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Some Notes on Nonrenormalizable Field Theory

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Some nonrecoil, derivative coupling theories, which are exactly soluble, are analyzed in order to discover what it is that produces the nonrenormalizable behavior typical of derivative coupling. This behavior is found to arise as the result of an essential singularity which the operators of the derivative theories possess at the origin in coordinate space and the resulting branch point the Fourier transforms of these operators possess at the origin in the complex coupling constant domain. This branch point causes the breakdown of the expansion of the Fourier transform of the operators in powers of the coupling constant and introduces the nonrenormalizable infinities. It is shown that a coordinate space coupling constant expansion is possible, and that a Fourier transform of the operators of the derivative theories may be defined by analytic continuations and made finite by renormalization.

I. INTRODUCTION

THE success of the renormalization program in identifying all of the infinite parts of quantum electrodynamics with a renormalization of the charge and mass of the electron has led to many attempts to duplicate this procedure with less tractable theories. Although these have sometimes succeeded, it has become clear that for many field theories it is not possible, with the usual methods, to remove the ambiguities by identifying the infinities with empirical constants. In particular, for derivative coupling theories, one finds that the usual renormalization procedure cannot be carried through because, no matter what subset of graphs is chosen as primitive, the degree of divergence increases indefinitely as one goes to higher order processes.

In what follows, several simple nonrecoil derivative coupling theories (all but one of which is nonrenormalizable by the usual methods) are analyzed in order to discover what it is that produces the nonrenormalizability which is characteristic of derivative coupling and whether, in spite of their very bad behavior in the power series expansion, something can still be done to make these theories finite and unambiguous. Perhaps the most striking result is that these theories in a sense are renormalizable; that is, it is possible to make all the operators of the theories finite by mass, Z_1 and Z_2 renormalizations.

The distinction between the renormalizable and nonrenormalizable theories considered occurs as follows: for the renormalizable theory, the renormalized propagator in coordinate space is analytic (in the finite plane) in the coordinate difference everywhere except at the origin, where it has a branch point. For small enough values of the coupling constant its Fourier transform exists and is analytic in the coupling constant in the neighborhood of $g^2/4\pi^2=0$. The renormalized propagators of all the nonrenormalizable theories, in coordinate space, are also analytic in the coordinate difference, in the finite plane, everywhere except at the origin. At the origin, however, they have essential singularities. Because of these essential singularities, the Fourier transform of the propagators for the nonrenormalizable theories exists only in one half of the complex coupling constant plane, and it sometimes turns out to be that half of the plane for which the Hamiltonian cannot be Hermitian. However, a Fourier transform can be defined by analytic continuation from the non-Hermitian half of the coupling constant plane. The Fourier transform in any case has a branch point at $g^2/4\pi^2=0$ which makes it a many-valued function of the coupling constant.

It then results that the set of progressively worse infinities, which in the power series expansion is associated with nonrenormalizability, is due solely to the failure of the expansion of the momentum space propagator about a branch point in coupling constant space.

The infinities which occur in the simple theories treated here, therefore, fall into two distinct classes: those which have nothing to do with the power series expansion and which occur in a similar fashion in both renormalizable and nonrenormalizable cases (renormalization constants), and those whose existence is due entirely to the coupling constant expansion (the nonrenormalizable infinities). This suggests that the behavior of the momentum space propagator, in the coupling constant, may divide field theories into three classes: those for which the propagator is analytic at the origin and is renormalizable; those for which the propagator is not analytic in the coupling constant at the origin, but a formal expansion of which gives an asymptotic series which is renormalizable; and those for which the behavior in the coupling constant at the origin is such that an expansion cannot be made at all.

It also turns out that the solutions of the nonrenormizable theories are so singular that it is not possible to freely interchange the order of such operations as differentiation and integration. The momentum space perturbation expansion sometimes implicitly interchanges these operations and, in so doing, may add

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spurious (sometimes infinite) terms to the correct solution, depending on the method of calculation. In the renormalizable case, the solutions are not so singular so that this difficulty does not occur, and the usual momentum space perturbation expansion gives the correct result.

In the second section, the propagators for several simple nonrecoil theories are obtained exactly by functional methods and are renormalized. The difference between the renormalizable and nonrenormalizable theories is then discussed in Sec. III. A perturbation method suitable for calculation with the nonrenormalizable theories is outlined in Sec. IV. In Sec. V, the operators of a nonrenormalizable theory are defined by analytic continuation. Appendix A gives the details of these analytic continuations and considers the multivalued behavior of the propagator as a function of the coupling constant. In Appendix B, the accuracy of a cutoff approximation to the nonrenormalizable theories is discussed.

II. THE PROPAGATOR FOR A CLASS OF NONRECOIL THEORIES

We shall, to illustrate the method, first obtain the propagator for a simple class of theories which has the following equations of motion for the nuclear field ψ, and the meson field ϕ:

$$\begin{aligned}\{-i(\partial/\partial t)+m+\delta m-g(\partial^n\phi/\partial t^n)\}\psi&=0\\ \{\Box^2-\mu^2\}\phi&=0.\end{aligned} \tag{1}$$

This can be thought of as a spinor field in the nonrecoil limit interacting with the nth derivative of a boson field with no vacuum polarization.[1] When $n=0$ we have the neutral scalar theory previously solved.[2,3] It will be seen that the method of solution would permit us to work out in a similar fashion any nonrecoil theory with a coupling term of the form $g\psi^*\alpha_i(D_i{}^n\phi)\psi$ where D^n is a differential operator and α is a commuting vector.

Since the technique used below is relatively new, it will be explained in some detail. Later, when the solutions are examined, it will be seen that a certain kind of perturbation expansion is possible even in the nonrenormalizable cases. This gives a series which can be summed when the α's commute. The reader who is not willing to follow the first method is referred to the second.

To find the propagator we employ the functional integral formulation of field theory first proposed by Feynman[4] and exploited recently by Edwards and Peierls[2] and Matthews and Salam.[5] Using this formulation the nucleon propagator for the fields above can be written as

$$S_{(n)}'(t-t')=\frac{\int\delta\phi\delta\bar\psi\delta\psi\psi(t)\bar\psi(t')\exp\{i\int\int\mathcal{L}^{(n)}(x_1)\delta(x_1-x_2)dx_1dx_2\}}{\int\delta\phi\delta\bar\psi\delta\psi\exp\{i\int\int\mathcal{L}^{(n)}(x_1)\delta(x_1-x_2)dx_1dx_2\}}=\langle T(\psi(t)\bar\psi(t'))\rangle_0, \tag{2}$$

where

$$\mathcal{L}^{(n)}=\frac{1}{2}\left[\bar\psi\left\{-i\frac{\partial}{\partial t}+m+\delta m-g\frac{\partial^n\phi}{\partial t^n}\right\}\psi+\text{Herm. conj.}-\{(\Box\phi)^2+\mu^2\phi^2\}\right].$$

Following Matthews and Salam the integration over ψ and $\bar\psi$ is performed, and, neglecting the effects of vacuum polarization, one gets an equation which (except for the factor $-i$) is identical to Eq. (21) of Edwards and Peierls:

$$S_{(n)}'(t-t')=-iN\int\delta\phi G_4{}^{(n)}(t,t';\phi)\times\exp\left\{-\frac{i}{2}\int\int\phi(\xi)\Delta^{-1}(\xi,\xi')\phi(\xi')d\xi d\xi'\right\}, \tag{3}$$

where

$$\Delta^{-1}(\xi,\xi')=\delta(\xi-\xi')(-\Box^2+\mu^2)_{\xi'},$$

$$N^{-1}=\int\delta\phi\exp\left\{-\frac{i}{2}\int\int\phi(\xi)\Delta^{-1}(\xi,\xi')\phi(\xi')d\xi d\xi'\right\},$$

and $G_4{}^{(n)}(t,t';\phi)$ is the solution of the equation

$$\left\{-i\frac{\partial}{\partial t}+m+\delta m-g\frac{\partial^n\phi}{\partial t^n}\right\}G_4{}^{(n)}(t,t';\phi)=\delta(t-t') \tag{4}$$

and is a functional of ϕ. The symbol $\int f[\phi]\delta\phi$ means a functional integration over all functions ϕ. This integral is still to be precisely defined, but in the simple cases considered it will be seen that the method of solution is to manipulate the variable function ϕ by linear transformations until the functional integration cancels out.

The solution of Eq. (4) is

$$G_4{}^{(n)}(t,t';\phi)=i\theta(t-t')\exp\left\{-i(m+\delta m)(t-t')+ig\int_{t'}^{t}\frac{\partial^n\phi}{\partial\xi^n}d\xi\right\} \tag{5}$$

where

$$\theta(t-t')\begin{matrix}=1 & t-t'>0\\ =0 & t-t'<0\end{matrix}.$$

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[1] Equations of this form have been solved and renormalized also by R. Arnowitt and S. Deser, preceding paper [Phys. Rev. **99**, 349 (1955)].

[2] S. F. Edwards and R. E. Peierls, Proc. Roy. Soc. (London) **A224**, 24 (1954).

[3] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

[4] R. P. Feynman, Revs. Modern Phys. **20**, 376 (1948).

[5] P. Matthews and A. Salam, Nuovo cimento **II**, 120 (1955).

Putting this into Eq. (3) one has, with $\tau\equiv t-t'$

$$S_{(n)}'(\tau)=\theta(\tau)e^{-i(m+\delta m)\tau}N\int\delta\phi\exp\left\{ig\int_{t'}^{t}\frac{\partial^n\phi(\xi)}{\partial\xi^n}d\xi-\frac{i}{2}\int\int\phi(\xi)\Delta^{-1}(\xi,\xi')\phi(\xi')d\xi d\xi'\right\}. \quad (6)$$

Now let

$$\phi(\xi)=\phi'(\xi)+g\int_{t'}^{t}\Delta^{(n)}(\xi,\eta)d\eta;\quad \delta\phi'=\delta\phi,$$

where

$$\int_{-\infty}^{\infty}d^4\xi\Delta^{-1}(x,\xi)\Delta^{(n)}(\xi,x')=\delta^{(n)}(t-t')\delta(\mathbf{r}-\mathbf{r}').$$

This gives

$$S_{(n)}'(\tau)=\theta(\tau)\exp\left\{-i(m+\delta m)\tau+\frac{ig^2}{2}\int_{t'}^{t}d\xi\int_{t'}^{t}d\xi'\frac{\partial^n}{\partial\xi^n}\Delta^{(n)}(\xi,\xi')\right\} \quad (7)$$

as the functional integral just cancels out.

$\Delta^{(n)}(\xi,\xi')$ can be represented by

$$\Delta^{(n)}(\xi,\xi')=\frac{\partial^n}{\partial\xi_0'^n}\Delta^{(0)}(\xi,\xi'). \quad (8)$$

Then the second part of the exponential of Eq. (7) becomes

$$\frac{ig^2}{2}\int_{t'}^{t}d\xi_0\int_{t'}^{t}d\xi_0'\frac{\partial^n}{\partial\xi_0^n}\frac{\partial^n}{\partial\xi_0'^n}\Delta^{(0)}(\xi,\xi'), \quad (9)$$

which at $\boldsymbol{\xi}=\boldsymbol{\xi}'$ is [letting $\omega=(p^2+\mu^2)^{\frac{1}{2}}$]

$$\frac{-g^2}{8\pi^2}\int_{t'}^{t}d\xi_0\int_{t'}^{t}d\xi_0'\int_0^\infty\omega^{2n-1}p^2dp\times\left(e^{-i\omega|\xi_0-\xi_0'|}+\begin{matrix}0\\ i\omega\delta(\xi_0-\xi_0')+\cdots\end{matrix}\right)\begin{matrix}n=0\\ n>0\end{matrix} \quad (10)$$

where

$\Delta^{(0)}(\xi,\xi')$ at $\boldsymbol{\xi}=\boldsymbol{\xi}'$ is $\Delta^{(0)}(0,\xi_0;0,\xi_0')$

$$=\frac{i}{4\pi^2}\int_0^\infty p^2dp\frac{e^{-i\omega|\xi_0-\xi_0'|}}{\omega},$$

and the terms omitted in (10) are derivatives of $\delta(\xi_0-\xi_0')$ which give zero when integrated from t' to t.

Doing the ξ_0 and ξ_0' integrations, (10) becomes

$$\frac{g^2}{4\pi^2}\int_0^\infty p^2dp\omega^{2n-1}\left[\frac{1}{\omega^2}(e^{-i\omega\tau}-1)+\begin{matrix}i\tau/\omega\\ 0\end{matrix}\right]\begin{matrix}n=0\\ n>0.\end{matrix} \quad (11)$$

This gives for the propagator:

$$S_{(n)}'(\tau)=\theta(\tau)e^{-im\tau}R^{(n)}(\tau),$$

where

$$R^{(n)}(\tau)=\exp\left\{\frac{g^2}{4\pi^2}\int_0^\infty p^2dp\omega^{2n-3}(e^{-i\omega\tau}-1)\right\} \quad (12)$$

and

$$\delta m=\frac{g^2}{4\pi^2}\int_0^\infty p^2dp\omega^{-2}\quad\text{for}\quad n=0;\quad \delta m=0\quad\text{for}\quad n>0.$$

The propagator is renormalized by a multiplicative constant $Z_2^{(n)}$:

$$S_{(n)}'(\tau)=Z_2^{(n)}S_c^{(n)}(\tau), \quad (13)$$

where

$$Z_2^{(n)}=\exp\left\{-\frac{g^2}{4\pi^2}\int_0^\infty p^2dp\omega^{2n-3}\right\}.$$

The Z_2 renormalization makes $S_c^{(n)}(\tau)$ finite for $\tau>0$. We see that this multiplication shifts the value of the propagator so that where the unrenormalized function is finite for $\tau=0$ and zero for $\tau\neq0$, the renormalized function is finite for $\tau>0$ and infinite for $\tau=0$.[6]

For $n>0$ it is convenient to investigate the behavior of the renormalized radiative correction term $R_c^{(n)}(\tau)$ for meson mass $\mu=0$, as this does not change the behavior of the function at its singularities. Then $R_c^{(n)}(\tau)$ becomes:

$$R_c^{(n)}(\tau)=\exp\left\{\frac{g^2}{4\pi^2}\int_0^\infty p^{2n-1}dpe^{-ip\tau}\right\}.$$

The integral in the exponent is well defined for τ in the lower half plane and can be continued to all points in the complex τ plane, except $\tau=0$, to give[7]:

$$R_c^{(n)}(\tau)=\exp\{(g^2/4\pi^2)(-1)^n(2n-1)!\tau^{-2n}\}.$$

Letting $\gamma=(-1)^n(g^2/4\pi^2)$, we have for $n>0$, the radiative correction term

$$R_c^{(n)}(\tau)=\exp\{\gamma(2n-1)!\tau^{-2n}\}, \quad (14)$$

where $(-1)^n\gamma=(-1)^n\gamma^*\geqslant0$ for Hermitian Hamiltonians.

With the interaction term $g\psi^*\alpha_i(D_i^n\phi)\psi$ (where $\alpha_i\alpha_j-\alpha_j\alpha_i=0$; $\alpha_1\alpha_1=\alpha_2\alpha_2=\cdots=1$) everything goes through almost exactly as above (except that one gets

[6] Although $Z_2^{(n)}$ above appears always to be smaller than one, actually this depends upon the way the limit $\tau\to0$ is taken. Since $Z_2^{(n)}$ in the $n>0$ cases is

$$Z_2^{(n)}=\lim_{\tau\to0}\exp(\gamma\tau^{-2n}),$$

the value of a function at its essential singularity, this value depends upon the path by which it was reached.

[7] If the meson mass was kept unequal to zero, the singular functions in the exponential would be Hankel functions of argument $\mu\tau$ and their derivatives.

a mass renormalization). Equation (7) becomes

$$S'(\tau)=\theta(\tau)\exp\left\{-i(m+\delta m)+\frac{ig^2}{2}\alpha_i\alpha_j\int\int_{t'}^{t}d\xi_0'd\xi_0 D_\xi{}^i D_{\xi'}{}^j\Delta^0(\xi,\xi')\right\}. \quad (7a)$$

For $D=\nabla$, (11) becomes

$$\frac{g^2}{4\pi^2}\int_0^\infty k^4dk\omega^{-3}(e^{-i\omega\tau}-1+i\omega\tau) \quad (11a)$$

and so, after renormalization—for meson mass $\mu=0$, this is just identical to $R_c^{(1)}(\tau)$ above. In general any differential operator D^n will give results which, with respect to the singularities of the solutions, are identical to those above.

III. COMPARISON OF THE RENORMALIZABLE AND NONRENORMALIZABLE THEORIES

The differences between the theories with interaction terms of the form $g\psi^*\alpha_i(D_i{}^n\phi)\psi$, which we shall designate as nonrenormalizable or $n>0$ theories, and the renormalizable or $n=0$ theory (with the interaction term $g\psi^*\phi\psi$), result almost entirely from the type of singularity the operators of these theories possess at the origin in coordinate space.

For $n=0$ the renormalized propagator is singular at $\tau=0$ for Hermitian Hamiltonians, but it is summable at this point for any complex value of γ, $|\gamma|<1$. For $n=0$, $S_c^{(0)}(\tau)$ is[8]

$$S_c^{(0)}(\tau)=\theta(\tau)\exp\left\{-im\tau+\gamma\int_0^\infty\frac{p^2dp}{\omega^3}e^{-i\omega\tau}\right\}. \quad (15)$$

In order to find the behavior at $\tau=0$ we write

$$\int_0^\infty dp\frac{p^2}{\omega^3}e^{-i\omega\tau}=\frac{\pi i}{2}H_0^{(1)}(\mu\tau)-\mu^2\int_0^\infty\frac{dp}{\omega^3}e^{-i\omega\tau}. \quad (16)$$

For $\tau\ll 1$ the second term can be neglected as we are interested only in the singular part at $\tau=0$. Then (16) becomes

$$\int_0^\infty dp\frac{p^2}{\omega^3}e^{-i\omega\tau}\sim -\ln\bar{\gamma}(\mu\tau)+\ln 2$$

where

$$\ln\bar{\gamma}=C\simeq 0.577\cdots.$$

Thus $R_c^{(0)}(\tau)$ for $\tau\ll 1$ is

$$R_c^{(0)}(\tau)\sim(\text{const})(\mu\tau)^{-\gamma}. \quad (17)$$

In coordinate space, for $\gamma>0$, $R_c^{(0)}(\tau)$ is singular at $\tau=0$. However, for $|\gamma|<1$, the Fourier transform of $S_c^{(0)}(\tau)$ exists and is analytic in γ at $\gamma=0$. One can reproduce the renormalized perturbation series by expanding $R_c^{(0)}(\tau)$ in powers of γ. Then each term of the expansion converges (involving integrals like $\int_0 \log^n\tau d\tau$) and, for $|\gamma|<1$, the series itself converges.

For $n>0$, as can be seen from Eq. (14), the radiative correction term $R_c^{(n)}(\tau)$ has an essential singularity at $\tau=0$. As a consequence of this, the propagator $S_c^{(n)}(\tau)$ is not summable at $\tau=0$ for γ in the right half plane and thus the Fourier transform of $S_c^{(n)}(\tau)$ is defined only for γ in the left half complex plane. The fact that the Fourier transform of the propagator exists for certain complex values of γ, however, suggests that a Fourier transform could be defined over the entire domain of γ by an analytic continuation. This possibility is exploited, in Sec. V, in order to define the momentum space operators of the $n>0$ theories. However, the Fourier transform of the propagator so defined has a branch point at $\gamma=0$. It is just this that makes the momentum space coupling constant expansion impossible and introduces a nonrenormalizable set of infinities into these theories. For the characteristic nonrenormalizability in which the degree of divergence at infinity (in the Fourier transform space) increases with each term of the series is reflected in the breakdown of the expansion $\int_0\exp[-(\gamma/\tau^{2n})-i(q+m)\tau]d\tau$ in powers of γ. The expansion, of course, breaks down independent of the sign of γ.

Another consequence of the high degree of singularity possessed by the $n>0$ theories, is the fact that in the calculation of the operators of these theories spurious terms may arise if the order of the operations of differentiation and integration are interchanged. This arises, in particular, for the nonrecoil theories treated above when the differential operators involve the time. In the nth derivative theories each meson propagation line is operated on to give

$$-\frac{1}{4\pi^3}\int_0^\infty p^2dp\frac{d^n}{d\xi_1{}^n}\frac{d^n}{d\xi_2{}^n}\int_{C_F}\frac{e^{+iE(\xi_1-\xi_2)}}{(E-\omega)(E+\omega)}dE.$$

Putting $(\pm i)^n g^n E^n$ at each vertex, as one would do in the usual perturbation expansion, is equivalent to reversing the orders of the operations and could introduce spurious terms even if the entire expansion did not break down.

In all the theories, infinite constants appear which can be absorbed by Z_2, Z_1 and mass renormalizations and whose existence is independent of the method of calculation. These "traditional" divergences do not arise because of the power series expansion, but their identification and absorption into physical constants is made difficult by the expansion. The nonrenormalizable divergences associated with the derivative coupling theories are produced by the incorrect expansion and thus disappear when the method of calculation is changed.

[8] The meson mass μ cannot be set equal to zero in the $n=0$ case because a logarithmic infrared divergence would result.

IV. A PERTURBATION EXPANSION FOR THE NONRENORMALIZABLE THEORIES

Because of the behavior of the momentum space propagator at $\tau=0$ for the $n>0$ theories, the usual coupling constant expansion becomes impossible. However, since the propagator in coordinate space for $\tau\neq 0$ is analytic in γ, it is possible to obtain all the operators even for the $n>0$ theories by a power series expansion in γ as long as one remains in coordinate space and keeps $\tau\neq 0$. The fact that such an expansion does exist for the nonrenormalizable cases would make it possible to explore more complicated theories for which it is not such a trivial matter to obtain an exact solution.[9] Also, this provides an alternative derivation of the results of Sec. II.

We will consider a general nonrecoil theory, with an interaction term $g\psi^*\alpha\cdot(D^n\phi)\psi$, in which the nucleon propagator (everything will be in coordinate space) is given by

$$S_F(t_2-t_1)=\theta(t_2-t_1)e^{-im(t_2-t_1)}, \tag{18}$$

the meson propagator by

$$\Delta_F(x_2-x_1)=\frac{1}{16\pi^3}\int_0^\infty d^3p e^{-ip(\mathbf{r}_2-\mathbf{r}_1)}\frac{e^{-i\omega|t_2-t_1|}}{\omega}, \tag{19}$$

and the vertex operators by

$$g\alpha_i D_i^n. \tag{20}$$

Then the $2l$-order propagator is given by a sum of diagrams in which a nucleon line, beginning at x' and ending at x, emits and absorbs mesons at the $2l$ points $x_1\cdots x_{2l}$, where the meson lines connect all permutations of the coordinate pairs $(x_1x_2)(x_3x_4)\cdots(x_{2l-1}x_{2l})$. $S^{(2l)}(\tau)$ is

$$S^{(2l)}(\tau)=\int_{-\infty}^{+\infty}dt_{2l}\cdots\int_{-\infty}^{+\infty}dt_1\theta(t_1-t')\cdots\theta(t-t_{2l})$$
$$\times(-1)^l\frac{g^{2l}}{l!2^l}e^{-im(t_1-t')-im(t_2-t_1)-\cdots-im(t-t_{2l})}$$
$$\times\alpha_{\mu1}D_{\mu1}{}^n\alpha_{\mu2}D_{\mu2}{}^n\cdots\alpha_{\mu2l}D_{\mu2l}{}^n\sum_{\substack{\text{Permutations}\\ \text{of } x_1\cdots x_{2l}}}\Delta_F(x_2-x_1)$$
$$\times\Delta_F(x_4-x_3)\cdots\Delta_F(x_{2l}-x_{2l-1}). \tag{21}$$

Here $D_{\mu j}{}^n$ is a differential operator which acts on the coordinate x_j. The factor $1/2^l l!$ is introduced because we sum over all permutations of the coordinates $x_1\cdots x_{2l}$ rather than all pairings. This just duplicates $l!2^l$ identical diagrams.

The θ functions combine to give limits of integration over the t variables:

$$t'\leqslant t_1\leqslant t_2\leqslant\cdots\leqslant t_{2l}\leqslant t,$$

and the exponentials give $e^{-im(t-t')}$. Associating the differential operators $D_{\mu j}{}^n$ with the coordinates on which they operate, $S^{(2l)}(\tau)$ becomes

$$S^{(2l)}(\tau)=(-1)^l\frac{g^{2l}}{2^l l!}\theta(\tau)e^{-im\tau}\int_{t'}^{t}dt_{2l}\int_{t'}^{t_{2l}}dt_{2l-1}\cdots$$
$$\times\int_{t'}^{t_2}dt_1\alpha_{\mu1}\alpha_{\mu2}\cdots\alpha_{\mu2l}\sum_{\text{Permutations}}D_{\mu1}{}^nD_{\mu2}{}^n\Delta_F(x_2-x_1)\cdots$$
$$\times D_{\mu2l}{}^nD_{\mu(2l-1)}{}^n\Delta_F(x_{2l}-x_{2l-1}). \tag{22}$$

If the α's commute the integrand is symmetric in $t_1\cdots t_{2l}$ and we may change the limits of integration to get,

$$S^{(2l)}(\tau)=\frac{g^{2l}}{2^l l!}\frac{(-1)^l}{(2l)!}\theta(\tau)e^{-im\tau}\alpha_{\mu1}\alpha_{\mu2}\cdots\alpha_{\mu2l}$$
$$\times\sum_{\text{Permutations}}\int_{t'}^{t}dt_{2l}\int_{t'}^{t}\cdots\int_{t'}^{t}dt_1D_{\mu1}{}^nD_{\mu2}{}^n\Delta_F(x_2-x_1)\cdots$$
$$\times D_{\mu2l}{}^nD_{\mu(2l-1)}{}^n\Delta_F(x_{2l}-x_{2l-1}). \tag{23}$$

Using the representation (19) for $\Delta_F(x_2-x_1)$ and assuming for an explicit example that $D^n=\nabla$, we can do the t integrations. Then we have for $S^{(2l)}(\tau)$:

$$S^{(2l)}(\tau)=\frac{g^{2l}}{l!}\frac{1}{(2l)!}\theta(\tau)e^{-im\tau}Q^l\alpha_{\mu1}\cdots$$
$$\alpha_{\mu2l}\sum_{\text{Permutations}}\delta_{\mu1,\mu2}\delta_{\mu3,\mu4}\cdots\delta_{\mu(2l-1),\mu2l}, \tag{24}$$

where

$$Q=\frac{1}{16\pi^3}\int\frac{k_{\mu i}k_{\mu j}d^3k}{\omega^3}\{e^{-i\omega\tau}-1+i\omega\tau\}$$
$$=\frac{1}{4\pi^2}\frac{1}{3}\int\frac{k^4dk}{\omega^3}\{e^{-i\omega\tau}-1+i\omega\tau\}$$

at $\mathbf{x}=\mathbf{x}'$ and is diagonal because of the antisymmetry of $k_{\mu i}k_{\mu j}$. Since the α's commute the sum over permutations gives $(2l)!$ identical factors $(\alpha_\mu\alpha_\mu)^l=3^l$ for a three dimentional ∇. Therefore, we have

$$S^{2l}(\tau)=g^{2l}3^lQ^l/l!\theta(\tau)e^{-im\tau} \tag{25}$$

and

$$S_F'(\tau)=\sum_{l=0}^{\infty}S^{(2l)}(\tau)=\theta(\tau)e^{-im\tau}$$
$$\times\exp\left\{\frac{g^2}{4\pi^2}\int_0^\infty\frac{k^4dk}{\omega^3}(e^{-i\omega\tau}-1+i\omega\tau)\right\},$$

which checks with Eqs. (7a) and (11a). Other operators of the theory such as the vertex function and the meson-nucleon Green function can be calculated in a similar fashion by inserting one or more external lines into the propagator and again summing over all possible permutations.

[9] Arnowitt and Deser (reference 1) have investigated the $ps(pv)$ theory, including recoil, with the use of an expansion in a parameter which is not related to the coupling constant but is proportional to the proper distance Mx_μ.

The crucial point in the calculation is the assertion that the α's commute. If they did not, then each pairing of the coordinates would have a different weight and there would no longer be symmetry in the $t_1\cdots t_{2l}$ variables. Then the t integrations cannot be done so easily, and the problem is no longer trivial.

It is of interest to inspect this perturbation expansion of the propagator for the nonrenormalizable theories. For $t\neq 0$ it is clear that term by term the infinite parts can be removed by mass and Z_2 renormalizations. For $t=0$ the renormalized expansion breaks down and this just is due to the singularity of the exact solution at this point. One also sees that the terms of the series correspond, as they must, to an expansion of the exact propagator in powers of γ.

V. DEFINITION OF THE OPERATORS FOR THE NONRENORMALIZABLE THEORIES

In order that the nonrenormalizable theories treated above be physically meaningful, it is at least necessary that the Fourier transforms of the various operators of the theories be defined, as these Fourier transforms are closely related to quantities, such as the scattering amplitudes, which are comparable with experiments. The scattering due to the diagram in Fig. 1, for example, is proportional to the Fourier transform of the propagator; more generally, the meson-nucleon scattering amplitude is given by the Fourier transform of the meson-nucleon Green's function which will involve, with other quantities, the Fourier transform of the propagator.

The renormalized propagators, for the n larger than zero theories, are given by

$$S_c^{(n)}(\tau)=\theta(\tau)e^{-im\tau}\exp\{\gamma(2n-1)!\tau^{-2n}\}, \quad (26)$$

and the Fourier transform of $S_c^{(n)}(\tau)$ is (suppressing the superscript n)

$$S(q)=\int_0^\infty e^{-i(m+q)\tau}\exp\{\gamma(2n-1)!\tau^{-2n}\}d\tau. \quad (27)$$

This is defined for $m+q$ in the lower half plane $(m+q-i\epsilon)$ and for γ in the left half plane. We want to extend the domain of definition of Eq. (27) to all complex values of γ. When this is done the Fourier transform of the propagator can be defined for those theories (n even; $n>0$) for which the requirement that the Lagrangian be Hermitian would place γ in the right half plane; also the singularity that the Fourier transform of the propagator possesses at $\gamma=0$ can be examined. This turns out to be a logarithmic branch point and introduces an ambiguity into the theory because the Fourier transforms of the operators become many-valued functions of the coupling constant.

$S(q)$ can be analytically continued from the left half plane into the entire domain of γ by the procedure given in Appendix A. Equation (A2) of this appendix

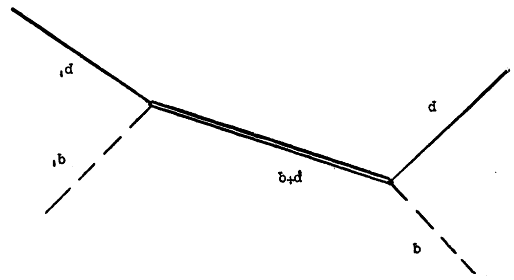

FIG. 1. A Feynman diagram for the scattering of a meson by a nucleon. The double line represents the nucleon propagator with all radiative connections: $S_c(p+q)$.

gives a representation of $S(q)$

$$S(q)=\int_{C\theta}\exp\{-i(m+q)\tau-\lambda(2n-1)!\tau^{-2n}e^{i\theta}\}d\tau, \quad (28)$$

where $\gamma=-\lambda e^{i\theta}$ and $\lambda=\lambda^*\geqslant 0$, which is defined for the entire domain of γ. From Eq. (28) one can see that $S(q)$ is a many valued function of γ, as there will be an infinite number of contours $C_{\theta+2l\pi}$ which correspond to a given $\gamma=-\lambda e^{i\theta}e^{i2l\pi}$, $l=0\pm1\pm2\cdots$.

In the $n=1$ case, where γ must be real and smaller than zero for the physically interesting Hamiltonians, the values of θ are restricted to even multiples of π, $\theta=2l\pi$, where $l=0, \pm1, \pm2\cdots$. We can define $S_0(q)$ (that function of q for which $l=0$) as the principle value of $S(q)$. It is $S_0(q)$ that would be reached in a "natural" way beginning with a cut-off theory and permitting the cutoff to go to infinity. (See Appendix B.) $S_l(q)$ is the sum of $S_0(q)$ and terms which are the contributions of the integrand of Eq. (28) when integrated over a contour which encloses the point $\tau=0$.

In a theory which contained scattering, if the same general results were valid, the extra terms, while not changing the residue of the propagator at $q+m=0$, would give an additional contribution to the scattering which did not necessarily have the same energy dependence as the scattering due to the principal value. Some explicit evaluations of these terms are given in Appendix A.

In the $n=2$ case γ must be real and larger than zero for Hermitian Lagrangians. Then θ is an odd multiple of π, $\theta=(2l+1)\pi$, where $l=0, \pm1, \pm2\cdots$. Again one gets a many-valued propagator but here the principal value $S_0(q)$ would not be approached by a cut-off solution. (The cut-off solution diverges as the cutoff approaches infinity.) The fact that two such cases occur seems to be a peculiarity of the nonrecoil theories. In the four-dimensional case, the results obtained by Arnowitt and Deser[1] show that due to the presence of functions like $\exp(-g^2/x^2)e^{-ikx}$ which must be integrated over all of space-time—positive and negative values of x^2—no cut-off solution would approach the continuation which gives the finite Fourier transform.

The point $\gamma=0$ is a finite branch point, for any value of n, and is shared by all the Riemann surfaces. Thus, as the interaction goes to zero (as γ approaches zero from any direction) the Fourier transform of the propagator becomes, unambiguously, the free propagator.

Once the Fourier transform of the propagator has been defined the other operators of the theory are also defined and can be made finite by renormalizations. The vertex function for example is:

$$\bar{\Gamma}^{(n)}(t,t';\xi)=N\int\delta\phi\phi(\xi)G_4^{(n)}(t,t';\phi)\times\exp\left\{-i/2\int\int\phi(\xi)\Delta^{-1}(\xi,\xi')\phi(\xi')d\xi d\xi'\right\}, \quad (29)$$

where $\bar{\Gamma}^{(n)}$ is related to the usual vertex function by

$$\int\Delta^{-1}(\xi,\xi')S^{-1}(y,t')S^{-1}(x,t)\bar{\Gamma}^{(n)}(x,y;\xi')dxdyd\xi'=\Gamma^{(n)}(t,t';\xi). \quad (30)$$

By a procedure entirely analogous to that of Sec. II, one gets for $\bar{\Gamma}^{(n)}$

$$\bar{\Gamma}^{(n)}(t,t';\xi)=gS_{(n)}'(\tau)\int_{t'}^{t}\Delta^{(n)}(\xi,\eta_0)d\eta_0. \quad (31)$$

The Fourier transform of this operator is well defined if the Fourier transform of $S'(\tau)$ is defined, and there are no additional ambiguities. Let

$$\bar{\Gamma}_c^{(n)}=Z_1^{(n)}\bar{\Gamma}^{(n)}. \quad (32)$$

Then $\bar{\Gamma}_c^{(n)}$ will be finite if

$$Z_1^{(n)}=Z_2^{(n)}. \quad (33)$$

The same kind of argument is applicable to the meson-nucleon Green function

$$G(t,t';\xi,\xi')=g^2\int_{t'}^{t}\Delta^{(n)}(\xi,\eta_0)d\eta_0\times\int_{t'}^{t}\Delta^{(n)}(\xi',\eta_0')d\eta_0'S_{(n)}'(\tau) \quad (34)$$

and to the other operators of these theories.

The matrix elements between free particle states of the operators which describe scattering amplitudes [such as the meson-nucleon Green function, Eq. (34)] are all zero. This is because the interactions of the theories treated can be transformed away and contain no scattering of mesons by nucleons.[10] The procedure outlined in the foregoing, however, makes it possible to define finite matrix elements for virtual transitions and thus to make all the operators finite for any value of n, in spite of the fact that for $n>0$ the operators, from the usual perturbation series point of view, are not renormalizable.

[10] G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949), p. 47.

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APPENDIX A. BEHAVIOR OF THE PROPAGATOR IN THE γ DOMAIN

The existence of the Fourier transform of $S_c^{(n)}(\tau)$ for part of the complex γ plane makes possible the definition of a Fourier transform of $S_c^{(n)}(\tau)$, for the entire domain of γ, by the process of analytic continuation.[11] If $-\gamma=\lambda e^{i\theta}$ where $\lambda=\lambda^*\geqslant 0$, the Fourier transform of $S_c^{(n)}(\tau)$ is

$$S(q)=\int_0^\infty\exp\{-i(m+q)\tau-\lambda(2n-1)!\tau^{-2n}e^{i\theta}\}d\tau, \quad (A1)$$

and this is well defined for $m+q$ in the open lower half plane and for γ in the left half plane $(-\pi/2\leqslant\theta\leqslant\pi/2)$. If the contour of integration in the τ plane is now deformed to C_θ, as shown in Fig. 2, we obtain the analytic continuation of Eq. (A1) into the entire domain of γ. $S(q)$ is then

$$S(q)=\int_{C\theta}\exp\{-i(m+q)\tau-\lambda(2n-1)!\tau^{-2n}e^{i\theta}\}d\tau. \quad (A2)$$

This is a many-valued function of γ since there are an

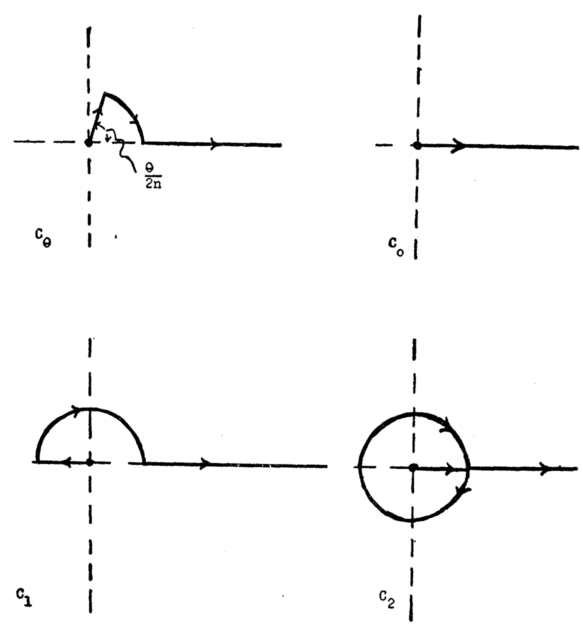

FIG. 2. Contours in the τ plane which define the Fourier transform of the propagator in various regions of the γ domain. C_θ applies for any value of n. C_0, C_1, and C_2 are the contours which give $S_0(q)$, $S_1(q)$, and $S_2(q)$ for the case $n=1$.

[11] The author wishes to acknowledge some very helpful discussions on this point with Professor A. Beurling and Dr. P. Malliavin of the Institute for Advanced Study.

infinite number of values of θ which correspond to a given γ and each of these give a different $S(q)$ due to the contour C_θ. Some of the contours that arise are shown in Fig. 2.

As an example of the kind of behavior $S(q)$ exhibits let us investigate in detail the case $n=1$. For this case $\gamma=-g^2/4\pi^2$ (in order that the Hamiltonian be Hermitian) so that the relevant values of θ are $\theta=2l\pi$ where $l=0, \pm 1, \pm 2\cdots$. Let $S_l(q)$ be the function $S(q)$ when $\theta=2l\pi$. Then

$$S_l(q)=S_0(q)+T_l(q), \tag{A3}$$

where $S_0(q)$ is defined as the principal value of $S(q)$ and $T_l(q)$ is the contribution that arises due to integrations, such as those in Fig. 2, which surround the point $\tau=0$. The procedure outlined above would also be applicable if the meson mass μ were not set equal to zero. However, the explicit evaluation of $T_l(q)$ then becomes much more difficult.

$T_2(q)$, which is the contribution of the integrand of Eq. (A2) in a complete, clockwise circuit of the origin, is (for $n=1$)

$$T_2(q)=2\pi\sum_{r=1}^{\infty}\frac{(\lambda)^r(m+q)^{2r-1}}{r!(2r-1)!}. \tag{A4}$$

$T_2(q)$ goes to zero as $(m+q)\to 0$ and as $\lambda\to 0$. As $m+q$ or λ become infinite along the real axis $T_2(q)$ also becomes infinite. This latter behavior is true for all odd n while for n even $(n>0)$ $T_2(q)$ oscillates and goes to zero as λ or $(m+q)$ become infinite. For the meson mass unequal to zero $T_2{}^\mu(q)$ remains finite as $m+q\to 0$ and is zero for $\lambda=0$. However, the explicit form of $T_2{}^\mu(q)$ would be much more complicated than that of Eq. (A4) and its dependence as $q\to\infty$ has not been ascertained as yet.

A peculiarity of the meson mass zero case is the logarithmic branch point $S(q)$ possesses, in the q domain, at the point $m+q=0$. This corresponds to the production of mesons with arbitrary energy. The value at $S(q)$ in the $m+q$ domain differs from branch to branch of the Riemann surface by just $T_2(q)$.

In examining the behavior of $S_l(q)$ for the various values of l it is convenient to separate off the singular part of $S_l(q)$ at $m+q=0$. We then have (for $n=1$)

$$S_0(q)=\frac{-i}{m+q}+\int_0^\infty e^{-i(m+q)\tau}h(\tau)d\tau, \tag{A5}$$

where $h(\tau)=\exp(-\lambda/\tau^2)-1$. The second term of Eq. (A5) contains the radiative correction to the propagator; it is finite for all values of q and goes to zero as $\lambda\to 0$. For $l\neq 0$ we get

$$S_1(q)=\frac{-i}{m+q}-\int_0^\infty e^{+i(m+q)\tau}h(\tau)d\tau+\theta(m+q)T_2(q)$$
$$S_2(q)=\frac{-i}{m+q}+\int^\infty e^{-i(m+q)\tau}h(\tau)d\tau+T_2(q),\text{ etc.} \tag{A6}$$

As $m+q\to 0$, $S(m+q)$ becomes

$$S_l(q)=-i/m+q+\text{finite terms}, \tag{A7}$$

and for $\lambda=0$, $S_l(q)=-i/m+q$.

We can obtain a crude idea of the behavior of these functions, for small values of $m+q$ and λ, by replacing $h(\tau)$ by the step function

$$h(\tau)\simeq-\theta[3/2(\lambda)^{\frac{1}{2}}-\tau]. \tag{A8}$$

Then neglecting terms of order $m+q$ or higher we have,

$$\begin{aligned} S_0&\simeq -i/m+q-3/2\sqrt{\lambda}+\cdots\\ S_1&\simeq -i/m+q+3/2\sqrt{\lambda}+\cdots\\ S_2&\simeq -i/m+q-3/2\sqrt{\lambda}+\cdots,\text{ etc.}\end{aligned} \tag{A9}$$

For larger values of $m+q$, $T_2(q)$ becomes dominant except in the odd l cases where the step function $\theta[\pm(m+q)]$ cuts off $T_2(q)$ as $m+q$ changes sign.

APPENDIX B. CUT-OFF APPROXIMATION

In spite of the impossibility of making a coupling constant expansion for the Fourier transforms of the $n>0$ theories discussed above, it is possible to approximate the exact theory by a cut-off theory, the error term which can be estimated when the solutions are known. The Fourier transform of the operators of the cutoff theory can be calculated in a perturbation expansion and this expansion will be practical if the coupling constant and cutoff are small enough.

As an example let us consider the $n=1$ theory with an interaction term $g\psi^*(\alpha\cdot\nabla\phi)\psi$. Then the renormalized propagator is

$$S_c(\tau)=\theta(\tau)e^{-im\tau}\exp\left\{\frac{g^2}{4\pi^2}\int_0^\infty\frac{k^4dk}{\omega^3}e^{-i\omega\tau}\right\}. \tag{B1}$$

A theory which cuts off the meson momentum at K would give for the renormalized propagator,

$$S_c{}^K(\tau)=\theta(\tau)e^{-im\tau}\exp\left\{\frac{g^2}{4\pi^2}\int_0^K\frac{k^4dk}{\omega^3}e^{-i\omega\tau}\right\}. \tag{B2}$$

The cut-off solution above is analytic in $g^2/4\pi^2$ for all values of τ and thus the Fourier transform is analytic in $g^2/4\pi^2$ at $g^2/4\pi^2=0$ and can be expanded in the coupling constant at the origin. As K increases and the cutoff solution approaches the $K=\infty$ solution the terms of the expansion of the Fourier transform of $S_c{}^K(\tau)$ grow increasingly large and oscillate. Thus the expansion appears to be completely dependent on the cutoff.

This is not necessarily so, however, for an examination of the exact solutions shows that the cutoff and $K=\infty$ theories differ mostly in the region where $\tau\to 0$, but this region itself contributes very little to the Fourier transform of the propagator because of the radiative correction:

$$R_c(\tau)=\exp\{-(g^2/4\pi^2)\tau^{-2}\}\quad\text{as}\quad\tau\to 0.$$

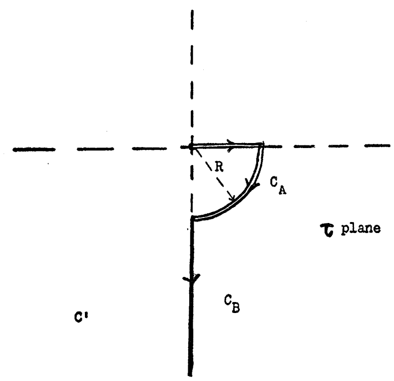

FIG. 3. Contour used in evaluating the error term $E(K,q)$.

To make this more precise we will examine the Fourier transform of $S_c(\tau)-S_c{}^K(\tau)$ and show that, for a large enough cutoff, the error term is small and, for a small coupling constant, a perturbation expansion of the Fourier transform of $S_c{}^K(\tau)$ can be made.

The Fourier transform of $S_c{}^K(\tau)$ is

$$S^K(q)=\int_0^\infty \exp\left\{-i(m+q)\tau+\frac{g^2}{4\pi^2}\int_0^K \frac{k^4dk}{\omega^3}e^{-i\omega\tau}d\tau\right\} \quad \text{(B3)}$$

and the principal value of that of $S_c(\tau)$ is

$$S(q)=\int_0^\infty \exp\left\{-i(m+q)\tau+\frac{g^2}{4\pi^2}\int_0^\infty \frac{k^4dk}{\omega^3}e^{-i\omega\tau}d\tau\right\}. \quad \text{(B4)}$$

Define the error term $E(K,q)$ to be

$$E(K,q)=[S(q)-S^K(q)]/S^K(q). \quad \text{(B5)}$$

If the contour of integration is shifted to C', shown in Fig. 3, we can divide the contribution to the error term into a portion which comes from the neighborhood of the origin C_A and that which comes from the major part of the path C_B. The contribution to $E(K,q)$ which results from C_B is

$$E^B(k,q)\simeq(\lambda/R)e^{-KR}\{1/R+K\} \quad \text{(B6)}$$

where $\lambda=g^2/4\pi^2$, while the contribution from C_A is of the order of

$$E^A(K,q)\simeq\exp(-\lambda/R^2)R+\frac{\pi R}{2}\lambda\exp(\lambda/R^2-KR)[KR-1]\frac{1}{S^K(q)}. \quad \text{(B7)}$$

Letting $R=r\sqrt{\lambda}$ and $K=k/\sqrt{\lambda}$, one can choose values of r and k so that the error term becomes arbitrarily small. However, this is limited in practice by the requirement that K be kept reasonably small in order that a perturbation expansion of $S^K(q)$ be possible. In any case one gets a better approximation close to the poles of $S^K(q)$ since the major contribution at the singularities comes from large values of τ where $S_c{}^K(\tau)$ is a very good approximation to $S_c(\tau)$.

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Agreement of Classical and Quantum Coulomb Excitation Integrals*

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A comparison is made between classical and quantum Coulomb excitation integrals. Use is made of a relation due to Breit and Daitch concerning close equality of classical and quantum density integrals. It is shown that the indefinite Coulomb excitation integrals show agreement at nearly the same distances as the corresponding density integrals. The agreement of the Coulomb excitation integrals for small excitations is therefore believed to be caused at least partly by the agreement of the density integrals.

THE fact that semiclassical and quantum calculations of Coulomb excitation are in close agreement has been discussed by Breit and Daitch,[1] by Daitch, Lazarus, Hull, Benedict and Breit,[2] and by Biedenharn and Class.[3] In a note to appear shortly, Breit and Daitch[4] show that there is a close agreement between integrals of classical and quantum densities integrated from $r=0$ to r. They show that if the integrals are taken to a node of the radial wave function F/r and if the JWKB approximation holds at the point r, there is an exact agreement between the two integrals. A redistribution of densities occurs as a result of going from the classical to the quantum theories. It is reasonable to expect, therefore, that the calculation of radial

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