Extended Isotopic Spin Invariance and Meson-Nucleon Coupling*

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The identities derived from a Lagrangian density that is invariant under transformations that depend on arbitrary functions are used to develop the restrictions on allowed forms of coupling that follow from assuming extended isotopic spin invariance. A proof is given that invariance under local isotopic spin rotations is not possible for pions and nucleons alone. The relations between the transformation law and the invariant Lagrangian are developed if (a) the pion-nucleon system is allowed to admit a more general extended isotopic spin group, or (b) additional fields are admitted in order to maintain the local isotopic spin rotational invariance. In either case, definite restrictions on the permissible transformation properties for the field quantities are obtained if pion-nucleon isotopic spin conservation is to be maintained.

I. INTRODUCTION

IN the isotopic spin formalism, nucleons and pions are described by spinors and vectors in isotopic spin space. Pion-nucleon phenomena and the charge independence of nuclear forces (at least at energies corresponding presumably to interactions via pions alone) suggest that if electromagnetic interactions are neglected, total isotopic spin is conserved in interactions involving pions and nucleons alone. This situation is described formally by stating that the pion-nucleon interaction is invariant with respect to the three-parameter group of rotations of the isotopic spin coordinate system.

This paper is concerned with the possibility of extending this invariance to isotopic spin rotations which are local in the sense that at different points in space-time different isotopic spin rotations are permitted. Such an extension is motivated[1] by the possibility that the meson-nucleon coupling may be suggested by some such extended physical symmetry principle. This is the case in electrodynamics where the assumption of local gauge invariance suggests the coupling between charged particles and the electromagnetic field, and in general relativity where general covariance leads to Bianchi identities which suggest how other fields are coupled into the gravitational field.

The principle to be emphasized here is that certain identities are present whenever the equations of motion are derived from a Lagrangian density that is invariant under transformations depending on arbitrary functions rather than arbitrary parameters. This important theorem, due originally to Noether,[2] is briefly derived in the next section: the new idea here consists in recognizing that these identities are applicable to coupled fields. The central results needed are Eqs. (17), (18), and (19).

In the third section, these identities are used to show that local isotopic spin rotational invariance is not possible for nucleons and scalar or pseudoscalar mesons. The possibility of allowing an additional "**b** field" in order to maintain local isotopic spin rotational invariance, as Yang and Mills have recently done,[3] is considered in the fourth section. It is shown that, as a consequence of the local isotopic spin invariance, independently of any special choice of equations of motion, this **b** field must obey nonlinear equations of motion and must contribute to the total isotopic spin.

The point of view implied here is that isotopic spin may be explicable as a consequence of some extended invariance principle in the same way as extended gauge invariance and general coordinate covariance illuminate, respectively, the nature of electric charge and the role of energy-momentum.

II. EXTENDED INVARIANCE

Identities and Conservation Laws

Suppose the equations of motion are obtained by the variation of an action integral

$$I=\int_\Omega \mathcal{L}(y^A(x),y^A{}_{,\mu}(x))d^4x. \tag{1}$$

The $_{,\mu}$ designate differentiation with respect to the coordinates x^μ. Y^A is the Ath field variable. The summation convention is assumed. Under the transformation considered, let the coordinates change by

$$\delta x^\mu = x^{\mu\prime}-x^\mu, \tag{2}$$

where x' designates the point x in the transformed system, and let

$$\delta\mathcal{L}=\mathcal{L}(y'^A(x'),y^A{}_{,\mu}'(x'))-\mathcal{L}(y^A(x),y_{,\mu}{}^A(x)) \tag{3}$$

be the local variation in the integrand $\mathcal{L}$. Then the change in the action integral is

$$\delta I=\int_\Omega[\delta\mathcal{L}+\mathcal{L}(\delta x^\mu)_{,\mu}]d^4x, \tag{4}$$

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[1] See, for example, the interchange between W. Pauli and A. Pais, Physica 19, 887 (1953).

[2] E. Noether, Nachr. Akad. Wiss. Göttingen, Math.-physik. Kl. 235 (1918). See also E. Bessel-Hagen, Math. Ann. 84, 258 (1921); P. G. Bergmann and R. Schiller, Phys. Rev. 89, 4 (1953).

[3] C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

where $(\delta x^\mu)_{,\mu}$ is the term in the Jacobian brought about by the transformation from x to x' coordinates. For the equations of motion to be invariant, the action integral must change at most by a boundary term on which the variations performed in obtaining the equation of motion vanish. Since the volume V is arbitrary, this requires[4] that the integrand in Eq. (4) change at most by a divergence:

$$\delta\mathfrak{L}+\mathfrak{L}(\delta x^\mu)_{,\mu}=(\delta\Omega^\mu)_{,\mu}. \tag{5}$$

Then $\mathfrak{L}-\Omega^\mu{}_{,\mu}$ would be an invariant Lagrangian density, since

$$\delta(\mathfrak{L}-\Omega^\mu{}_{,\mu})+(\mathfrak{L}-\Omega^\mu{}_{,\mu})(\delta x^\mu)_{,\mu}=0. \tag{6}$$

In the following, Ω^μ is dropped and it is assumed that one always has to do with an invariant Lagrangian density. This possibility of changing the Lagrangian density by a suitable divergence can introduce a certain ambiguity in the densities that are differentially conserved, but will not affect the conclusions of this paper about the possibility of admitting a local isotopic spin group for the pion-nucleon system. If the substantial change in the form of y^A as a function of its argument is designated by

$$\delta^* y^A=y'^A(x)-y^A(x), \tag{7}$$

then

$$\delta\mathfrak{L}=\delta^*\mathfrak{L}+\mathfrak{L}_{,\mu}\delta x^\mu, \tag{8}$$

and

$$\delta^*\mathfrak{L}=\mathfrak{L}^A\delta^* y^A+\left(\frac{\partial\mathfrak{L}}{\partial y^A{}_{,\mu}}\delta^* y^A\right)_{,\mu}. \tag{9}$$

The Lagrangian derivative,

$$[\mathfrak{L}]_A\equiv\partial\mathfrak{L}/\partial y^A-(\partial\mathfrak{L}/\partial y^A{}_{,\mu})_{,\mu}, \tag{10}$$

vanishes where the field equations hold.

Equation (6), the invariance of the Lagrangian density, gives

$$[\mathfrak{L}]_A\delta^* y^A+\left(\frac{\partial\mathfrak{L}}{\partial y^A{}_{,\mu}}\delta^* y^A+\mathfrak{L}\delta x^\mu\right)_{,\mu}=0. \tag{11}$$

This is Noether's first result: If the equations of motion are invariant under the variations (2) and (7), $[\mathfrak{L}]_A\delta^* y^A$ must be of the form of a divergence.

Now suppose that the transformation is described by p arbitrary parameters or functions of the coordinates ξ^i $(i=1,2\cdots p)$ and that the transformation induces the change

$$\delta x^\mu=c_i{}^\mu\xi^i \tag{12}$$

in the coordinates and the change

$$\delta^* y^A=a_i{}^A\xi^i+b_i{}^{A\mu}\xi^i{}_{,\mu} \tag{13}$$

in the field variables. The $a_i{}^A$, $b_i{}^{A\mu}$ are definite functions of the field variables and their derivatives. Where no confusion will result the index i is omitted below. Since $[\mathfrak{L}]_A\delta^* y^A$ must be a divergence, the p identities

$$[\mathfrak{L}]_A a_i{}^A\equiv([\mathfrak{L}]_A b_i{}^{A\mu})_{,\mu} \tag{14}$$

must obtain among the field equations $[\mathfrak{L}]_A=0$. These identities, which characterize theories that depend on arbitrary functions rather than arbitrary parameters, constitute Noether's second result.

Since the field quantities need not all be of the same kind, these identities can be used to obtain relations among coupled fields.

Equation (11) now states that

$$(T^\mu\xi+U^{\mu\lambda}\xi_{,\lambda})_{,\mu}=0, \tag{15}$$

where

$$\begin{aligned}-T^\mu&\equiv[\mathfrak{L}]_A b^{A\mu}+(\partial\mathfrak{L}/\partial y^A{}_{,\mu})a^A+\mathfrak{L}c^\mu,\\ -U^{\mu\lambda}&\equiv(\partial\mathfrak{L}/\partial y^A{}_{,\mu})b^{A\lambda}.\end{aligned} \tag{16}$$

Since ξ, $\xi_{,\mu}$, and $\xi_{,\mu\lambda}$ may each have arbitrary values, Eq. (15) implies

$$U^{\mu\lambda}+U^{\lambda\mu}=0, \tag{17}$$

$$T^\mu=U^{\mu\lambda}{}_{,\mu}, \tag{18}$$

$$T^\mu{}_{,\mu}=0. \tag{19}$$

Now a conservation law like Eq. (19) could be obtained directly from Eq. (11) by using the equations of motion $[\mathfrak{L}]_A=0$, even if the ξ^i were arbitrary parameters rather than arbitrary functions. The crucial point is, however, that when as in this paper there is an *extended* invariance, Eqs. (17)–(19) are obtained as identities in form which which are not dependent on the applicability of the field equations.

The next two subsections illustrate these identities, briefly for the gravitational field, and in more detail for the electromagnetic field, which bears a closer resemblance to the meson-nucleon application to be made.

Gravitational Field

The four (Bianchi) identities (14) which follow in general relativity from the assumed covariance of the theory with respect to arbitrary transformations on the four coordinates, make $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$ have vanishing covariant divergence. (The $R_{\mu\nu}$ and R are the contractions of the curvature tensor.) This suggests that in the presence of nongravitational fields described by an energy-momentum tensor $T_{\mu\nu}$, the field equations be taken to be

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-kT_{\mu\nu}. \tag{20}$$

In fact, in simple enough field theories, this principle that a field's energy-momentum density should be capable of being the source of the gravitational field does away with any ambiguity in the definition of the energy-momentum density.

[4] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik* (Verlag Julius Springer, Berlin, 1931), second edition, p. 165.

Electromagnetic Field

Consider from now on transformations on the field variables alone

$$\delta x^{\mu}=0. \tag{21}$$

Generalizing a nomenclature of Pauli,[5] one might now define transformations of the first kind for which $\delta y^A=a^A\xi$ ($b^{A\mu}=0$), and transformations of the second kind for which $\delta y^A=b^{A\mu}\xi_{,\mu}$ ($a^A=0$). Then Eq. (19) states that a "current" $J^{\mu}\equiv-T^{\mu}$ of fields undergoing transformations of the first kind is conserved. Equations (18) and (17) state that this current is the divergence of an antisymmetric "superpotential"[6] $U^{\mu\lambda}$ consisting of fields undergoing transformations of the second kind.

The assumption of extended gauge invariance,

$$\delta\mathfrak{L}=0 \tag{22}$$

for

$$\delta\psi=ie\xi\psi,\quad \delta\bar{\psi}=-ie\xi\bar{\psi},\quad \delta A_{\mu}=\xi_{,\mu}, \tag{23}$$

requires by Eq. (19) that the charged-particle current

$$J_{\mu}=ie\left(\bar{\psi}\frac{\partial\mathfrak{L}}{\partial\bar{\psi}_{,\mu}}-\frac{\partial\mathfrak{L}}{\partial\psi_{,\mu}}\psi\right) \tag{24}$$

be conserved. Equation (17) in this case requires

$$\frac{\partial\mathfrak{L}}{\partial A_{\mu,\nu}}+\frac{\partial\mathfrak{L}}{\partial A_{\nu,\mu}}=0, \tag{25}$$

namely that the derivatives of the vector potential occur only in the antisymmetric combination $F_{\mu\nu}=A_{\mu,\nu}-A_{\nu,\mu}$. Equation (18) requires that

$$J_{\mu}=(\partial\mathfrak{L}/\partial A_{\mu,\nu})_{,\nu}=\partial\mathfrak{L}/\partial A_{\mu}, \tag{26}$$

or that the vector potential occur in the Lagrangian density only through an interaction term $J_{\mu}A_{\mu}$. In this way the assumption of extended gauge invariance leads to the form of coupling between the charged-particle and electromagnetic fields.

III. PION-NUCLEON SYSTEM

Under infinitesimal rotations of the isotopic-spin coordinate system, described by the three parameters ξ^i, the nucleon and (pseudoscalar) meson variables transform as isotopic spinors and vectors:

$$\delta\psi=\tfrac{1}{2}i\xi^i\tau^i\psi,\quad \delta\bar{\psi}=-\tfrac{1}{2}i\bar{\psi}\xi^i\tau^i,$$
$$\delta\phi^i=\epsilon^{ijk}\phi^j\xi^k. \tag{27}$$

Here the τ^i are the Pauli matrices in isotopic spin space and the ϵ^{ijk} are the alternating symbols. It is natural to assume that the Lagrangian density $\mathfrak{L}$ is invariant even when the ξ^i in Eq. (27) are allowed to be arbitrary functions of space-time. Defining this as local isotopic rotational spin invariance for the nucleon isotopic spinors and meson isotopic vectors,

$$0=\delta\mathfrak{L}=\left(\frac{\partial\mathfrak{L}}{\partial\phi^i_{,\mu}}\delta\phi^i+\frac{\partial\mathfrak{L}}{\partial\psi_{,\mu}}\delta\psi+\delta\bar{\psi}\frac{\partial\mathfrak{L}}{\partial\bar{\psi}_{,\mu}}\right)_{,\mu}$$
$$+[\mathfrak{L}]_{\phi^i}\,\delta\phi^i+[\mathfrak{L}]_{\psi}\delta\psi+\delta\bar{\psi}[\mathfrak{L}]_{\bar{\psi}}. \tag{28}$$

The identities (14) in this theory are

$$\tfrac{1}{2}i([\mathfrak{L}]_{\psi}\,\tau^i\psi-\bar{\psi}\tau^i[\mathfrak{L}]_{\bar{\psi}})=\epsilon^{ijk}\phi^j[\mathfrak{L}]_{\phi^k}. \tag{29}$$

The existence of these three identities is connected with the assumption that the local isotopic-spin coordinate system can be chosen arbitrarily.

Substituting from Eq. (27) into Eq. (28), one obtains, from the vanishing of the coefficient of ξ^i,

$$J_{\mu}{}^{i}{}_{,\mu}=0, \tag{30}$$

where

$$J_{\mu}{}^{i}=-\tfrac{1}{2}i\left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\mu}}\tau^i\psi-\bar{\psi}\tau^i\frac{\partial\mathfrak{L}}{\partial\bar{\psi}_{,\mu}}\right)+\epsilon^{ijk}\phi^j\frac{\partial\mathfrak{L}}{\partial\phi^k_{,\mu}} \tag{31}$$

is an isotopic spin current consisting of spinor and orbital parts. However, from the vanishing of the coefficient of $\xi^i_{,\mu}$,

$$J_{\mu}{}^{i}=0. \tag{32}$$

If the meson and the nucleon fields transform locally as isotopic vectors and spinors, then all components of the isotopic-spin current vanish everywhere. This result follows whenever the two fields ψ and ϕ both undergo transformations of the first kind only: a coupling between pions and nucleons cannot be brought about as a consequence of local isotopic spin invariance.

In the face of this result, two possibilities present themselves:

(1) According to Eqs. (18) and (19) if the nucleon and pion fields transformed by

$$\delta\psi=\tfrac{1}{2}i\xi^i\tau^i\psi+b_{\psi}{}^{\mu}\,\xi_{,\mu}$$
$$\delta\bar{\psi}=-\tfrac{1}{2}i\bar{\psi}\xi^i\tau^i+b_{\bar{\psi}}{}^{\mu}\,\xi_{,\mu} \tag{33}$$
$$\delta\phi^i=\epsilon^{ijk}\phi^j\xi^k+b_{\phi^i}{}^{\mu}\,\xi_{,\mu}$$

and

$$J_{\mu}=U^{\mu\lambda}{}_{,\lambda},$$
$$U^{\mu\lambda}=-U^{\lambda\mu}\equiv\frac{\partial\mathfrak{L}}{\partial\psi_{,\lambda}}\,b_{\psi}{}^{\mu}+b_{\bar{\psi}}{}^{\mu}\,\frac{\partial\mathfrak{L}}{\partial\bar{\psi}_{,\lambda}}+\frac{\partial\mathfrak{L}}{\partial\phi^l_{,\lambda}}\,b_{\phi^l}{}^{\mu}, \tag{34}$$

the current $J_{\mu}{}^{i}$ in Eq. (31) would be conserved and nonvanishing. This possibility of extending the constant isotopic spin rotations to something *different* from the local rotations (27) is being investigated.

(2) Alternatively one might insist on the local isotopic spin invariance of this section, i.e., $\xi^i=\xi^i(x)$ but no $\xi^i_{,\mu}$ appearing in the transformation law (27). Then the result of this section leads one to maintain this invariance by introducing additional fields which transform suitably. The addition of new fields does not,

[5] W. Pauli, Revs. Modern. Phys. **13**, 203 (1941).
[6] P. G. Bergmann and R. Schiller, reference 2.

however, illuminate in any simple way phenomena between pions and nucleons for which isotopic spin invariance was demanded in the first place.

IV. ADDITIONAL FIELDS

Yang and Mills[3] have recently sought to maintain local isotopic spin invariance by positing the existance of a certain "**b** field." This field is a vector in space-time and in isotopic spin space, and transforms according to the law

$$\delta b_\mu{}^i = \epsilon^{ijk} b_\mu{}^j \xi^k + \frac{1}{2\epsilon}\xi^i{}_{,\mu}. \tag{35}$$

It is not difficult to show from the identities (17)–(19) that this field carries isotopic spin itself and obeys nonlinear equations of motion. These properties (which correspond to no field yet observed) cannot then be avoided, as one might conceivably have hoped, by employing a different Lagrangian than the one used by Yang and Mills.

The identities corresponding to Eqs. (17) and (18) are

$$(\partial\mathcal{L}/\partial b_\mu{}^i{}_{,\nu}) + (\partial\mathcal{L}/\partial b_\nu{}^i{}_{,\mu}) = 0, \tag{36}$$

$$\epsilon^{ijk} b_\lambda{}^j (\partial\mathcal{L}/\partial b_\lambda{}^k{}_{,\mu}) = \partial\mathcal{L}/\partial b_\mu{}^i. \tag{37}$$

These relations indicate that the derivatives of the **b** field occur in the Lagrangian only in the antisymmetric combination $b_\mu{}^i{}_{,\nu} - b_\nu{}^i{}_{,\mu}$ and that the $b_\mu{}^i$ themselves occur only via the combination $f_{\mu\nu}{}^i = b_\mu{}^i{}_{,\nu} - b_\nu{}^i{}_{,\mu} - 2\epsilon\epsilon^{ijk} b_\mu{}^j b_\nu{}^k$. Since the Lagrangian is to be a scalar in space-time and in isotopic spin space, the most general form for the **b** field Lagrangian is

$$\mathcal{L} = \mathcal{L}(f_{\mu\nu}{}^i f_{\mu\nu}{}^i). \tag{38}$$

Corresponding to Eq. (19), a current

$$J_\mu{}^i = \epsilon^{ijk} b_\lambda{}^j (\partial\mathcal{L}/\partial b_\lambda{}^k{}_{,\mu}) \tag{39}$$

is conserved. This isotopic spin current contributed by the **b** field violates, wherever real **b** quanta are involved, the observed conservation of isotopic spin for pions and nucleons alone.

That the equations of motion of the **b** field are nonlinear follows, for example, from the observation that for an extended isotopic spin vector like $f_{\mu\nu}{}^i$, the covariant derivative is

$$D_\nu f_{\mu\nu}{}^i \equiv \partial_\nu f_{\mu\nu}{}^i + 2\epsilon\epsilon^{ijk} b_\nu{}^j f_{\mu\nu}{}^k.$$

These consequences can be avoided by conceiving of a field which undergoes transformations of the second kind only, so that according to Eq. (16) it makes no contribution to the isotopic current, yet does act as a right-hand side in Eq. (18).

The formalism of this paper is useful in deriving identities and conservation laws in more general situations. If the **b** field is coupled to nucleons and pions transforming according to Eq. (27), then from Eqs. (14) and (16) the three identities

$$\frac{1}{2\epsilon}([\mathcal{L}]_{b_\mu{}^i})_{,\mu} = \tfrac{1}{2}i([\mathcal{L}]_\psi\, \tau^i\psi - \bar{\psi}\tau^i[\mathcal{L}]_{\bar{\psi}}) - \epsilon^{ijk}(\phi^j[\mathcal{L}]_{\phi^k} + b_\lambda{}^j[\mathcal{L}]_{b_\lambda{}^k}) \tag{40}$$

must hold among the equations of motion, and the conserved current is

$$J_\mu{}^i = -\frac{i}{2}\left(\frac{\partial\mathcal{L}}{\partial\psi_{,\mu}}\tau^i\psi - \bar{\psi}\tau^i\frac{\partial\mathcal{L}}{\partial\bar{\psi}_{,\mu}}\right) + \epsilon^{ijk}\left(\phi^j\frac{\partial\mathcal{L}}{\partial\phi^k{}_{,\mu}} + b_\lambda{}^j\frac{\partial\mathcal{L}}{\partial b_\lambda{}^k{}_{,\mu}}\right). \tag{41}$$

V. CONCLUSIONS

The admissible couplings between meson-nucleon fields have been investigated from the invariant Lagrangian point of view. A distinction has been made between transformations linear in the arbitrary functions and transformations linear in the gradients of the arbitrary functions. Fields transforming in the first way contribute to a current which is conserved; fields transforming in the second way contribute to the superpotentials from which this current is obtained.

Assuming invariance under local isotopic spin rotations for the pion-nucleon system leads to the result that the total isotopic current density must vanish identically. Since this is inadmissible physically, one is led to consider (a) more general extended isotopic spin invariance principles for pions and nucleons, or (b) additional fields constructed to maintain invariance under local isotopic spin rotations. In either case, if the ordinary pion-nucleon isotopic current density is to be conserved, the most general extension of the restricted transformation law (27) admits only transformations of the second kind on the field variables. For the pion-nucleon fields this leads to relations (34) between the extended transformation contemplated and the invariant Lagrangian.

The **b** field introduced by Yang and Mills, on the other hand, since it undergoes in part transformations of the first kind, contributes to the isotopic spin current. This result, which prevents conservation of isotopic spin between nucleons and pions whenever the **b** field is present, is not avoidable by conceiving of different equations of motion.

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