between the

[17] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), Appendix B.

[18] S. P. Lloyd, Phys. Rev. 81, 161 (1951).

[19] E. P. Wigner, *Gruppentheorie* (Friedrich Vieweg und Sohn, Braunschweig, 1931).

various angular momenta of the bombarding particles. For magnetic transitions, we have:

$$\left\langle \frac{F_{l'}(\eta_2,k_2r)}{k_2r} Y_{l'}^{m'}(\theta,\phi) \times \left| [L(L+1)]^{-\frac{1}{2}} h_L^{(1)}(kr)\,\mathbf{j}_P\cdot\mathbf{L}Y_L^M(\theta,\phi) \right| \times \frac{F_l(\eta_1,k_1r)}{k_1r} Y_l^m(\theta,\phi) \right\rangle \equiv Q(l',l,Lm)C(Ll';Mmm'), \quad (10)$$

and for electric transitions:

$$\left\langle \frac{F_l(\eta_2,k_2r)}{k_2r} Y_{l'}^{m'}(\theta,\phi) \right| k^{-1}[L(L+1)]^{-\frac{1}{2}} h_L^{(1)}(kr) \times \mathbf{j}_P\cdot\nabla\times\mathbf{L}Y_L^M(\theta,\phi) \left| \frac{F_l(\eta_1,k_1r)}{k_1r} Y_l^m(\theta,\phi) \right\rangle \equiv Q(l',l;Le)C(Ll';Mmm'). \quad (11)$$

The evaluation of these reduced matrix elements is given in Appendix I.

The indicated operations in Eq. (3) can now be carried out. After some manipulation, the total cross section is found to be:

$$\sigma_L^{(e,m)} = \left(\frac{k_2}{k_1}\right)\frac{2J_f+1}{2J_i+1}\frac{(f\|(e,m)L\|i)^2}{2L+1}\cdot\left(\frac{32\pi^2mk}{\hbar^2}\right)^2 \times\sum_{l,l'}(2l'+1)|Q(l'l;L(e,m))|^2. \quad (12)$$

In the long-wavelength approximation the reduced multipole matrix elements assume a simple form. The total cross section has a more recognizable form in this limit, and for the electric quadrupole transitions we have the explicit result that:

$$\sigma_2^{(e)} = 4\pi\left(\frac{k_2}{k_1}\right)\left(\frac{2J_f+1}{2J_i+1}\right)(f\|r^2\|i)^2\left(\frac{8\pi z_1z_2me^2}{5k_1k_2\hbar^2}\right)^2 \times\sum_l\left\{\frac{l(l+1)(2l+1)}{(2l-1)(2l+3)}\left[\int_0^\infty r^{-3}drF_l(\eta_1,k_1r)F_l(\eta_2,k_2r)\right]^2 +\frac{3l(l-1)}{2(2l-1)}\left[\int_0^\infty r^{-3}drF_l(\eta_1,k_1r)F_{l-2}(\eta_2,k_2r)\right]^2 +\frac{3(l+1)(l+2)}{2(2l+3)}\left[\int_0^\infty r^{-3}drF_l(\eta_1,k_1r)F_{l+2}(\eta_2,k_2r)\right]^2\right\}. \quad (13)$$

Directional Correlation

If the nuclei excited by the bombarding particles upon returning to their ground states emit gamma rays with a half-life short compared to the reorientation times of external perturbing fields, these gamma rays will be correlated with the direction of the incident beam, and the emergent beam as well. In practice this latter direction is unobserved and corresponds to the directional correlations averaged over this direction. Calculations for such correlations, using classical trajectories, were given by Alder and Winther.[7]

The calculation of this directional correlation is greatly facilitated by the result[20] that if one knows the $\gamma-\gamma$ correlation for the nuclear transitions characterized by the angular momenta $J_1\rightarrow J_2\rightarrow J_3$ to be,

$$W(\theta)=\sum_\nu A_\nu P_\nu(\cos\theta),$$

then the directional correlation for particle emission replacing the γ transition $J_1\rightarrow J_2$ is given by

$$W(\theta)=\sum_\nu A_\nu a_\nu P_\nu(\cos\theta),$$

where the a_ν are "particle parameters" independent of the nuclear transitions. The problem for the directional correlation is thus reduced to solution by standard techniques. The solution is, however, even more immediate if one notes the close formal similarity between this Coulomb excitation problem and the internal conversion problem. Except for the fact that the present problem involves both initial and final particle states in the continuum, the two problems are precisely the inverse of each other. The calculation given below exploits this similarity, and does not reproduce details given by Biedenharn and Rose.[20] As discussed there, the particle parameters are the ratio of the tensor parameters of the projectile to the tensor parameters of the gamma ray it replaces, normalized so that $a_0=1$. To obtain the tensor parameters of the projectile we couple the tensor parameters of the observed initial state and the tensor parameters of the unobserved final state. The latter, being random, are scalar.

The tensor parameters of the initial state[21] are:

$$R_i(\nu q;ll')=4\pi[(2l+1)(2l'+1)]^{\frac{1}{2}}(-)^{l-q}i^{l-l'} \times C(ll'\nu;000)e^{i(\sigma_l-\sigma_{l'})}D_{q0}^{\nu}(\mathfrak{k}_1). \quad (14)$$

For the final state we have the simple result:

$$R_f(\nu q)=\delta_{0\nu}\delta_{0q}.$$

Using the previous definition, Eq. (10), for the reduced multipole matrix elements, one finds that the coupled tensor parameters, for a pure multipole transition are:

$$R(\nu q)=\sum_{ll'\bar{l}} Q(l\bar{l};L\pi)Q^*(l'\bar{l};L\pi)[(2l+1)(2l'+1)]^{\frac{1}{2}} \times W(l\bar{l}\nu L;Ll')R_i(\nu q;ll'). \quad (15)$$

[20] L. C. Biedenharn and M. E. Rose, Revs. Modern Phys. **25**, 729 (1953). This result is originally due to S. P. Lloyd, thesis, University of Illinois, 1951 (unpublished).

[21] *Note added in Proof.*—In the preliminary work of Biedenharn and Rose[11] the phase factor $i^{l-l'}$ in Eq. (14) was accidentally omitted. This error was propagated in the work of Biedenharn and Class[10] but affected only Eq. (6) and Fig. 2 of that paper (an erratum has been submitted). The equations in this paper have been corrected accordingly.

Unfortunately an equivalent error (of different origin, however) was contained in the calculations of Alder and Winther[5,7] invalidating their results for a_2 and a_4 (but not b_0). This led to a spurious agreement between the classical limit for a_2 in reference 10 and the no energy loss result for a_2 in reference 7.

The tensor parameters of the gamma transition whose observation is replaced by the particle transition are:

$$R_\gamma(\nu q) \sim (-)^{q+L+1} C(L\,L\,\nu;\,1\,-1\,0) D_{q0}{}^{\nu}(\mathfrak{k}). \quad (16)$$

Thus one immediately obtains for the particle parameters, the result:

$$a_\nu(L,\pi) = \{\sum_{ll'\bar{l}} (2l+1)(2l'+1)(-)^{l} C(ll'\nu;\,000) i^{l-l'}$$
$$\times e^{i(\sigma_l - \sigma_{l'})} Q(l\bar{l};\,L\pi) Q^*(l'\bar{l};\,L\pi) W(l\bar{l}\nu L;\,Ll')\}$$
$$\times \{(-)^{L+1} C(LL\nu;\,1-10) \sum_{l\bar{l}} (2l+1) |Q(l\bar{l};\,L\pi)|^2\}^{-1}. \quad (17)$$

The index π denotes the type of multipole, i.e., electric (e) or magnetic (m).

By virtue of the vector coupling relationship, the values of l, l' are restricted to lie between $\bar{l}+L$ and $|\bar{l}-L|$. Hence in the triple sum over l, l' and $\bar{l}$ only one of these represents an infinite summation. Moreover, parity eliminates approximately half of the values available for l or l'. Since the total cross section itself was a singly infinite summation, it follows that both the total cross section and the directional correlation are tasks of about the same level of difficulty, with the primary task the rapid evaluation of the reduced multipole matrix elements for Coulomb waves.

For the long wavelength limit, the particle parameters assume a much simpler form. Confining attention to the experimentally interesting electric quadrupole case, the explicit results are found to be

$$a_2(2e) = b_2/b_0, \quad (18)$$

$$a_4(2e) = b_4/b_0, \quad (19)$$

with

$$b_0 = \sum_l \left[\frac{l(l+1)(2l+1)}{(2l-1)(2l+3)} I^2(l,l) + \frac{3l(l-1)}{2(2l-1)} I^2(l-2,\,l) + \frac{3}{2}\frac{(l+1)(l+2)}{2l+3} I^2(l+2,\,l) \right], \quad (20)$$

$$b_2 = \sum_l \left[\frac{3(l-2)(l-1)(l)}{(2l-1)^2} I^2(l-2,\,l) - \frac{l(l+1)(2l+1)(2l-3)(2l+5)}{(2l-1)^2(2l+3)^2} I^2(l,l) + \frac{3(l+1)(l+2)(l+3)}{(2l+3)^2} I^2(l+2,\,l) - 6\cos(\sigma_l - \sigma_{l-2}) \frac{l(l-1)(l+1)}{(2l-1)^2} I(l-2,\,l) I(l,l) - 6\cos(\sigma_l - \sigma_{l+2}) \frac{l(l+1)(l+2)}{(2l+3)^2} I(l+2,\,l) I(l,l) \right], \quad (21)$$

$$b_4 = -\frac{1}{16} \sum_l \left[\frac{9l(l-1)(l-2)(l-3)}{(2l+1)(2l-1)^2} I^2(l-2,\,l) + \frac{36(2l+1)(l+2)(l+1)(l)(l-1)}{(2l+3)^2(2l-1)^2} I^2(l,l) + \frac{9(l+4)(l+3)(l+2)(l+1)}{(2l+1)(2l+3)^2} I^2(l+2,\,l) \right.$$
$$- 60\cos(\sigma_{l-2} - \sigma_l) I(l,l) I(l-2,\,l) \times \frac{(l+1)(l)(l-1)(l-2)}{(2l-1)^2(2l+3)} + 210\cos(\sigma_{l-2} - \sigma_{l+2}) \times \frac{(l+2)(l+1)(l)(l-1)}{(2l-1)(2l+1)(2l+3)} I(l-2,\,l) I(l+2,\,l)$$
$$\left. - 60\cos(\sigma_l - \sigma_{l+2}) \frac{(l+3)(l+2)(l+1)(l)}{(2l-1)(2l+3)^2} \times I(l,l) I(l+2,\,l) \right]. \quad (22)$$

The σ_l in Eqs. (20)–(22) are the Coulomb phase shifts, $\sigma_l = \arg[\Gamma(l+1+i\eta_1)]$, and the radial integrals are defined as

$$I(l,l') \equiv \int_0^\infty r^{-3} dr F_l(\eta_1, k_1 r) F_{l'}(\eta_2, k_2 r). \quad (23)$$

The sums are over all positive values of l including zero, with the understanding that $I(l,l') = 0$ if $l + l' < 1$.

Limitations of the Present Treatment

In this treatment there are several approximations, and it is essential to make clear the errors involved. First there is the question of replacing the multipole moments by the long-wavelength approximation. The parameter involved here can, for an order of magnitude orientation, be taken to be $(r_{\text{t.p.}}/\lambda\!\!\!{}^{-}_{\text{rad}})$, where $r_{\text{t.p.}}$ is the classical turning point radius given approximately by $(L^2+\eta^2)^{\frac{1}{2}}$. For low angular momenta, this parameter ranges from $\sim 1/100$ (for 4-Mev protons exciting 100-kev radiation on $Z=50$) to $\sim 1/5$ (for 4-Mev alphas exciting 500-kev radiation on $Z=50$). Except for the radial current contributions in electric multipoles, this parameter enters as the square; it is clear, however, that there will exist cases where the error is not negligible. For sufficiently high angular momentum the error is always large, but since the dominant contributions are from $L \sim \eta$ (except for $E1$) this need not introduce appreciable error. For cases where retardation is appreciable, the contribution of the higher terms gives rise to matrix elements which can be evaluated by the methods given below, and, in fact, these corrections are generally simpler to evaluate than the lower order terms. Because retardation effects may be important, the accurate form for the multipoles has been given in Appendix I.

The neglect of electronic shielding is another possible source of error. The parameter involved here is the ratio of the Bohr K shell radius to the turning point radius. For 2-Mev protons on $Z=50$, this parameter is $\sim 1/30$ for $L=0$. For increasing L, however, the comparison is less favorable, and, in fact, becomes sizable for $L\sim 20$. Drell and Huang[22] have carried out an estimate of the screening effect for 2-Mev proton bremsstrahlung on $Z=50$ and find the correction negligible (<0.1 percent). Since the large angular momenta contribute far more prominently to dipole bremsstrahlung than for quadrupole excitation[23] their result indicates that shielding may be safely neglected.

The effects of penetration of the nucleus have been neglected, and this is possibly important for S wave particles. For 5-Mev protons on $Z=50$, the nuclear radius is about one half the turning point radius and the penetration of the order of a few percent or less. This is a rather extreme case, but it indicates that, in the vicinity of resonances at least, the $L=0$ contributions might have to be altered by the addition of irregular components. This constitutes a refinement of the theory that seems somewhat premature, but it is well to bear in mind the possibility of such effects.

The assumption that the nucleus defines the center of mass may be a serious source of error. Although it is easy to calculate the particle multipole moments in terms of the true center of mass, this is no real help since the nuclear absorption takes place with multipoles based upon its own center of mass, and the error caused by the non-coincidence of these two centers still arises. The effect is particularly bad for alpha particles on light nuclei. For such cases, in the center-of-mass system, the effective dipole moment will almost vanish, yet the translation of the remaining multipoles to the nuclear center-of-mass system will reintroduce dipole terms. The approximation is thus poor in some cases of interest. Although corrections can be made for this effect, attention will be restricted to larger $A(\sim 100)$ where the error should be of the order of one percent or less.

Finally it should be mentioned, for completeness, that spinless particles have been assumed. For the case of protons, the spin magnetization should enter the magnetic multipoles significantly; for electric multipoles the change is generally insignificant. This effect may be treated by using the spin magnetization current in the magnetic multipoles, as discussed in reference 17.

II. RADIAL COULOMB INTEGRALS

In this section, explicit integration of the general Coulomb radial matrix element $(m,n;L)$ is discussed. $(m,n;L)$ is defined as

$$(m,n;L)\equiv\int_0^\infty drr^{-n}F_L(\eta_1,k_1r)F_{L+m}(\eta_2,k_2r), \quad (24)$$

where k_1 and k_2 are respectively the "incoming" and "outgoing" wave numbers. The parameter η is given by

$$\eta\equiv z_1z_2e^2/\hbar v,$$

and the Coulomb wave function[24] is defined in terms of the confluent hypergeometric function to be

$$F_L(\eta,r)=C_L(\eta)r^{L+1}e^{-ir}\,{}_1F_1(L+1-i\eta;2L+2;2ir),$$

with the normalizing factor

$$C_L(\eta)=2^Le^{-\pi\eta/2}|\Gamma(L+1+i\eta)|/\Gamma(2L+2).$$

A more convenient form in which to treat the integral of Eq. (24) is

$$(m,n;L)=k_2^{n-1}\int_0^\infty drr^{-n}F_L(\eta,\rho r)F_{L+m}(\rho\eta,r), \quad (25)$$

where $0<\rho\equiv k_1/k_2<\infty$, and $\eta=\eta_1$. Although on physical grounds ρ is restricted to values greater than or equal to unity, it is convenient to discuss also cases with ρ less than unity. This enables one to restrict m to be zero or a positive integer. Since the integral must converge at the origin, the inequality $2L+3+m-n>0$ must hold. In addition from the vector addition rule one has generally that $m\leqslant n-1$, but if retardation corrections enter, m may exceed $n-1$.

In order to transform the integral into more useful forms, it is necessary at this point to introduce explicitly into the integral the convergence factor e^{-sr}. For all values of s such that $\mathrm{Re}\,s>0$, the resulting integral is then uniformly convergent. The limit $s\to 0$ will be taken in the final answer, and it is this limiting operation that introduces the discontinuous behavior at $\rho=1$ discussed in the introduction. The integral with the convergence factor s is denoted by $(m,n;L;s)$. Employment of the Euler-type integral representation for the confluent hypergeometric functions,[25] interchange of the order of integration, and integration over the r-coordinate yields a double contour integral form for the matrix element:

$$(m,n;L;s)=K\int_0^{(1+)}du\int_0^{(1+)}\times dv[s+i(1+\rho)-2i(v+\rho u)]^{n-2L-m-3}\times u^{L-i\eta}(u-1)^{L+i\eta}v^{L+m-i\rho\eta}(v-1)^{L+m+i\rho\eta}, \quad (26)$$

[22] S. D. Drell and K. Huang, Phys. Rev. **99**, 686 (1955). A preliminary account of this work appears in Bull. Am. Phys. Soc. **30**, No. 3, 28 (1955).

[23] Similar considerations show that estimating errors in WKB calculations by comparison to the Sommerfeld bremsstrahlung results [see K. Alder and A. Winther, Phys. Rev. **96**, 237 (1954)] should not necessarily be conclusive.

[24] *Tables of Coulomb Wave Functions*, U. S. National Bureau of Standards, Applied Mathematics Series 17 (U. S. Government Printing Office, Washington, D. C., 1952).

[25] Erdelyi, Magnus, Oberhettinger, and Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 272. We shall in the sequel denote this reference as HTF-1.

with

$$K=(-1)^m k_2^{n-1}\rho^{L+1}2^{2L+m-2}e^{-\frac{1}{2}\pi\eta(1+\rho)}\times\frac{\Gamma(2L+3+m-n)}{\sinh\pi\eta\,\sinh\pi\rho\eta|\Gamma(L+1+i\eta)||\Gamma(L+m+1+i\rho\eta)|}.$$

The many-valued functions appearing in the integrand are made analytic by taking their principal values. The symbol $\int_0^{(1+)}$ signifies a loop which begins and ends at the origin and encircles the point 1 counter-clockwise. In the case at hand, both loops can be considered closed contours encircling the points 0 and 1 of the integration variables.

The double integral appearing in Eq. (26) is readily expressed in terms of Appell's double hypergeometric series[26] $F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y)$.

The result is

$$\begin{aligned}(m,n;L;s)=&-(k_2^{n-1}/4\pi^2)\rho^{L+1}C_L(\eta)C_{L+m}(\rho\eta)\\&\times\Gamma(2L+m+3-n)[s+i(1+\rho)]^{n-2L-m-3}\\&\times F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y),\end{aligned}\quad(27)$$

where $\alpha=2L+m+3-n$, $\beta=L+1-i\eta$, $\beta'=L+m+1-i\rho\eta$, $\gamma=2L+2$, $\gamma'=2L+2m+2$, $x=2i\rho/[s+i(1+\rho)]$ and $y=2i/[s+i(1+\rho)]$. For calculational purposes, the double series (see Appendix II) for F_2 might be thought to be useful. Since the series form of F_2 converges absolutely only for

$$|x|+|y|=|2i\rho/[s+i(1+\rho)]|+|2i/[s+i(1+\rho)]|<1,$$

it is evident, however, that the limit $s\rightarrow 0$ can never taken and the series is, therefore, only of formal interest. Any transformation of the series which does not have the character of an analytic continuation cannot alter this conclusion.

For analytic continuation in a single variable, e.g., y, the theory of the Gauss hypergeometric function can be used to advantage. Such transformations, however, are insufficient to treat the case $\rho\sim 1$. For the parameters given in Eq. (27) with the specialization $m=n-1$, the desired analytic continuation has been given by Appell and Kampe de Feriet.[27] It will be shown below that the specialization, $m=n-1$, is not necessary, and that all cases of interest for Coulomb excitation can be worked out directly.

The special case $m=n-1$ with $m=0$, was treated by Sommerfeld,[12] who obtained the result:

$$\begin{aligned}(0,1,L;s)&=-\frac{1}{4\pi^2}C_L(\eta)C_L(\rho\eta)\Gamma(2L+2)[s+i(1+\rho)]^{-2L-2}\\&\times\left(\frac{s+i(1-\rho)}{s+i(1+\rho)}\right)^{-L-1+i\eta}\left(\frac{s+i(\rho-1)}{s+i(\rho+1)}\right)^{-L-1+i\rho\eta}\\&\times\rho^{L+1}{}_2F_1\left(\beta,\beta';\alpha;\frac{-4\rho}{s^2+(1-\rho)^2}\right).\end{aligned}\quad(28)$$

Here α, β, β', are as defined in Eq. (27), with $m=n-1=0$. For $\rho\sim 1$ the usual theory of the hypergeometric function can be used to obtain series in the variable $1-\rho$.

Series Applicable for Small Energy Loss

The analytic continuation of Eq. (27) is now obtained in a form which readily permits the confluences required in the classical limit. For convenience the order of integration in Eq. (26) is inverted. It is seen that the integrand as a function of u has branch points at 0 and 1, and a pole at $[s+i(1+\rho)]/2i\rho-(v/\rho)$. For $m<n-1$ there is also a pole at $u=\infty$. The integral over u can be evaluated in the usual way in terms of the two poles outside the original contour encircling the singularities at $u=0$ and $u=1$. The contribution due to the pole at infinity is called $f_1(v)$, while the contribution of the other pole is called $f_2(v)$. Thus,

$$\int_0^{(1+)}du\,u^{L-i\eta}(u-1)^{L+i\eta}\times[s+i(1+\rho)-2i(v+\rho u)]^{-2L-m-3+n}\equiv f_1(v)+f_2(v).\quad(29)$$

The evaluation of $f_1(v)$ yields

$$\begin{aligned}f_1(v)=&2\pi i(2i\rho)^{n-2L-m-3}\\&\times\frac{\Gamma(L+1+i\eta)}{\Gamma(L+m+3-n+i\eta)\Gamma(n-m-1)}\\&\times\sum_\lambda\frac{(2L+m+3-n)_\lambda(m+2-n)_\lambda}{\lambda!(L+m+3-n+i\eta)_\lambda}\\&\times\left[\frac{s+i(1+\rho)-2iv}{2i\rho}\right]^\lambda.\end{aligned}\quad(30)$$

The notation

$$(a)_q\equiv a(a+1)(a+2)\cdots(a+q-1)=\Gamma(a+q)/\Gamma(a)$$

is used in the series above. It will be noted that $f_1(v)=0$ for $n\leqslant m-1$, and is a terminating series for $n>m-1$.

The integration over v (for the $f_1(v)$ contribution) is carried out next. This is found to be

$$\begin{aligned}&\int_0^{(1+)}dv\,f_1(v)v^{L+m-i\rho\eta}(v-1)^{L+m+i\rho\eta}\\&=4\pi i(-)^{L+m}\sinh\pi\rho\eta(2i\rho)^{n-2L-m-3}\\&\times\frac{\Gamma(L+1+i\eta)|\Gamma(L+m+1+i\rho\eta)|^2}{\Gamma(L+m+3-n+i\eta)\Gamma(n-m-1)\Gamma(2L+2+2m)}\\&\times\sum_{\lambda,\mu}\frac{(2L+m+3-n)_\lambda(m+2-n)_\lambda(L+m+1-i\rho\eta)_\mu}{\mu!(\lambda-\mu)!(L+m+3-n+i\eta)_\lambda(2L+2+2m)_\mu}\\&\times\left(\frac{s+i(1+\rho)}{2i\rho}\right)^\lambda\left(\frac{-2i}{s+i(1+\rho)}\right)^\mu.\end{aligned}\quad(31)$$

---

[26] HTF-1, p. 224. Some properties of the Appell functions are given in Appendix II.

[27] P. Appell and M. J. Kampe de Feriet, *Fonctions hypergeometriques et hyperspheriques* (Gauthier-Villars, Paris, 1926), p. 43.

The double series in Eq. (31) is a terminating series in both λ and μ with $0\leqslant\mu\leqslant\lambda$ and $0\leqslant\lambda\leqslant n-2-m$.

The evaluation of the contribution due to the remaining pole is somewhat more difficult, and, in contrast to the foregoing, does not lead solely to a polynomial. It is readily shown that

$$f_2(v)=(2\pi i)(2i\rho)^{n-2L-m-3}\times\frac{\Gamma(L+1-i\eta)}{\Gamma(2L+3+m-n)\Gamma(-L-i\eta+n-m-1)}\times\left[\frac{s+i(1+\rho)-2iv}{2i\rho}\right]^{-L-m+n-2-i\eta}\times\left[\frac{s+i(1-\rho)-2iv}{2i\rho}\right]^{-L-m+n-2+i\eta}\times{}_2F_1\left(-m+n-1,\,-2L-2-m+n;\;-L-i\eta-m+n-1;\frac{s+i(1+\rho)-2iv}{2i\rho}\right).\quad(32)$$

For $m<n-1$, the ${}_2F_1$ function above is represented by a terminating series consisting of $(2L+3+m-n)$ terms. In the special case $m=n-1$ the series consists of a single term equal to one.

In order to carry out the remaining integration over v, it proves useful to employ the series form of Eq. (32):

$$\int_0^{(1+)}dv f_2(v)v^{L+m-i\rho\eta}(v-1)^{L+m+i\rho\eta}\equiv\sum_\lambda P_\lambda J_\lambda,\quad(33)$$

where

$$P_\lambda=\pi(2i)^{n-2L-m-2}\rho^{m+1-n-\lambda}(-)^\lambda\times\frac{\Gamma(L+1-i\eta)}{\Gamma(2L+3+m-n)\Gamma(-L-i\eta+n-m-1)}\times\frac{(n-m-1)_\lambda(n-2L-m-2)_\lambda}{\lambda!(n-L-i\eta-m-1)_\lambda}\quad(34)$$

and

$$J_\lambda=\int_0^{(1+)}dv v^{L+m-i\rho\eta}(v-1)^{L+m+i\rho\eta}\times\left(v-\frac{1+\rho}{2}+\frac{is}{2}\right)^{-L-m-2+n-i\eta+\lambda}\times\left(v-\frac{1-\rho}{2}+\frac{is}{2}\right)^{-L-m-2+n+i\eta}.\quad(35)$$

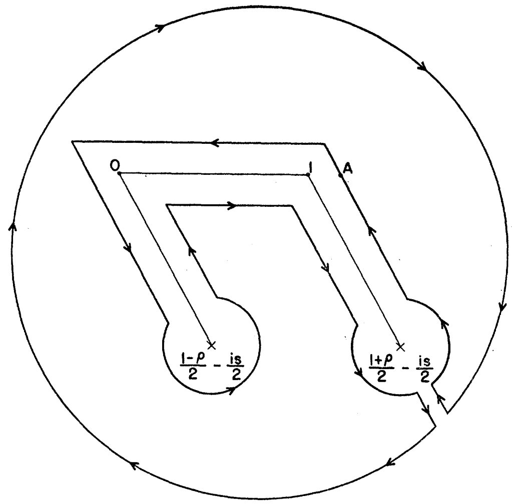

FIG. 1. The null contour D on which J_λ is defined. At the point A, $\arg v=0$. The outer portion of the contour schematically represents a circle of infinite radius.

The contour on which J_λ is defined is a loop encircling the points $v=0$ and $v=1$ in the positive sense. Consideration of the null contour D illustrated in Fig. 1 shows that the integral J_λ may be expressed in the form,

$$e^{-\pi\eta}\left(\frac{\sinh\pi\eta(\rho-1)}{\sinh\pi\eta\rho}\right)J_\lambda=\oint_\infty dv[\cdots]-\int_0^{[(\frac{1}{2}-\frac{1}{2}\rho-\frac{1}{2}is)+]}dv[\cdots]-\int_1^{[(\frac{1}{2}+\frac{1}{2}\rho-\frac{1}{2}is)+]}dv[\cdots],\quad(36)$$

where

$$[\cdots]=v^{L+m-i\rho\eta}(v-1)^{L+m+i\rho\eta}\times\left(v-\frac{1+\rho}{2}+\frac{is}{2}\right)^{-L-m-2+n-i\eta+\lambda}\times\left(v-\frac{1-\rho}{2}+\frac{is}{2}\right)^{-L-m-2+n+i\eta}.\quad(37)$$

The first integral on the right-hand side of Eq. (36) is understood to be a positive circle of large radius. The phases are determined by the requirement that the integrand be real and positive as v approaches infinity along the positive real axis, with the cuts as shown in Fig. 1. The second and third contours on the right-hand side of Eq. (36) terminate at 0 and 1, respectively.

Evaluation of the integral over the infinite circle is readily accomplished, and leads to a polynomial. Writing the result in terms of an Appell function for brevity, we find that

$$\oint_{\infty} dv[\cdots]=2\sinh[\pi\eta(1-\rho)](-)^{n+\lambda}$$
$$\times\frac{\Gamma(n+\lambda-1-i\eta(1-\rho))\Gamma(n-1+i\eta(1-\rho))}{\Gamma(2n+\lambda-2)}$$
$$\times F_2([3-\lambda-2n],[L+m-n-\lambda+2+i\eta],$$
$$[L+m-n-i\eta+2];[2-n-\lambda+i\eta(1-\rho)],$$
$$[2-n-i\eta(1-\rho)];[\tfrac{1}{2}is+\tfrac{1}{2}-\tfrac{1}{2}\rho],$$
$$[-\tfrac{1}{2}is+\tfrac{1}{2}-\tfrac{1}{2}\rho]). \quad (38)$$

We must now perform the integrations

$$\int_{0,1}^{[(\frac{1}{2}\pm\frac{1}{2}\rho-\frac{1}{2}is)+]}.$$

A simplification results from the fact that

$$\int_0^{[(\frac{1}{2}-\frac{1}{2}\rho-\frac{1}{2}is)+]} dv f_2(v)v^{L+m-i\rho\eta}(v-1)^{L+m+i\rho\eta}$$
$$=\text{complex conjugate}$$
$$\int_1^{[(\frac{1}{2}+\frac{1}{2}\rho-\frac{1}{2}is)+]} dv f_2(v)v^{L+m-i\rho\eta}(v-1)^{L+m+i\rho\eta}. \quad (39)$$

It is, therefore, necessary to perform only one of these integrations, e.g., $\int_0^{[(\frac{1}{2}-\frac{1}{2}\rho-\frac{1}{2}is)+]}$, which we shall denote as $J_\lambda^{(2)}$. In a straightforward manner we finally arrive at the result:

$$J_\lambda^{(2)}=2\pi i(-)^{\lambda+1}\left(\frac{\rho-1}{2}+\frac{is}{2}\right)^{n-1+i\eta(1-\rho)}$$
$$\times\frac{\Gamma(L+m+1-i\rho\eta)}{\Gamma(L+m+2-n-i\eta)\Gamma[n+i\eta(1-\rho)]}$$
$$\times F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y), \quad (40)$$

where

$$\alpha=2-n-\lambda+i\eta(1-\rho),\quad \beta=L+m+2-n+i\eta-\lambda,$$
$$\beta'=L+m+1-i\rho\eta,\quad \gamma=2-n-\lambda+i\eta(1-\rho),$$
$$\gamma'=n+i\eta(1-\rho),\quad x=\tfrac{1}{2}-\tfrac{1}{2}\rho+\tfrac{1}{2}is,\quad y=\tfrac{1}{2}-\tfrac{1}{2}\rho-\tfrac{1}{2}is.$$

In passing to the limit $s\to 0$ in the integrals obtained above only $J_\lambda^{(2)}$ need be examined critically. In this case, the factor $(\frac{1}{2}\rho-\frac{1}{2}+\frac{1}{2}is)^{n-1+i\eta(1-\rho)}$ has a different limit for $\rho>1$ than for $\rho<1$. The two limits are:

$$\lim_{s\to 0}(\tfrac{1}{2}\rho-\tfrac{1}{2}+\tfrac{1}{2}is)^{n-1+i\eta(1-\rho)}=\begin{cases}|\tfrac{1}{2}\rho-\tfrac{1}{2}|^{n-1+i\eta(1-\rho)}, & \text{for } \rho>1\\ |\tfrac{1}{2}\rho-\tfrac{1}{2}|^{n-1+i\eta(1-\rho)}(-)^{n+1}e^{-\pi\eta(1-\rho)}, & \text{for } \rho<1.\end{cases} \quad (41)$$

The polynomials given by Eqs. (31) and (38), together with the nonterminating double series given by Eq. (40) constitute, when introduced into Eqs. (26), (33) through (36), the desired results for $(m,n;L)$ valid in the vicinity of $\rho\sim 1$. Before explicitly exhibiting this final result, it is useful to note that the polynomials of Eqs. (31) and (38) are not, however, expressed in the most convenient form (although the present form does have the advantage of a more straightforward derivation). The present form, for example, does not exhibit the barrier penetration in a simple way.

A more useful result is obtained by utilizing in the integral over the infinite circle, Eq. (38), not the function $f_2(v)$ alone, but rather the sum $f_1(v)+f_2(v)$. The latter function is expressible as a hypergeometric function regular in the vicinity of infinity, and thus greatly simplifies the results. To compensate for the addition of the term in $f_1(v)$, we must alter the coefficient of the polynomial, Eq. (31), into which this term integrates.

It is convenient to give separate designations to these various contributions to the matrix element $(m,n;L)$. Let us define:

$$(m,n;L)=A(m,n;L)+B(m,n;L)+C(m,n;L), \quad (42)$$

where A and B are the polynomials discussed above. A arises from $f_1(v)$ and B arises from $f_1(v)+f_2(v)$. C is the nonterminating series part. Collecting the various constants that enter, one finds explicitly that:

$$A(m,n;L)=\pi i^{n-m}2^{n-3}\rho^{n-L-m-2}k_2^{n-1}\frac{e^{\frac{1}{2}\pi\eta(\rho-1)}}{\sinh\pi\eta(\rho-1)}\left|\frac{\Gamma(L+m+1+i\rho\eta)}{\Gamma(L+1+i\eta)}\right|$$
$$\times\frac{\Gamma(L+1+i\eta)}{\Gamma(L+m+3-n+i\eta)}\frac{\Gamma(2L+3+m-n)}{\Gamma(2L+2m+2)\Gamma(n-m-1)}$$
$$\times\sum_{\lambda,\mu}\frac{(2L+m+3-n)_\lambda(m+2-n)_\lambda(L+m+1-i\rho\eta)_\mu}{\mu!(\lambda-\mu)!(L+m+3-n+i\eta)_\lambda(2L+2m+2)_\mu}\left(\frac{1+\rho}{2\rho}\right)^\lambda\left(\frac{-2}{1+\rho}\right)^\mu. \quad (43)$$

$$B(m,n;L)=\pi k_2^{n-1}e^{\frac{1}{2}\pi\eta(1-\rho)}\left|\frac{\Gamma(L+1+i\eta)}{\Gamma(L+m+1+i\rho\eta)}\right|\rho^{L+1}2^{n-3}i^{-n-m}\frac{\Gamma(L+1+m-i\rho\eta)}{\Gamma(L+3-n-i\rho\eta)}\frac{\Gamma(2L+m+3-n)}{\Gamma(2L+2)\Gamma(m-1+n)}$$
$$\times\frac{1}{\sinh\pi\eta(\rho-1)}\sum_{\lambda,\mu}\frac{(L+1-i\eta)_\lambda(2L+m+3-n)_{\lambda+\mu}(2-m-n)_{\lambda+\mu}}{\lambda!(2L+2)_\lambda\,\mu!(L+3-n-i\rho\eta)_{\lambda+\mu}}\rho^\lambda\left(\frac{1-\rho}{2}\right)^\mu. \quad (44)$$

$$C(m,n;L)=\frac{\pi(k_2-k_1)^{n-1}\rho^{L+m+2-n}e^{-\frac{1}{2}\pi\eta|1-\rho|}}{2\sinh\pi\eta(\rho-1)}\times\mathrm{Re}\left\{\frac{i^{n-m}|(\rho-1)/2|^{i\eta(1-\rho)}}{\Gamma(n+i\eta(1-\rho))}\exp i\{2\sigma_{L+m+1-n}(\eta)-\sigma_L(\eta)-\sigma_{L+m}(\rho\eta)\}\right.$$
$$\times\sum_\lambda\frac{(n-m-1)_\lambda(n-2L-m-2)_\lambda}{\lambda!(n-L-i\eta-m-1)_\lambda}\rho^{-\lambda}\times F_2\left(2-n-\lambda+i\eta(1-\rho),\,L+m+2-n-\lambda+i\eta,\right.$$
$$\left.\left.L+m+1-i\rho\eta;\,2-n-\lambda+i\eta(1-\rho),\,n+i\eta(1-\rho);\,\frac{1-\rho}{2},\frac{1-\rho}{2}\right)\right\}. \quad (45)$$

It will be noted that although $A(m,n;L)$ and $B(m,n;L)$ are written in the form of terminating double series, $C(m,n;L)$ is in the form of a *triple* series, except in the case $m=n-1$, when the terminating sum over λ reduces to a single term. The advantage of the triple series form is that it permits the confluence required to obtain the classical limit. This point will be discussed in detail in a later section. An alternate expansion for $f_2(v)$ leads to a *double* series for $C(m,n;L)$, which does not, however, exhibit the confluence required for the classical limit. Since the procedure is very similar to that already used, we give only the result, Appendix III. For cases where L, η are not too large, this double series form is of value for computational purposes.

Results for Special Cases

The polynomials $A(m,n;L)$, $B(m,n;L)$ defined by the terminating series given in Eqs. (43) and (44) are much simpler than would at first appear. Consider, for example, the terms that enter in the quadrupole excitation. Here $n=3$ and $m=0$ or 2. For $m=2$ and $n=3$, the term $A(2,3;L)$ vanishes, and we have only $B(2,3;L)$ entering. This is found to be

$$B(2,3;L)=\frac{k_2^2}{6}\left|\frac{\Gamma(L+1+i\eta)}{\Gamma(L+3+i\rho\eta)}\right|e^{\frac{1}{2}\pi\eta(1-\rho)}\frac{\pi\eta(1-\rho^2)\rho^{L+2}}{2\sinh\pi\eta(\rho-1)}. \quad (46)$$

For the matrix element with $m=0$, $n=3$ both A and B enter. These are found to be

$$A(0,3;L)=k_2^2\rho^{1-L}e^{-\frac{1}{2}\pi\eta(1-\rho)}\left|\frac{\Gamma(L+1+i\rho\eta)}{\Gamma(L+1+i\eta)}\right|\times\frac{\pi\eta}{\sinh\pi\eta(\rho-1)}\frac{1}{2L(L+1)(2L+1)}, \quad (47)$$

$$B(0,3;L)=-k_2^2e^{\frac{1}{2}\pi\eta(1-\rho)}\left|\frac{\Gamma(L+1+i\eta)}{\Gamma(L+1+i\rho\eta)}\right|\times\frac{\pi\eta\rho^{L+2}}{\sinh\pi\eta(\rho-1)}\frac{1}{2L(L+1)(2L+1)}. \quad (48)$$

The special case for no energy loss ($\rho=1$) is a limit that is of some interest.[9,10] Unfortunately, however, it is apparently not possible to sum the required series (which are $_3F_2$ functions of unit argument) in general. It is, nevertheless, relatively easy to get the desired results for definite choices of m and n.

The particular case $m=n-1$ is, as usual, exceptional. Here the general result for $\rho=1$ is immediate. If one puts $\rho=1$ into the arguments (but not the parameters) of the equation for B, then the required sums can be done by the Gauss formula for the hypergeometric function. The result is:

$$\lim_{\rho\to1}(n-1,\,n;\,L)=k^{n-1}\cdot2^{n-3}\times\frac{[\Gamma(n-1)]^2}{\Gamma(2n-2)}\left|\frac{\Gamma(L+1+i\eta)}{\Gamma(L+n+i\eta)}\right|. \quad (49)$$

In the remaining cases where $m<n-1$, the result for $\rho=1$ must be obtained by a limiting process. This can be illustrated by the example for the matrix element $(0,3;L)$. The terms A and B, given above in Eqs. (47) and (48), cancel for $\rho=1$, if the $\sinh\pi\eta(1-\rho)$ in the denominator is disregarded. The required limit then yields:

$$\lim_{\rho\to1}(0,3;L)=\frac{k^2}{2L(L+1)(2L+1)}[2L+1-\pi\eta+i\eta\Psi(L+1-i\eta)-i\eta\Psi(L+1+i\eta)], \quad (50)$$

where $\Psi(x)=d/dx[\log\Gamma(x)]$.

It should be pointed out that the use of the recursion formulas given below provides an alternative procedure for obtaining explicit results for $\rho=1$.

As will be shown in general in the concluding section, the Coulomb matrix elements go over into the classical orbit integrals in the limit η, $L\to\infty$, $\rho\to1$, $\eta(1-\rho)\to\xi=$ finite. The results given for the quadrupole matrix elements provide a nice example of this. Thus:

$$4(\eta/k)^{-2}(2,3;L)\to\epsilon^{-2}/6,$$
$$4(\eta/k)^{-2}(0,3;L)\to\tfrac{1}{2}(\epsilon^2-1)^{-1}[1-(\epsilon^2-1)^{-\frac{1}{2}}\tan^{-1}(\epsilon^2-1)^{\frac{1}{2}}], \quad (51)$$

where $\epsilon=[1+L^2/\eta^2]^{\frac{1}{2}}\equiv$ eccentricity of the Kepler orbits. The terms on the right side are exactly the classical orbit integrals for $\xi=0$.[5,7]

The electric quadrupole matrix elements may be given in the following form suitable for calculation in the vicinity of no energy loss:

$$(2,3;L)=\frac{\pi\eta k_2^2\rho^{L+2}(1-\rho^2)e^{\frac{1}{2}\pi\eta(1-\rho)}}{12\sinh\pi\eta(1-\rho)}\left|\frac{\Gamma(L+1+i\eta)}{\Gamma(L+3+i\rho\eta)}\right| +\frac{\pi k_2^2(1-\rho)^2\rho^{L+1}e^{-\frac{1}{2}\pi\eta|1-\rho|}}{2\sinh\pi\eta(1-\rho)} \times\mathrm{Im}\left\{\frac{|\frac{1}{2}\rho-\frac{1}{2}|^{i\eta(1-\rho)}}{\Gamma[3+i\eta(1-\rho)]}\exp[i\sigma_L(\eta)-i\sigma_{L+2}(\rho\eta)]\times F_2(-1+i\eta(1-\rho), L+1+i\eta\; L+3-i\rho\eta;\,-1+i\eta(1-\rho), 3+i\eta(1-\rho);\,\tfrac{1}{2}-\tfrac{1}{2}\rho,\,\tfrac{1}{2}-\tfrac{1}{2}\rho)\right\}, \quad (52)$$

and

$$(0,3,L)=\frac{\pi\rho\eta k_2^2}{2L(L+1)(2L+1)\sinh\pi\eta(1-\rho)}\times\left[\rho^{-L}e^{-\frac{1}{2}\pi\eta(1-\rho)}\left|\frac{\Gamma(L+1+i\rho\eta)}{\Gamma(L+1+i\eta)}\right| -\rho^{L+1}e^{\frac{1}{2}\pi\eta(1-\rho)}\left|\frac{\Gamma(L+1+i\eta)}{\Gamma(L+1+i\rho\eta)}\right|\right] -\frac{\pi(k_2-k_1)^2e^{-\frac{1}{2}\pi\eta|1-\rho|}}{2\sinh\pi\eta(1-\rho)} \times\mathrm{Im}\left\{\frac{\rho^{-i\eta}|\frac{1}{2}\rho-\frac{1}{2}|^{i\eta(1-\rho)}\exp\{i\sigma_L(\eta)-i\sigma_L(\rho\eta)\}}{\Gamma(3+i\eta(1-\rho))} \times F_3\left(-L+i\eta,\,-L-i\rho\eta,\,L+1+i\eta, L+1-i\rho\eta;\,3+i\eta(1-\rho);\,\frac{\rho-1}{2\rho},\,\frac{1-\rho}{2}\right)\right\}. \quad (53)$$

Equation (53) uses the double series form given in Appendix III. This form is most useful for values of L and η which are not too large. (A rough criterion is that $|L+i\eta|^2(1-\rho)/2\sim 1$.) For larger values of L and η the triple series form given by Eq. (45) is to be preferred.

Recursion Relations for the Radial Matrix Elements

Recursion relations in the variable L are of great value both as a primary means of generating radial matrix elements from a few initial values, and, in cases when such a procedure is not advisable because of cumulative error, in checking values obtained by other methods. One can obtain recursion relations in many ways, either from the power series or from the various integral representations given earlier.[28] It proves most convenient to use the basic defining relation, Eq. (24), directly. The fundamental relations for the Coulomb wave functions[24] are:

$$Ld/d\rho[F_L(\eta,\rho)]=[L^2+\eta^2]^{\frac{1}{2}}F_{L-1}(\eta,\rho) -[(L^2/\rho)+\eta]F_L(\eta,\rho), \quad (54)$$

$$(L+1)\frac{d}{d\rho}F_L(\eta,\rho)=\left(\frac{(L+1)^2}{\rho}+\eta\right)F_L(\eta,\rho) -[(L+1)^2+\eta^2]^{\frac{1}{2}}F_{L+1}(\eta,\rho). \quad (55)$$

These relations imply the three-term recursion relation for the $F_L(\eta,\rho)$, and thus the three-term relation provides no new information. By partial integration of Eq. (24), and the use of Eqs. (54) and (55), four equations satisfied by the $(m,n;L)$ are obtained. Only three of these are independent and one thus obtains:

$$(2-m-n)(m,n;L)+k_1\eta_1\left(\frac{1}{L+1}-\frac{1}{L+m}\right)(m,\,n-1;L) +\frac{k_2}{L+m}[(L+m)^2+\eta_2^2]^{\frac{1}{2}}(m-1,\,n-1;L) -\frac{k_1}{L+1}[(L+1)^2+\eta_1^2]^{\frac{1}{2}}(m-1,\,n-1;L+1)=0, \quad (56)$$

$$(2+m-n)(m,n;L) +k_1\eta_1\left(\frac{1}{L+m+1}-\frac{1}{L}\right)(m,\,n-1;L) +\frac{k_1}{L}[L^2+\eta_1^2]^{\frac{1}{2}}(m+1,\,n-1;L-1) -\frac{k_2}{L+m+1}[(L+m+1)^2+\eta_2^2]^{\frac{1}{2}} \times(m+1,\,n-1;L)=0, \quad (57)$$

and

$$(2L+3+m-n)(m,n;L) +k_1\eta_1\left(\frac{1}{L+1}+\frac{1}{L+m+1}\right)(m,\,n-1;L) =\frac{k_1}{L+1}[(L+1)^2+\eta_1^2]^{\frac{1}{2}}(m-1,\,n-1;L+1) +\frac{k_2}{L+m+1}[(L+m+1)^2+\eta_2^2]^{\frac{1}{2}}(m+1,\,n-1;L). \quad (58)$$

In obtaining Eqs. (56) to (58), the integrated contribution from both the origin and infinity has been discarded. This implies that $2L+m+1-n>0$.

One may regard Eqs. (56) to (58) as expressing inter-relations among the three operations of changing the indices, m, n and L by integers. The desired recursion relations will result upon suitably eliminating the

[28] P. Appell and M. J. Kampe de Feriet (reference 27), give various contiguous relations for the Appell functions, from which recursion relations may be derived.

unwanted changes in the indices m and n. Two general relations involving a common value of n, but different values of m and L, may be obtained immediately from these equations. These relations are:

$$\frac{k_2(1+m-n)}{L+m}[(L+m)^2+\eta_2^2]^{\frac{1}{2}}(m-1,n;L)-\frac{k_1(1+m-n)}{L+1}[(L+1)^2+\eta_1^2]^{\frac{1}{2}}(m-1,n;L+1)+k_1\eta_1\left[\frac{(2L+1)(1-n)-m}{L(L+1)}-\frac{(2L+2m+1)(1-n)-m}{(L+m)(L+m+1)}\right](m,n,L)+\frac{k_2(1-m-n)}{L+m+1}[(L+m+1)^2+\eta_2^2]^{\frac{1}{2}}(m+1,n;L)-\frac{k_1(1-m-n)}{L}[L^2+\eta_1^2]^{\frac{1}{2}}(m+1,n;L-1)=0, \quad (59)$$

and

$$\frac{k_2}{L+m}[(L+m)^2+\eta_2^2]^{\frac{1}{2}}\left[\frac{1+m-n}{L+1}+\frac{2L+2+m-n}{L}-\frac{2L+1}{L+m+1}\right](m-1,n;L)-\frac{k_1}{L+1}[(L+1)^2+\eta_1^2]^{\frac{1}{2}}\times\left[\frac{1+m-n}{L+m}+\frac{2L+2m+1}{L}-\frac{2L+m+n}{L+m+1}\right]\times(m-1,n;L+1)-\frac{k_1}{L}[L^2+\eta_1^2]^{\frac{1}{2}}\times\left[\frac{1-m-n}{L+m+1}+\frac{2L+2+m-n}{L+m}-\frac{2L+2m+1}{L+1}\right]\times(m+1,n;L-1)+\frac{k_2}{L+m+1}[(L+m+1)^2+\eta_2^2]^{\frac{1}{2}}\times\left[\frac{1-m-n}{L}+\frac{2L+1}{L+m}-\frac{2L+m+n}{L+1}\right]\times(m+1,n;L)=0. \quad (60)$$

The latter relation is of particular interest for Coulomb excitation since it relates matrix elements differing by two units in m, which is just the condition imposed by conservation of parity.

Consider now some special cases of the above relations. For $m=0$, Eq. (58) gives a relation for $(0,n;L)$ and $(0,n-1;L)$ in terms of $(\pm1,n-1;L)$. But introducing $m=1$ into Eq. (56) and $m=-1$ into Eq. (57) enables one to eliminate the $(\pm1,n-1,L)$ elements in terms of elements with $m=0$. That is, elements with $m=0$ can be expressed in terms of other elements with $m=0$, but differing in n and L. Two independent relations can thus be obtained by this type of procedure, namely:

$$(2-n)^2(0,n;L)+\frac{(n-2)k_1\eta_1}{L(L+1)}(0,n-1;L)=\frac{k_1k_2}{(L+1)^2}[(L+1)^2+\eta_1^2]^{\frac{1}{2}}[(L+1)^2+\eta_2^2]^{\frac{1}{2}}\times(0,n-2;L+1)-\left[k_1^2+k_2^2+k_1^2\eta_1^2\left(\frac{1}{L^2}+\frac{1}{(L+1)^2}\right)\right](0,n-2;L)+\frac{k_1k_2}{L^2}[L^2+\eta_1^2]^{\frac{1}{2}}[L^2+\eta_2^2]^{\frac{1}{2}}(0,n-2;L-1), \quad (61)$$

and

$$(2-n)(2L+3-n)(0,n;L)+\frac{2k_1\eta_1(2-n)}{L+1}(0,n-1;L)=\frac{2k_1k_2}{(L+1)^2}[(L+1)^2+\eta_1^2]^{\frac{1}{2}}[(L+1)^2+\eta_2^2]^{\frac{1}{2}}\times(0,n-2;L+1)-\left(k_1^2+k_2^2+\frac{2k_1^2\eta_1^2}{(L+1)^2}\right)(0,n-2;L). \quad (62)$$

For the particular case of $n=3$, it will be noted that the left-hand sides of Eqs. (61) and (62) are the same to within a factor. In this case, one therefore finds the three terms recursion relation:

$$\frac{2k_1k_2}{L+1}[(L+1)^2+\eta_1^2]^{\frac{1}{2}}[(L+1)^2+\eta_2^2]^{\frac{1}{2}}(0,1;L+1)-\frac{2L+1}{L(L+1)}[(k_1^2+k_2^2)(L)(L+1)+2k_1^2\eta_1^2](0,1;L)+\frac{2k_1k_2}{L}[L^2+\eta_1^2]^{\frac{1}{2}}[L^2+\eta_2^2]^{\frac{1}{2}}(0,1;L-1)=0. \quad (63)$$

This simple relation merely expresses the fact that this matrix element may be written in terms of an ordinary hypergeometric function. This is the Sommerfeld result given previously. The above recursion relation will prove useful in the work that follows.

For $n>3$, Eqs. (61) and (62) in general yield a complicated five-term recursion formula, which, however, will not be given here.

In the electric dipole case the matrix elements that enter are $(\pm1,2;L)$. Although it might at first appear that these are also Appell functions, Eqs. (56) and (57) show they are expressible in terms of ordinary hypergeometric functions. For example, Eq. (56) with $m=1$ and $n=2$ gives:

$$(1,2;L)=\frac{k_2}{L+1}[(L+1)^2+\eta_2^2]^{\frac{1}{2}}(0,1;L)-\frac{k_1}{L+1}[(L+1)^2+\eta_1^2]^{\frac{1}{2}}(0,1;L+1). \quad (64)$$

The case $(-1, 2; L)$ is obtained by interchanging k_1 and k_2 in Eq. (64). Once again the electric dipole case shows great simplicity.

For the electric quadrupole case one has in addition to the $(0,3;L)$ matrix element discussed above, the matrix elements $(\pm 2, 3; L)$. A four-term recursion relation can be found in this case. Since the subsidiary relations that lead to this result are also of interest, the derivation will be given in more detail than would otherwise be the case. The desired element has $m=2$ and $n=3$ and if these values are introduced into Eq. (56) terms with $n=2$ but $m=2$ or 1 arise. Equation (57) shows that $m=2$, $n=2$ is expressible in terms of $m=1$, $n=2$ which, as given in Eq. (64) above, thus implies that the desired element can be related to the $(0,1;L)$ element. This relation is found to be

$$\begin{aligned}4(L+1)k_2[(L+2)^2+\eta_2^2]^{\frac{1}{2}}\{k_2[(L+3)^2+\eta_2^2]^{\frac{1}{2}}(2,3;L+1)\\ -k_1[(L+1)^2+\eta_1^2]^{\frac{1}{2}}(2,3;L)\}\\ =(k_1^2-k_2^2)\{[2k_1^2\eta_1^2+k_1^2(L+1)(2L+3)\\ -k_2^2(L+1)](0,1;L+1)-2k_1k_2[(L+1)^2+\eta_1^2]^{\frac{1}{2}}\\ \times[(L+1)^2+\eta_2^2]^{\frac{1}{2}}(0,1;L)\}. \quad (65)\end{aligned}$$

This result is itself of interest. For the special case $k_1=k_2$ (no energy loss) the right hand side vanishes and the resulting two term relation for $(2,3;L)$ can be solved to give, within a constant, the result previously obtained in Eq. (49). For $k_1\neq k_2$ the relation yields $(2,3;L)$ in terms of a finite sum over L of the $(0,1;L)$. Since the latter are more readily calculated than the $(2,3;L)$ themselves, this may provide a useful calculational procedure.

Using Eq. (63) in Eq. (65) yields the desired four term relation satisfied by the $(2,3;L)$:

$$\begin{aligned}2k_1k_2^2[(L+3)^2+\eta_1^2]^{\frac{1}{2}}[(L+4)^2+\eta_2^2]^{\frac{1}{2}}\\ \times[(L+5)^2+\eta_2^2]^{\frac{1}{2}}(2,3;L+3)-k_2[(L+4)^2+\eta_2^2]^{\frac{1}{2}}\\ \times[6k_1^2\eta_1^2+3(k_1^2+k_2^2)(L+3)^2\\ +(k_1^2-k_2^2)(L+3)(L+4)](2,3;L+2)\\ +k_1[(L+2)^2+\eta_1^2]^{\frac{1}{2}}[6k_1^2\eta_1^2+3(k_1^2+k_2^2)(L+3)^2\\ -(k_1^2-k_2^2)(L+2)(L+3)](2,3;L+1)\\ -2k_1^2k_2[(L+1)^2+\eta_1^2]^{\frac{1}{2}}[(L+2)^2+\eta_1^2]^{\frac{1}{2}}\\ \times[(L+3)^2+\eta_2^2]^{\frac{1}{2}}(2,3;L)=0. \quad (66)\end{aligned}$$

The result for $(-2, 3; L)$ is obtained by interchanging k_1 and k_2 in Eq. (66).

An equation for $(0,3;L)$ that is the analog of Eq. (65) can be obtained. The result is

$$\begin{aligned}(L+1)(L+3)(2L+5)k_1k_2[(L+2)^2+\eta_1^2]^{\frac{1}{2}}\\ \times[(L+2)^2+\eta_2^2]^{\frac{1}{2}}(0,3;L+2)\\ -(2L+3)[k_1^2\eta_1^2((L+1)^2+(L+2)^2)\\ +(k_1^2+k_2^2)(L+1)^2(L+2)^2](0,3;L+1)\\ +L(L+2)(2L+1)k_1k_2[(L+1)^2+\eta_1^2]^{\frac{1}{2}}\\ \times[(L+1)^2+\eta_2^2]^{\frac{1}{2}}(0,3;L)\\ =-(3/4)(L+1)(L+2)(2L+3)(k_1^2-k_2^2)^2\\ \times(0,1;L+1). \quad (67)\end{aligned}$$

This agrees with the earlier results obtained in Eq. (50) for the special case $k_1=k_2$, and, just as mentioned in connection with Eq. (65), may provide the basis for a useful calculational procedure.

It is clear that the relations, Eqs. (56) to (58), provide a basis for determining recursion relations in L for the $(m,n;L)$. The above results, however, are sufficient for the electric dipole and electric quadrupole cases. As a calculational tool the recursion relations are only useful over a limited spread in L, since the unavoidable cumulative error will eventually dominate. For good accuracy in the final result contributions for $L\sim 100$ may be expected to enter, so that it is clear that a straightforward use of recursion relations is inadvisable.

Classical Limit and Quantum Corrections

It is obvious that classical results will be obtained from the quantum results given above by letting $\hbar\rightarrow 0$. In the case at hand, this reduction applies not only to the complete answer but termwise to the various parts of the calculation as well. This is not unexpected since it is well known that the various Racah functions that enter the formulas for the a_ν go over into the Legendre polynomials of the classical result. The limit $\hbar\rightarrow 0$ which carries the Coulomb integrals into the classical Fourier integrals over the orbits appears here as the limiting process $1-\rho\rightarrow 0$, $L,\eta\rightarrow\infty$ and $\lim\eta(1-\rho)\equiv\xi$ is finite. This limit process cannot be carried out directly for the Coulomb integrals in the form given by Eq. (24); rather it requires the use of the analytic continuation into the region $\rho\sim 1$. By utilizing Eqs. (43)–(45), the limit process can be carried out immediately, to yield series which can be identified as the classical orbit integrals. These series, which are confluent forms of the Appell functions, are given in Appendix IV.

It is more direct, however, to proceed from the double integral Eq. (26) and thus obtain an expression for the complete classical limit[29] which is readily generalized to include first order quantum corrections. Instead of expanding the u contour to infinity in Eq. (26) and considering the contributions from the two poles as was done earlier, the u integration is performed

[29] Sommerfeld, reference 12, Appendix 16, has carried out a similar transformation (employing steepest descents) to compare proton bremsstrahlung with Kramer's semiclassical calculation (reference 16).

on the original contour to give

$$(m;n;L;s)=(-)^L 2\sinh\pi\eta(-2i)^{n-2L-m-3}K$$
$$\times\frac{|\Gamma(L+i\eta+1)|^2}{\Gamma(2L+2)}\sum_{\lambda=0}\frac{(-\rho)^\lambda(L+1+i\eta)_\lambda(n-m-1)_\lambda}{\lambda!(2L+2)_\lambda}$$
$$\times\int_0^{(1+)}dv\,v^{L+m-i\rho\eta}(v-1)^{L+m+i\rho\eta}$$
$$\times\left[v+\frac{is}{2}-\tfrac{1}{2}(1+\rho)\right]^{-L-1-\lambda-i\eta}$$
$$\times\left[v+\frac{is}{2}-\tfrac{1}{2}(1-\rho)\right]^{-L+i\eta-m+n-2}\quad(69)$$

with the use of a Kummer transformation. In order to obtain an expression suitable for the confluences required in the classical limit, the v contour is transformed to a contour starting at infinity and encircling the origin counterclockwise by the substitution $v=1/(z+1)$, with the result

$$\int_0^{(1+)}dv(\cdots)$$
$$=-(2)^{2L+m-n+3+\lambda}\int_\infty^{(0+)}dz(-z)^{L+m+i\rho\eta}$$
$$\times(z+1)^{1-m-n+\lambda}[1-\rho+is-z(1+\rho-is)]^{-L-i\eta-1-\lambda}$$
$$\times[1+\rho+is-z(1-\rho-is)]^{-L+i\eta-m+n-2}.\quad(70)$$

Since there is no pole at the origin and $(-z)$ is taken real and positive on the negative real axis, this integral is immediately expressible as a real integral,[30] *viz.*,

$$\int_\infty^{(0+)}dz(\cdots)$$
$$=(-)^{m-\lambda}2e^{\pi\eta}\sinh\pi\rho\eta(1+\rho-is)^{-L-i\eta-1-\lambda}$$
$$\times(1+\rho+is)^{-L+i\eta-m+n-2}\int_0^\infty dz\,z^{-i\eta(1-\rho)-1-\lambda+m}$$
$$\times(z+1)^{1-m-n+\lambda}\left[1-\frac{1-\rho+is}{1+\rho-is}\frac{1}{z}\right]^{-L-i\eta-1-\lambda}$$
$$\times\left[1-\frac{1-\rho-is}{1+\rho+is}z\right]^{-L+i\eta-m+n-2}.\quad(71)$$

The phase $e^{\pi\eta}$ is determined by the cut

$$|\arg[1-\rho+is-z(1+\rho-is)]|\leqslant\pi.$$

The integral (71) is now in a form for confluence to the classical limit. Equations (70) and (71) are substituted in (69) and, since (71) converges uniformly in λ for the classical limit, the summation and integration in (69) may be interchanged in the limit.

There results,

$$(m,n;L;s)=(-i)^{n-m-1}k_2^{n-1}2^{n-3}e^{\frac{1}{2}\pi\eta(1-\rho)}\rho^{L+1}$$
$$\times\left|\frac{\Gamma(L+i\eta+1)}{\Gamma(L+m+1+i\rho\eta)}\right|\frac{\Gamma(2L+m-n+3)}{\Gamma(2L+2)}2^{2L+m-n+3}$$
$$\times(1+\rho-is)^{-L-i\eta-1}(1+\rho+is)^{-L+i\eta-m+n-2}$$
$$\times\int_0^\infty dz\,z^{-i\eta(1-\rho)-1+m}(z+1)^{1-m-n}$$
$$\times\left[1-\frac{1-\rho+is}{1+\rho-is}\frac{1}{z}\right]^{-L-i\eta-1}\left[1-\frac{1-\rho-is}{1+\rho+is}z\right]^{-L+i\eta-m+n-2}$$
$$\times\sum_\lambda\frac{(L+1+i\eta)_\lambda(n-m-1)_\lambda}{\lambda!(2L+2)_\lambda}$$
$$\times\left[\frac{\rho(z+1)}{z-(1-\rho+is)/(1+\rho-is)}\right]^\lambda\left[\frac{2}{1+\rho-is}\right]^\lambda.\quad(72)$$

In the limit L, η very large, $\eta(1-\rho)\equiv\xi$ finite, and $s\rightarrow 0$, the series can be summed if $(L+1+i\eta)_\lambda$ is replaced by the first term in the expansion,

$$(L+1+i\eta)_\lambda=(L+1+i\eta)^\lambda+(L+1+i\eta)^{\lambda-1}\sum_1^{\lambda-1}j$$
$$+(L+1+i\eta)^{\lambda-2}\sum_1^{\lambda-1}j\sum_1^{j-1}i+\cdots,\quad(73)$$

with unity neglected in comparison with $L+i\eta$. The expression $(2L+2)_\lambda$ is treated similarly, with the result that

$$\sum_\lambda(\cdots)\sim\left[\frac{(L-i\eta)z-(L+i\eta)}{2Lz}\right]^{-n+m+1}$$
$$\times\left[1+\frac{L(1-\rho)(z-1)}{(L-i\eta)z-(L+i\eta)}\Big/1-\frac{1-\rho}{2}\frac{1}{z}\right]^{-n+m+1}.\quad(74)$$

The second factor on the right in Eq. (74) is a quantum correction and is neglected in the passage to the classical limit. In the classical limit the integral becomes

$$\int_0^\infty dz(\cdots)=\left(\frac{L-i\eta}{2L}\right)^{-n+m+1}\int_0^\infty dz\,z^{-i\xi+n-2}$$
$$\times(z+1)^{1-m-n}\left(z-\frac{L+i\eta}{L-i\eta}\right)^{-n+m+1}$$
$$\times\exp\left[(L+i\eta)\left(\frac{1-\rho}{2}\right)\frac{1}{z}+(L-i\eta)\left(\frac{1-\rho}{2}\right)z\right].\quad(75)$$

The exponentials result from expanding the factors concerned and approximating by the first term in expansions of the type given in Eq. (73).[31] If the classical

[30] See, for example, Hankel's contour for the Γ function, E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, London, 1945), p. 244.

[31] It is not obvious that this procedure is valid, but it may in fact be justified by a more detailed argument employing Hankel's form for the gamma function.

variables

$$1+(iL/\eta)=\epsilon e^{i\vartheta} \qquad (76)$$

are introduced in Eq. (75), there results

$$\int_0^\infty dz(\cdots)=\left(\frac{-i\epsilon\eta}{2L}\right)^{-n+m+1}\int_0^\infty dz z^{-i\xi+n-2} \times(z+1)^{1-m-n}(ze^{i\vartheta}+e^{-i\vartheta})^{-n+m+1} \times\exp\left[\frac{i\epsilon\xi}{2}\left(\frac{e^{-i\vartheta}}{z}-e^{i\vartheta}z\right)\right]. \qquad (77)$$

The substitution $z\rightarrow 1/z$ in the integral is equivalent to complex conjugation, hence the integral is real. When Eq. (77) is substituted into Eq. (72) and the limit is taken for the coefficient, the classical answer is obtained. Thus,

$$\lim(m,n;L)=\frac{1}{4}\left(\frac{2k}{\epsilon\eta}\right)^{n-1}e^{\xi\vartheta}\int_0^\infty dz z^{-i\xi+n-2} \times(1+z)^{2-2n}\left[\frac{ze^{i\vartheta}+e^{-i\vartheta}}{1+z}\right]^{-n+m+1} \times\exp\left[\frac{i\epsilon\xi}{2}\left(\frac{e^{-i\vartheta}}{z}-e^{i\vartheta}z\right)\right]. \qquad (78)$$

Equation (78) can be put in a more recognizable form by making the substitution $z=e^{t-i\vartheta}$, and noting that

$$\frac{e^{t-i\vartheta}}{(1+e^{t-i\vartheta})^2}=\frac{\epsilon}{2}\frac{[\epsilon+\cosh t+i(\epsilon^2-1)^{\frac{1}{2}}\sinh t]}{[\epsilon\cosh t+1]^2}, \qquad (79)$$

and

$$\frac{(e^t+e^{-i\vartheta})}{(1+e^{t-i\vartheta})}=\frac{\epsilon+\cosh t+i(\epsilon^2-1)^{\frac{1}{2}}\sinh t}{\epsilon\cosh t+1}. \qquad (80)$$

As a result of these considerations, Eq. (78) becomes

$$\lim(m,n;L)=\frac{1}{4}\left(\frac{k}{\eta}\right)^{n-1}\int_{-\infty}^{+\infty}dt e^{-i\xi(\epsilon\sinh t+t)} \times\frac{[\epsilon+\cosh t+i(\epsilon^2-1)^{\frac{1}{2}}\sinh t]^m}{[\epsilon\cosh t+1]^{m+n-1}}. \qquad (81)$$

Here, the substitution $t\rightarrow -t$ results in complex conjugation. This integral, to within the factor $\frac{1}{4}(k/\eta)^{n-1}$, is the general Fourier integral over classical orbits of the multipole matrix elements as given, for example, by Ter-Martirosyan and by Alder and Winther.[5]

The advantage of the procedure outlined above is that it enables one to obtain easily integral forms for the first order quantum correction to the classical limit. In order to avoid undue complication, the quantum correction is illustrated only for the case $m=n-1$. It is clear, however, that the procedure for the general case $m\leqslant n-1$ is the same except for additional corrections arising from summing the series [see Eqs. (73) and (74)].

For the special case $m=n-1$, Eq. (72) gives the result,

$$(n-1,n;L)=k_2^{n-1}2^{n-3}\left[\frac{4\rho}{(1+\rho)^2}\right]^{L+1}e^{\frac{1}{2}\pi\eta(1-\rho)} \times\left|\frac{\Gamma(L+1+i\eta)}{\Gamma(L+n+i\rho\eta)}\right|\int_0^\infty dz z^{-i\eta(1-\rho)+n-2} \times(z+1)^{2-2n}\left[1-\frac{1-\rho}{1+\rho}\frac{1}{z}\right]^{-L-i\eta-1} \times\left[1-\frac{1-\rho}{1+\rho}z\right]^{-L+i\eta-1}. \qquad (82)$$

The first-order quantum corrections for this integral arise from neglect of the second term in expansions of the form of Eq. (73) in the approximation by exponentials, and the neglect of unity compared with $L+i\eta$ and $L-i\eta$. If the first two terms in expansion of $(-1)^\lambda(L+i\eta+1)_\lambda$ and of $(-1)^\lambda(L-i\eta+1)_\lambda$ are kept one obtains,

$$\left[1-\left(\frac{1-\rho}{1+\rho}\right)z\right]^{-L+i\eta-1}=\exp\left[(L+1-i\eta)\left(\frac{1-\rho}{1+\rho}\right)z\right] \times\left\{1+\frac{1}{2}\frac{\left[(L+1-i\eta)\left(\frac{1-\rho}{1+\rho}\right)z\right]^2}{L+1-i\eta}+\cdots\right\}, \qquad (83)$$

and

$$\left[1-\left(\frac{1-\rho}{1+\rho}\right)\frac{1}{z}\right]^{-L-i\eta-1}=\exp\left[(L+i\eta+1)\left(\frac{1-\rho}{1+\rho}\right)\frac{1}{z}\right] \times\left\{1+\frac{1}{2}\frac{\left[(L+i\eta+1)\left(\frac{1-\rho}{1+\rho}\right)\frac{1}{z}\right]^2}{L+i\eta+1}+\cdots\right\}, \qquad (84)$$

Thus, it is found that the desired Coulomb integral, accurate through terms of order $(1-\rho)$, is given by

$$(n-1,n;L)=k_2^{n-1}2^{n-3}\left[\frac{4\rho}{(1+\rho)^2}\right]^{L+1}e^{\frac{1}{2}\pi\eta(1-\rho)} \times\left|\frac{\Gamma(L+1+i\eta)}{\Gamma(L+n+i\rho\eta)}\right|\int_0^\infty dz\, z^{-i\eta(1-\rho)+n-2}(z+1)^{2-2n} \times\exp\left[(L+1-i\eta)\left(\frac{1-\rho}{1+\rho}\right)z+(L+i\eta+1)\left(\frac{1-\rho}{1+\rho}\right)\frac{1}{z}\right] \times\left\{1+\frac{1}{2}\frac{z^2}{L-i\eta+1}\left[(L-i\eta+1)\left(\frac{1-\rho}{1+\rho}\right)\right]^2 +\frac{1}{2}\frac{z^{-2}}{L+i\eta+1}\left[(L+i\eta+1)\left(\frac{1-\rho}{1+\rho}\right)\right]^2\right\}. \qquad (85)$$

As it is written, the classical variables ξ and ϵ have not been introduced above, since their introduction complicates the formulas.

For the cases of physical interest, $(1-\rho)/(1+\rho)$ is generally small, $\sim 1/20$. The difference between $(1-\rho)/(1+\rho)$ and $\frac{1}{2}(1-\rho)$ is hence insignificant. On the other hand, η is not too large, since $3\leqslant\eta\leqslant 8$ is usual. Since the quantum correction is most significant for the low angular momenta, it is thus seen from Eq. (85) that the zeroth order (classical) approximation should be improved by the use of $\epsilon^2=1+(L+1)^2/\eta^2$ to replace $\epsilon^2=1+L^2/\eta^2$.

ACKNOWLEDGMENTS

The authors are indebted to Dr. F. K. McGowan and Dr. P. H. Stelson for many helpful discussions, which in large part stimulated the present work. In addition, the help and encouragement of Dr. R. A. Charpie and Dr. M. E. Rose and the courtesies extended by the Oak Ridge National Laboratory to one of the authors (L.C.B.) while he was a visitor during the summer of 1954 are gratefully acknowledged.

APPENDIX I. REDUCED COULOMB MULTIPOLE MATRIX ELEMENTS

Magnetic Multipoles

$$Q(l'l;Lm)C(Ll l';Mmm')\equiv \left\langle \frac{F_{l'}(\eta_2,k_2r)}{k_2r}Y_{l'}^{m'}(\theta,\phi)\right|[L(L+1)]^{-\frac{1}{2}}h_L^{(1)}(kr)\times \mathbf{j}_P\cdot\mathbf{L}Y_L^M(\theta,\phi)\left|\frac{F_l(\eta_1,k_1r)}{k_1r}Y_l^m(\theta,\phi)\right\rangle. \quad (86)$$

Introducing the definition of the current operator, in spherical coordinates, one finds two terms, of which the first is

$$[L(L+1)]^{-\frac{1}{2}}\frac{\hbar z_1e}{2imc}\times\left\langle\frac{F_{l'}}{k_2r}Y_{l'}^{m'}\right|h_L^{(1)}(\mathbf{L}Y_L^m)\cdot\frac{\mathbf{r}\times\mathbf{L}}{r^2}\left|\frac{F_l}{k_1r}Y_l^m\right\rangle.$$

The second term has the $\mathbf{r}\times\mathbf{L}$ operator acting on the $F_{l'}Y_{l'}^{m'}$, and can be shown to be equal to the term above. The angular function $\mathbf{L}Y_L^m$ and $\mathbf{r}\times\mathbf{L}Y_l^m$ can be expressed as vector spherical harmonics[32] and this facilitates performing the angular integrals.

The result, after some manipulation, is found to be

$$Q(l'l;Lm)=\left(\frac{\hbar z_1e}{imc}\right)\frac{(2L+1)(2l+1)}{\surd(4\pi)}\left[\frac{(l+1)(2l+3)}{2l'+1}\right]^{\frac{1}{2}}\times C(Ll+1l';000)W(LLll+1;1l')\frac{1}{k_1k_2}\times\int_0^\infty r^{-1}drF_l(\eta_1,k_1r)F_{l'}(\eta_2,k_2r)h_L(kr). \quad (87)$$

[32] H. C. Corben and J. Schwinger, Phys. Rev. 58, 967 (1940).

Electric Multipoles

$$Q(l'l;Le)C(Ll l';Mmm')\equiv\left\langle\frac{F_{l'}(\eta_2,k_2r)}{k_2r}Y_{l'}^{m'}(\theta,\phi)\right|k^{-1}[L(L+1)]^{-\frac{1}{2}}h_L(kr)\times(\mathbf{j}_P\cdot\nabla\times\mathbf{L}Y_L^M(\theta,\phi))\left|\frac{F_l(\eta_1,k_1r)}{k_1r}Y_l^m(\theta,\phi)\right\rangle. \quad (88)$$

The matrix element on the right can be put in an alternative, more convenient, form by using the expansion for $\nabla\times\mathbf{L}$ and partial integration. One finds that:

$$\langle\cdots\rangle=[L(L+1)]^{-\frac{1}{2}}\left\langle\frac{F_{l'}(\eta_2,k_2r)}{k_2r}Y_{l'}^{m'}\right.\times\left|\varrho_P\frac{d}{dr}(rh_L)Y_L^M+ikY_L^Mh_L\mathbf{r}\cdot\mathbf{j}_P\right|\left.\frac{F_l}{k_1r}Y_l^m\right\rangle. \quad (89)$$

Hence the reduced matrix element is

$$Q(l'l;Le)=\left[\frac{(2L+1)(2l+1)}{4\pi(L)(L+1)(2l'+1)}\right]^{\frac{1}{2}}\times C(Ll l';000)(k_1k_2)^{-1}z_1e\times\int_0^\infty dr\left\{F_l(\eta_1,k_1r)F_{l'}(\eta_2,k_2r)d/dr[rh_L(kr)]+\frac{\hbar k}{2mc}h_L(kr)(k_1rF_{l'}(\eta_2,k_2r)F_l'(\eta_1,k_1r)-k_2rF_l(\eta_1,k_1r)F_{l'}'(\eta_2,k_2r)\right\}. \quad (90)$$

The long wavelength limit for these reduced matrix elements replaces $h_L(kr)$ by $-i(2L-1)!!(kr)^{-L-1}$, and discards the radial current in the electric multipoles. For the electric quadrupole case we obtain:

$$Q(l'l;2e)\rightarrow ik^{-3}z_1e[30(2l+1)/4\pi(2l'+1)]^{\frac{1}{2}}\times C(2ll';000)\frac{1}{k_1k_2}\int_0^\infty r^{-3}drF_lF_{l'}. \quad (91)$$

APPENDIX II

The Appell functions used in this report have the series definitions:

$$F_1(\alpha,\beta,\beta';\gamma;x,y)=\sum\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^my^n, \quad (92)$$

$$F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y)=\sum\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_nm!n!}x^my^n, \quad (93)$$

$$F_3(\alpha,\alpha',\beta,\beta';\gamma;x,y)=\sum\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^my^n. \quad (94)$$

The series for F_1 and F_3 are absolutely convergent for $|x|$, $|y|<1$, while F_2 is absolutely convergent for $|x|+|y|<1$.

The confluent function Ψ_2 is given by the series:

$$\Psi_2(\alpha;\gamma,\gamma';x,y)=\sum\frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n m!n!}x^m y^n. \quad (95)$$

The transformations and reductions that are useful for the present work are

$$F_2(\alpha,\beta,\beta';\alpha,\alpha;x,y)=(1-x)^{-\beta}(1-y)^{-\beta'}\times{}_2F_1\left(\beta,\beta';\alpha;\frac{xy}{(1-x)(1-y)}\right). \quad (96)$$

[This is the reduction used in obtaining Sommerfeld's integral, Eq. (28).]

$$F_2(\alpha,\beta,\beta';\alpha,\gamma';x,y) =(1-x)^{-\beta}F_1[\beta',\alpha-\beta,\beta;\gamma';y,(y/1-x)]. \quad (97)$$

[This transformation reduces the case with $m=n-1$ to a single contour integral.]

$$F_3(\alpha,\alpha',\beta,\beta';\alpha+\alpha';x,y) =(1-y)^{-\beta'}F_1[\alpha,\beta,\beta';\alpha+\alpha';x,(y/y-1)]. \quad (98)$$

APPENDIX III

A double series formulation is derived here for $C(m,n;L)$ in the general case where $m\neq n-1$.

$$C(m,n;L)=\frac{2Ke^{\pi\eta}\sinh\pi\rho\eta}{\sinh\pi\eta(1-\rho)}\mathrm{Re}\int_0^{[(\frac{1}{2}-\frac{1}{2}\rho-\frac{1}{2}is)+]}dv \times v^{L+m-i\rho\eta}(v-1)^{L+m+i\rho\eta}f_2(v). \quad (99)$$

If the alternative form for $f_2(v)$ is employed, *viz.*,

$$f_2(v)=2\pi i(-2i\rho)^{n-2L-m-3}\times\frac{\Gamma(L+1+i\eta)}{\Gamma(2L+m+3-n)\Gamma(-L+i\eta+n-m-1)}\times\left(\frac{s+i(1-\rho)-2iv}{2i\rho}\right)^{-L-m+n-2+i\eta}\times{}_2F_1\left(-L+i\eta,L+1+i\eta;-L+i\eta-m+n-1;\frac{s+i(1-\rho)-2iv}{-2i\rho}\right), \quad (100)$$

one finds, using now the series for the ${}_2F_1$ function, that:

$$C(m,n;L)=\frac{\pi}{2}\frac{e^{-\frac{1}{2}\pi\eta|1-\rho|}}{\sinh\pi\eta(1-\rho)}(k_2-k_1)^{n-1}\times\mathrm{Re}\left\{i^{m-n}\rho^{-i\eta}\left|\frac{\rho-1}{2}\right|^{i\eta(1-\rho)}\times\exp[i\sigma_L(\eta)-i\sigma_{L+m}(\rho\eta)]\frac{1}{\Gamma(n+i\eta(1-\rho))}\times F_3\left(-L+i\eta,-L-m-i\rho\eta,L+1+i\eta,L+m+1-i\rho\eta;n+i\eta(1-\rho);\frac{\rho-1}{2\rho},\frac{1-\rho}{2}\right)\right\}. \quad (101)$$

By using the reduction formulas given in Appendix II, one can show that in the particular case $m=n-1$ this reduces properly to a single F_2, as given in Eq. (45).

APPENDIX IV

A series representation for the integrals in the classical limit can be obtained by taking the appropriate limit for $A(m,n;L)$, $B(m,n;L)$, and $C(m,n;L)$, as given in Eqs. (43)–(45), respectively. Consider first $C(m,n;L)$. In the classical limit, one finds that

$$F_2(2-n-\lambda+i\eta(1-\rho),L+m+2-n-\lambda+i\eta,L+m+1-i\rho\eta;2-n-\lambda+i\eta(1-\rho),n+i\eta(1-\rho);\tfrac{1}{2}(1-\rho),\tfrac{1}{2}(1-\rho))\rightarrow\psi_2(2-n-\lambda+i\xi;2-n-\lambda+i\xi,n+i\xi;\tfrac{1}{2}i\epsilon\xi e^{-i\vartheta},-\tfrac{1}{2}i\epsilon\xi e^{i\vartheta}), \quad (102)$$

where[25]

$$\psi_2(\alpha;\gamma,\gamma';x,y)\equiv\sum\{(\alpha)_{m+n}/(\gamma)_m(\gamma')_n m!n!\}x^m y^n,$$

and

$$\exp[i(2\sigma_{L+m+1-n}(\eta)-\sigma_L(\eta)-\sigma_{L+m}(\rho\eta))]\rightarrow(\eta\epsilon)^{i\xi}e^{i\vartheta(2n-2-m)}(-)^{n-m-1}i^{-m},$$

with the final result that

$$C(m,n;L)\rightarrow\frac{\pi}{2}\left(\frac{k\xi}{\eta}\right)^{n-1}\frac{e^{-\epsilon\xi\sin\vartheta-\frac{1}{2}\pi|\xi|}}{\sinh\pi\xi}\times\mathrm{Re}\left\{i^{-n}\left|\frac{\epsilon\xi}{2}\right|^{i\xi}\frac{e^{i\vartheta(2n-2-m)}}{\Gamma(n+i\xi)}\sum_\lambda\frac{(n-m-1)_\lambda}{\lambda!}\times(1-e^{-2i\vartheta})^\lambda\psi_2(\alpha;\gamma,\gamma';x,y),\right. \quad (103)$$

where the parameters of ψ_2 are the same as those occurring in Eq. (102).

To obtain the limiting forms for $A(m,n;L)$ and $B(m,n;L)$ it is most expedient to use alternative forms of Eqs. (43)–(44), which are obtained by Kummer transformations of $f_1(v)$ and $f_2(v)$ in the integrands.

The result is

$$A(m,n;L)=k_2^{n-1}\left[\frac{4\rho}{(1+\rho)^2}\right]^{L+1}\left(\frac{2}{1+\rho}\right)^{m-3}(1+\rho)^{n-4}$$
$$\times i^{-1-m-n}\frac{e^{\frac{1}{2}\pi\xi}}{\sinh\pi\xi}\left|\frac{\Gamma(L+m+1+i\rho\eta)}{\Gamma(L+1+i\eta)}\right|$$
$$\times\frac{\Gamma(2L+3+m-n)}{\Gamma(2L+2m+2)}\sum_\lambda\frac{(m+n-1)_\lambda}{\lambda!}$$
$$\times\frac{(L+m+1+i\rho\eta)_\lambda}{(2L+2m+2)_\lambda}\left(\frac{2}{1+\rho}\right)^\lambda\oint_{-1}dww^{n-2-i\xi}$$
$$\times[1+w]^{m+1+\lambda-n}\left[1+\frac{1-\rho}{1+\rho}w\right]^{-L-m-1-i\rho\eta-\lambda}$$
$$\times\left[1+\frac{1-\rho}{1+\rho}w^{-1}\right]^{-L-2+n+i\rho\eta}. \quad (104)$$

The limiting form is then found to be

$$A(m,n;L)\to-\frac{1}{8}\left(\frac{2k}{\eta\epsilon}\right)^{n-1}\frac{e^{\pi\xi-\xi\vartheta}}{\sinh\pi\xi}\oint_{-1}dww^{n-2-i\xi}$$
$$\times[we^{-i\vartheta}+e^{i\vartheta}]^{2-2n}\left[\frac{1+w}{we^{-i\vartheta}+e^{i\vartheta}}\right]^{m+1-n}$$
$$\times\exp\left[-\frac{i\epsilon\xi}{2}(we^{-i\vartheta}-w^{-1}e^{i\vartheta})\right]$$
$$=(-)^{m+1}\frac{\pi i}{4}\left(\frac{2k}{\eta\epsilon}\right)^{n-1}(2i\sin\vartheta)^{1-m-n}$$
$$\times\frac{e^{-\xi\vartheta}}{\sinh\pi\xi}\sum_{\lambda,\mu,\nu}\frac{(m+n-1)_\lambda}{\lambda!\mu!\nu!}$$
$$\times\frac{\Gamma(n-1-i\xi+\mu-\nu)}{\Gamma(n-m-1-\lambda)\Gamma(m+1-i\xi+\mu+\lambda-\nu)}$$
$$\times(1-e^{2i\vartheta})^{-\lambda}\left(\frac{i\xi\epsilon}{2}e^{-i\vartheta}\right)^\mu\left(\frac{-i\xi\epsilon}{2}e^{i\vartheta}\right)^\nu. \quad (105)$$

Similarly, the results for B are

$$B(m,n;L)=k_2^{n-1}\left[\frac{4\rho}{(1+\rho)^2}\right]^{L+1}\left(\frac{2}{1+\rho}\right)^{m-3}(1+\rho)^{n-4}$$
$$\times i^{m+1-n}\frac{e^{-\frac{1}{2}\pi\xi}}{\sinh\pi\xi}\left|\frac{\Gamma(L+1+i\eta)}{\Gamma(L+m+1+i\rho\eta)}\right|$$
$$\times\frac{\Gamma(2L+m+3-n)}{\Gamma(2L+2)}\sum_\lambda\frac{(m+n-1)_\lambda}{\lambda!}$$
$$\times\frac{(L+m+1+i\rho\eta)_\lambda}{(2L+2m+2)_\lambda}\left(\frac{2}{1+\rho}\right)^\lambda\oint_{-1}dww^{n-2-i\xi}$$
$$\times[1+w]^{m+1+\lambda-n}\left[1-\frac{1-\rho}{1+\rho}w\right]^{-L-m-1-i\rho\eta-\lambda}$$
$$\times\left[1+\frac{1-\rho}{1+\rho}w^{-1}\right]^{-L-2+n+i\rho\eta}. \quad (106)$$

The above results show the symmetry between A and B, namely that $A\rightleftharpoons B$ under the interchange of (k_1,η_1,L_1) with (k_2,η_2,L_2). The limiting form of B is then

$$B(m,n;L)\to\frac{1}{8}\left(\frac{2k}{\epsilon\eta}\right)^{n-1}\frac{e^{-\xi(\pi-\vartheta)}}{\sinh\pi\xi}\oint_{-1}dww^{n-2+i\xi}$$
$$\times\exp\left[\frac{i\xi\epsilon}{2}(e^{i\vartheta}w-e^{i\vartheta}w^{-1})\right]$$
$$\times[1+w]^{2-2n}\left[\frac{we^{-i\vartheta}+e^{i\vartheta}}{1+w}\right]^{m+1-n}$$
$$=(-)^m\frac{1}{8}\left(\frac{2k}{\epsilon\eta}\right)^{n-1}(2i\sin\vartheta)^{1+m-n}$$
$$\times\frac{e^{\xi\vartheta}}{\sinh\pi\xi}\sum_{\lambda,\mu,\nu}\frac{(n-m-1)_\lambda}{\lambda!\mu!\nu!}$$
$$\times\frac{\Gamma(n-1+i\xi+\mu-\nu)}{\Gamma(m-n-1-\lambda)\Gamma(1-m+i\xi+\lambda+\mu-\nu)}$$
$$\times(e^{2i\vartheta}-1)^{-\lambda}\left(\frac{-i\xi\epsilon}{2}e^{-i\vartheta}\right)^\mu\left(\frac{i\xi\epsilon}{2}e^{i\vartheta}\right)^\nu. \quad (107)$$