

after these corrections and that for compound nucleus contribution a similar discrepancy yet remains, it is in the sense to correspond to a greater reduced width for neutrons (in C^{13}) than for protons (in N^{13}); i.e., "the neutrons stick out further than the protons." Such an effect has been suggested for heavier nuclei, though it would be very surprising to find it holding for so light a nucleus as $A=13$.

An estimate of the course of the cross section for the reaction $C^{12}(d,t)C^{11}$ was made on the basis of compound nucleus formation by assuming, as before, that the whole of the cross section for $C^{12}(d,n)N^{13}$ at low deuteron energies involves compound nucleus formation. On the assumption that the reduced width for triton emission is as great as that for neutron emission (the assumption of "preformed" tritons), we predict the dashed line of Fig. 2—in which the coming into play of successive residual states of C^{11} has been allowed for and the associated irregularities smoothed out. It is seen that even under the very unpalatable assumption of the existence of preformed tritons, compound nucleus theory fails by an order of magnitude to explain the observed C^{11} formation. We are forced then to assume that this (d,t) reaction proceeds by some pickup mechanism and that we are indeed measuring the relative

probability of the deuteron's losing a nucleon to the nucleus and removing one from it. As yet no sufficiently reliable theory of (d,t) pickup exists to warrant a comparison being made with these results. It is interesting to note that, at $E_d=3.3$ Mev, the angular distribution of the reaction $C^{13}(d,t)C^{12}$ is such as to suggest that a direct mechanism already predominates.⁴

It is interesting to compare these results with those of Cohen and Handley⁹ on (p,t) reactions. These authors suggest that triton emission from a compound nucleus state has an inherent probability comparable with that for single nucleon emission. They base this argument on the rather flat angular distributions sometimes obtained which, they remark, tell against a pickup process. However, this conclusion is no longer valid when the energy of one or both the charged particles concerned is of the order of or below the Coulomb barrier; here a direct mechanism can give a sensibly isotropic angular distribution. It appears that considerable interest attaches to the resolution of this question of the mechanism by which tritons and similar complicated particles are emitted from nuclei in events of moderate to high energy.

⁹ B. L. Cohen and T. H. Handley, Phys. Rev. **93**, 514 (1954)

Many-Body Problem for Strongly Interacting Particles. II. Linked Cluster Expansion*

K. A. BRUECKNER

Indiana University, Bloomington, Indiana

(Received April 28, 1955)

An approximation method developed previously to deal with many particles in strong interaction is examined in further detail. It is shown that the series giving the interaction energy is a development in a sequence of linked or irreducible cluster terms each of which gives a contribution to the energy proportional to the total number of particles. Consequently the convergence of the expansion is independent of the total number of particles. The origin of this simple feature is illustrated by showing that a similar situation exists in the expansion of standard perturbation theory. The numerical convergence of the expansion is quantitatively discussed for the nuclear problem where it is shown that the correction arising from the first cluster term involving three particles is less than the leading term by a factor of about 10^{-4} . The smallness of the correction is largely a result of the action of the exclusion principle.

I. INTRODUCTION

IN a previous paper¹ (to be referred to as I) we have given a method for reducing approximately the many body problem for strongly interacting particles to a problem of self-consistent fields. Some of the physical content and origin of the method were discussed there and the nature of certain correction terms which can be neglected for very many particles was discussed. We shall in this paper examine the structure of another type of correction term which arises from interaction

of clusters of particles and in so doing exhibit the general structure of the expansion involved. This will also allow us to draw some general conclusions about the convergence and accuracy of the method.

In Sec. II, we shall briefly summarize the relevant formulas from I and describe some difficulties which appear in high-order terms in the expansion for the energy which can be removed by a simple modification of the many-body propagation function. In Sec. III, we show how similar terms appear to arise in the usual perturbation theory but that they cancel identically, in a manner simply related to the cancellation discussed in Sec. II. In Sec. IV, we summarize these results and show how they may be generalized into a simple pre-

* Supported in part by a grant from the National Science Foundation.

¹ K. A. Brueckner and C. A. Levinson, Phys. Rev. **97**, 1344 (1955).

scription for evaluating matrix elements or, equivalently, used to redefine the many-body propagator for the system. In Sec. V, we take as an example the nuclear case and discuss quantitatively the convergence of the method. In Sec. VI, we discuss the equation for the two-body reaction matrices which provide the leading approximation to the energy. Finally in Sec. VII, we summarize our results and make some concluding remarks.

II. FORMALISM

A. Structure of the Expansion for the Energy

In this section, we shall summarize the results of I and discuss some of the formal properties of the expansion. We use the formalism developed in I for dealing with identical particles. We shall compress the notation used there by introducing the following definitions:

$$\begin{aligned} \mathbf{T}_i &= T \eta_i^* \eta_i, \\ \mathbf{v}_{ij, kl} &= \eta_i^* \eta_j^* v_{ij, kl} \eta \eta_k, \\ \mathbf{t}_{ij, kl} &= \eta_i^* \eta_j^* t_{ij, kl} \eta \eta_k, \\ \mathbf{I}_{ij, kl} &= \mathbf{t}_{ij, kl} - \mathbf{t}_{ij, ij} \delta_{kl} - \mathbf{t}_{ij, ji} \delta_{kl}. \end{aligned} \quad (1)$$

In this notation the equation for the operator t is

$$\mathbf{t}_{ij, kl} = \mathbf{v}_{ij, kl} + \sum_{mn} \frac{1}{e} \mathbf{v}_{ij, mn} \mathbf{t}_{mn, kl}, \quad (2)$$

where $1/e$ is the propagator for the many-body system. We have previously defined e to be the operator

$$e = E - \sum_i \mathbf{T}_i - \sum_{ij} \mathbf{t}_{e, ij}, \quad (3)$$

where

$$\mathbf{t}_{e, ij} = \frac{1}{2} (\mathbf{t}_{ij, ij} + \mathbf{t}_{ij, ji}), \quad (4)$$

which we shall still use for the present; in Sec. IV we shall show how a different definition may be advantageously introduced to simplify the structure of the expansion for the energy. In terms of these operators, the energy of the system is

$$E = (\phi_0, \sum_i \mathbf{T}_i + \mathbf{t}_{e, ij} \phi_0) + \frac{1}{2} \sum_{ijkl} (\phi_0, \mathbf{I}_{ij, kl} F_{kl} \phi_0), \quad (5)$$

where

$$\begin{aligned} F &= 1 + \frac{1}{2e} \sum_{ijkl} \mathbf{I}_{ijkl} F_{kl}, \\ F_{kl} &= 1 + \frac{1}{2e} \sum_{mn \neq kl} \mathbf{I}_{mnr} F_{rs}. \end{aligned} \quad (6)$$

Equation (5) for the energy may be written more explicitly as an expansion in the incoherent or non-diagonal operators I . We use Eq. (6) for the F 's and

find for the leading terms in the expansion

$$\begin{aligned} E - (\phi_0, \sum_i \mathbf{T}_i \phi_0) &= \sum_{ij} (\phi_0, \mathbf{t}_{e, ij} \phi_0) \\ &+ \frac{1}{2} \sum \left(\phi_0, \mathbf{I}_{ij, kl} \frac{1}{e} \mathbf{I}_{mn, rs} \phi_0 \right) \\ &+ \frac{1}{4} \sum \left(\phi_0, \mathbf{I}_{ij, kl} \frac{1}{e} \mathbf{I}_{mn, rs} \frac{1}{e} \mathbf{I}_{pq, tv} \phi_0 \right) \\ &+ \frac{1}{8} \sum \left(\phi_0, \mathbf{I}_{ij, kl} \frac{1}{e} \mathbf{I}_{nr, rs} \frac{1}{e} \mathbf{I}_{pq, tv} \frac{1}{e} \mathbf{I}_{ab, cd} \phi_0 \right). \end{aligned} \quad (7)$$

In all of these terms the condition must be imposed on the summations that consecutive pairs of indices not be matched, this condition arising from the similar condition in the defining equation for F_{kl} . Let us now examine these terms in detail. The first term on the right hand side of Eq. (7) is that previously evaluated in detail elsewhere; it gives by far the largest contribution to the energy of a highly degenerate Fermi gas. We shall return to a discussion of its structure in more detail in Sec. VI. The second term vanishes identically since the product of the two nondiagonal operators with $mn \neq kl$ has no diagonal matrix element. The third term is the first correction term to the energy as given by the first term alone; it involves a cluster of three particles. A typical pairing of indices gives a term

$$\Delta E_3 = \frac{1}{8} \sum \left(\phi_0, \mathbf{I}_{ij, i'j'} \frac{1}{e} \mathbf{I}_{j'k, jk'} \frac{1}{e} \mathbf{I}_{i'k', ik} \phi_0 \right). \quad (8)$$

This term has what we shall call a typical linked or irreducible structure; each I is linked to the other two. As a consequence of this structure, this term cannot be written as a product of simpler terms; this result is a general feature of all linked terms. As we shall see it is criterion of linkage which characterizes the expansion and determines its convergence.

To determine the order of magnitude of these typical terms, it is convenient at this stage to introduce a method of counting the dependence on the total number of particles, N . We observe that the operators I contain a factor of $v^{-1} \sim N^{-1}$ which comes from the normalization of the wave functions. Each free summation over momentum states gives a factor of N since

$$\sum_{\mathbf{k}} \rightarrow \frac{v}{(2\pi)^3} \int d\mathbf{k} \sim N. \quad (9)$$

In determining the number of free summations we must of course take proper account of the Kronecker delta function on the total momentum contained in each operator I . Thus we can determine the N -dependence of a given term by a simple counting procedure. Application of these rules to the terms so far

discussed gives

$$E_1 = \sum_{ij} (\phi_0, \mathbf{t}_{e, ij} \phi_0) \sim N^2(1/N) = N \quad (10)$$

as we expect. Similarly

$$E_3 \sim N^4/N^3 = N, \quad (11)$$

with three factors of N coming from the summation over i, j, k and one factor of N from the summation over i' .

In the next term of fourth order in the I 's, we first find terms of a new structure arising.² In addition to a variety of linked cluster terms, we now find reducible or unlinked terms. These have the form

$$E_4^{(\tau)} = \sum \left(\phi_0, \mathbf{I}_{ij} \frac{1}{e} \mathbf{I}_{k'l, k'l} \frac{1}{e} \mathbf{I}_{i'j', i'j'} \frac{1}{e} \mathbf{I}_{k'l, k'l} \phi_0 \right). \quad (12)$$

If ij and $k'l$ are summed independently, application of the counting rule gives

$$E_4^{(\tau)} \sim N^2. \quad (13)$$

On the other hand, for a typical linked term such as

$$\sum \left(\phi_0, \mathbf{I}_{ij} \frac{1}{e} \mathbf{I}_{i'j', i'j'} \frac{1}{e} \mathbf{I}_{k'l, k'l} \frac{1}{e} \mathbf{I}_{i'j', i'j'} \frac{1}{e} \mathbf{I}_{k'l, k'l} \phi_0 \right), \quad (14)$$

the counting rule shows a linear dependence on N . This result holds generally for a linked cluster; these are always of the order N or lower and hence depend at most on the same power of N as the leading term. Consequently the convergence of this series does not depend on N (as long as N is very large) except through the appearance of unlinked clusters. Before examining these in more detail, we write down some typical unlinked clusters of higher order which also depend on higher powers of N than the first. These are, for example,

$$E_5^{(\tau)} = \left(\phi_0, \mathbf{I}_{ij} \frac{1}{e} \mathbf{I}_{i'j', i'j'} \frac{1}{e} \mathbf{I}_{k'l, k'l} \frac{1}{e} \mathbf{I}_{mn, mn} \frac{1}{e} \mathbf{I}_{m'n', m'n'} \right. \\ \left. \times \frac{1}{e} \mathbf{I}_{k'l, k'l} \frac{1}{e} \mathbf{I}_{m'n', m'n'} \phi_0 \right) \sim N^2,$$

$$E_6^{(\tau)} = \left(\phi_0, \mathbf{I}_{ij} \frac{1}{e} \mathbf{I}_{i'j', i'j'} \frac{1}{e} \mathbf{I}_{mn, mn} \frac{1}{e} \mathbf{I}_{i'j', i'j'} \frac{1}{e} \mathbf{I}_{rs, rs} \frac{1}{e} \mathbf{I}_{m'n', m'n'} \right. \\ \left. \times \frac{1}{e} \mathbf{I}_{m'n', m'n'} \frac{1}{e} \mathbf{I}_{r's', r's'} \phi_0 \right) \sim N^3. \quad (15)$$

B. Reducible Clusters

The terms which cause difficulty since they are not linear in N all have a similar unlinked structure. Consequently they do not represent actual interaction energy; for example, in the fourth order term, the two pairs

² The author is indebted to Professor J. Bardeen for pointing out the lack of a satisfactory treatment of these terms in other theories of the many-body system.

ij and $k'l$ are independent of each other with no interaction term linking them. They therefore arise in the expansion only because of an inadequacy in this approximation scheme. It is easy, however, to show that they may be eliminated by a very simple and natural redefinition of the propagator $1/e$. To see this, let us look elsewhere for similar unlinked terms in the energy. It is easy to see that these first appear in the second order term $t_{ij, ij}$. Let us examine a typical term

$$\mathbf{t}_{ij, ij} = \mathbf{v}_{ij, ij} + \mathbf{v}_{ij, i'j'} \frac{1}{e} \mathbf{t}_{i'j', ij} \\ = \mathbf{v}_{ij, ij} + \mathbf{v}_{ij, i'j'} (E - \sum_k \mathbf{T}_k - \sum_{kl} \mathbf{t}_{c, kl})^{-1} \mathbf{t}_{i'j', ij}. \quad (16)$$

We can evaluate the propagator by making use of the fact that to a good approximation

$$E = \sum \mathbf{T}_k^{(0)} + \sum_{kl} \mathbf{t}_{c, kl}^{(0)}, \quad (17)$$

where by the superscript zero we mean evaluated in the ground state. Consequently the energy denominator is

$$E - \sum \mathbf{T}_i - \sum_{kl} \mathbf{t}_{c, kl} = e_{i'j'} + \sum_{kl \neq i'j'} (t_{c, kl}^{(0)} - t_{c, kl}), \quad (18)$$

where

$$e_{i'j'} = \mathbf{T}_i + \mathbf{T}_j + V_c(i) + V_c(j) \\ - \mathbf{T}_{i'} - \mathbf{T}_{j'} - V_c(i') - V_c(j'), \quad (19)$$

and

$$V_c(i) = \sum_j \mathbf{t}_{c, ij}, \quad (20)$$

etc. The terms in $t_{c, kl}$, which do not refer explicitly to the excited particles i', j' have changed because of an implicit dependence of the operator for the $k'l$ pair on the excitation state of the unlinked ij pair. Explicitly, neglecting the second-order change in t ,

$$\mathbf{t}_{c, kl}^{(0)} - \mathbf{t}_{c, kl} = \mathbf{v}_{kl, k'l} \left(\frac{1}{e_{k'l}} - \frac{1}{e_{k'l, i'j'}} \right) \mathbf{t}_{k'l, kl} \\ = \mathbf{v}_{kl, k'l} \frac{1}{e_{k'l}} \frac{1}{e_{k'l, i'j'}} (e_{k'l, i'j'} - e_{k'l}) \mathbf{t}_{k'l, kl} \\ = \mathbf{v}_{kl, k'l} \frac{1}{e_{k'l}} \frac{1}{e_{k'l, i'j'}} \mathbf{t}_{k'l, kl} e_{i'j'}. \quad (21)$$

Inserting this change into the propagator for the ij pair and expanding the propagator, the expression for $t_{ij, ij}$ breaks up into two terms

$$\mathbf{t}_{ij, ij} = \mathbf{v}_{ij, ij} + \mathbf{v}_{ij, i'j'} \frac{1}{e_{i'j'}} \mathbf{t}_{i'j', ij} \\ - \mathbf{v}_{ij, i'j'} \frac{1}{e_{i'j'}} \mathbf{v}_{kl, k'l} \frac{1}{e_{i'j', k'l}} \frac{1}{e_{k'l}} \mathbf{t}_{k'l, kl} e_{i'j', ij}. \quad (22)$$

The last term is identical except for sign with $E_4^{(\tau)}$

except that two of the I operators have been replaced by v operators. These, however, are equal to first order; thus we see that the N^2 terms cancel, leaving only the linked terms for irreducible clusters which are of no higher order than N . The two-body operator defined by the equation

$$\mathbf{K}_{ij, ij} = v_{ij, ij} + v_{ij, ij'} - \frac{1}{e_{ij'}} \mathbf{K}_{ij', ij}, \quad (23)$$

which differs from $t_{ij, ij}$ in that the unlinked terms are omitted from the propagator, is thus, although much simpler, a much better approximation to the leading term in the energy. It is similar to the result used in applications of these methods to the nuclear problem.^{1,3}

Before going on to show that is possible to generalize this definition of the propagator for a general matrix element, we shall show how a similar simplification occurs in ordinary perturbation theory.

III. RELATION TO PERTURBATION THEORY

For simplicity in this section, we shall assume that the particles are distinguishable; the more general discussion closely follows the treatment we give here.

In the usual perturbation theory⁴ the Hamiltonian is separated into the unperturbed Hamiltonian

$$H_0 = \sum_i T_i, \quad (24)$$

and the perturbation

$$h = \sum_{ij} v_{ij} \quad (25)$$

(for simplicity we absorb a factor of $\frac{1}{2}$ into the v 's). We take for the zeroth order wave functions a product of the eigenfunctions of H_0 , for example plane waves normalized in a large box of volume v . The perturbation series for the energy then is

$$E = \sum_n E_n. \quad (26)$$

We introduce a compact notation to simplify the form of the perturbation expansion; we define

$$(\phi_0, \theta \phi_0) = \langle \theta \rangle \quad (27)$$

for any operator. We also let

$$\left(\phi_n, \frac{1}{\epsilon_0 - \epsilon_n} \phi_n \right) = \left(\phi_n, \frac{1}{\epsilon_0 - H_0} \phi_n \right) = \left(\phi_n, \frac{1}{a} \phi_n \right), \quad (28)$$

where the operator $1/a$ is

$$1/a = 1/(\epsilon_0 - H_0). \quad (29)$$

Using this notation it is then easy to show that the perturbation series for the energy is given by the development

$$E_n = \langle h S_{n-1} \rangle - \sum_{m=1}^{n-1} E_m \langle S_{n-m} \rangle, \quad (30)$$

where

$$S_n = -\frac{1}{a} h S_{n-1} - \frac{1}{a} \sum_{m=1}^{n-1} E_m S_{n-m}. \quad (31)$$

The diagonal elements of S_n are not needed explicitly; they can, however, be fixed by the unit normalization condition

$$\sum_{m=0}^n S_m S_{n-m} = 0 \quad n \neq 0 \quad (32)$$

together with

$$S_0 = 1. \quad (33)$$

We also introduce

$$u = h - \langle h \rangle, \quad (34)$$

which has vanishing matrix elements in the ground state. The expansion can then be written

$$E_0 = \epsilon_0 = \langle H_0 \rangle,$$

$$E_1 = \langle h \rangle,$$

$$E_2 = \left\langle \frac{1}{h-h} \right\rangle = \left\langle \frac{1}{u-u} \right\rangle,$$

$$E_3 = \left\langle \frac{1}{h-u-h} \right\rangle,$$

$$E_4 = \left\langle \frac{1}{h} \left[\frac{1}{u-u} - \left\langle \frac{1}{u-u} \right\rangle \right] \frac{1}{-h} \right\rangle,$$

$$E_5 = \left\langle \frac{1}{h} \left[\frac{1}{u-u} \frac{1}{a} - \left\langle \frac{1}{u-u} \right\rangle \frac{1}{a} \right] - \frac{1}{a} \left\langle \frac{1}{u-u} \right\rangle - \left\langle \frac{1}{u-u} \right\rangle \frac{1}{a} \left[\frac{1}{-h} \right] \right\rangle. \quad (35)$$

The structure of these terms is extremely simple; we can obtain them by taking the n th order iterate of u

$$\left\langle a \left(\frac{1}{-u} \right)^n \right\rangle, \quad (36)$$

and subtracting all matrix elements of the same order in which matrix elements in the ground state of any subset of the u 's appear, always remembering that a matrix element of the form

$$\left\langle \frac{1}{-a} \quad \text{or} \quad \frac{1}{a} \right\rangle \quad (37)$$

is to be set equal to zero. This simple prescription suffices to define the full perturbation series for the energy.

We next note some simple properties of the expansion by examining the first few terms; we shall later generalize the results. Inserting the series for h and for u , the

³ K. A. Brueckner, Phys. Rev. **97**, 1353 (1955).

⁴ See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., 1949), Chap. VII.

expression for E_2 becomes

$$\sum_{n \neq 0} \sum_{ij} \sum_{kl} (\phi_0, v_{ij} \phi_n) \frac{1}{\epsilon_0 - \epsilon_n} (\phi_n, v_{kl} \phi_0). \quad (38)$$

In the double sum over ij, kl , since matrix elements to the ground state are omitted, the matrix element will vanish unless $ij = kl$. This result also is apparent if we note that v_{kl} is in effect an operator exciting the pair kl from the ground state; this pair must be returned by v_{ij} . Thus

$$E_2 = \sum_{n \neq 0} \sum_{ij} (\phi_0, v_{ij} \phi_n) \frac{1}{\epsilon_0 - \epsilon_n} (\phi_n, v_{ij} \phi_0) = \sum_{ij} \left\langle \frac{1}{a} v_{ij} v_{ij} \right\rangle, \quad (39)$$

which is the second term in the expansion for t_{ij} . We break down the third term

$$E_3 = \sum_{ijklmn} \left\langle \frac{1}{a} v_{ij} (v_{kl} - \langle v_{kl} \rangle) \frac{1}{a} v_{mn} \right\rangle \quad (40)$$

in a similar way. In this term we can combine the indices ij, kl, mn in several ways. The simplest possibility is with all pairs the same, giving

$$E_3^{(1)} = \sum_{ijk} \left\langle \frac{1}{a} v_{ij} (v_{ij} - \langle v_{ij} \rangle) \frac{1}{a} v_{ij} \right\rangle. \quad (41)$$

We can also pair ij and mn and leave kl free; this gives

$$\left\langle \frac{1}{a} v_{ij} (v_{kl} - \langle v_{kl} \rangle) \frac{1}{a} v_{ij} \right\rangle. \quad (42)$$

This can be further reduced since unless k or l is equal to i or j , only the diagonal elements of $v_{kl} - \langle v_{kl} \rangle$ with respect to the ground state enter, giving zero. Thus we have a restriction on k, l so that

$$E_2^{(2)} = \sum_{ijk} \left\langle \frac{1}{a} v_{ij} (v_{ik} - \langle v_{ik} \rangle + v_{jk} - \langle v_{jk} \rangle) \frac{1}{a} v_{ij} \right\rangle. \quad (43)$$

Writing the matrix element out more explicitly, this is

$$E_3^{(2)} = \sum_{ijn} (\phi_0, v_{ij} \phi_n) \frac{1}{\epsilon_0 - \epsilon_n} \times \sum_k [(\phi_n, v_{ik} \phi_n) - (\phi_0, v_{ik} \phi_0) + (i \rightarrow j)] \frac{1}{\epsilon_0 - \epsilon_n} \times (\phi_n, v_{ij} \phi_0). \quad (44)$$

We now note that this term in E_3 can be combined with the second order term E_2 to give

$$E_2 + E_3^{(2)} = \sum_{ijn} (\phi_0, v_{ij} \phi_n) \left\{ \frac{1}{\epsilon_0 - \epsilon_n} + \frac{1}{\epsilon_0 - \epsilon_n} \times \sum_k [(\phi_n, v_{ik} \phi_n) - (\phi_0, v_{ik} \phi_0) + (i \rightarrow j)] \times \frac{1}{\epsilon_0 - \epsilon_n} \right\} (\phi_n, v_{ij} \phi_0) \\ \cong \sum_{ij} (\phi_0, v_{ij} \phi_n) \{ \epsilon_0 + \sum_k (\phi_0, (v_{ik} + v_{jk}) \phi_0) - \epsilon_n - \sum_k (\phi_n, (v_{ik} + v_{jk}) \phi_n) \}^{-1} (\phi_n, v_{ij} \phi_0). \quad (45)$$

Thus the effect of $E_3^{(2)}$ is to give the first correction term in the nonperturbation propagator $1/e$ in E_2 .

The remaining term in E_3 arises from combining three different indices, corresponding to the involvement of three particles. This term is

$$E_3^{(3)} = \sum_{ijk} \left\langle \frac{1}{a} v_{ij} v_{jk} v_{ki} \right\rangle, \quad (46)$$

which is the first term in the expansion of the three body cluster term.

The next term in the energy E_4 contains a variety of types of combinations of the matrix element of the v 's; we shall consider here in detail only those terms which are similar to the unlinked clusters discussed in Sec. II. The general reductions are given in the Appendix. The terms of fourth order which we shall not discuss here are the following:

- (1) fourth order interaction of v_{ij} (for a single pair).
- (2) changes in the propagators of lower order terms.
- (3) linked 4-body cluster.

The unlinked or reducible terms in E_4 are:

$$E_4^{(r)} = \sum_{ijkl} \left\{ \left\langle \frac{1}{a_{ij}} v_{ij} v_{kl} \frac{1}{a_{ijkl}} \left(v_{kl} \frac{1}{a_{ij}} v_{ij} + v_{ij} \frac{1}{a_{kl}} v_{kl} \right) \right\rangle - \sum_{ijkl} \left\langle \frac{1}{a_{ij}^2} v_{ij} v_{ij} \right\rangle \left\langle \frac{1}{a_{kl}} v_{kl} v_{kl} \right\rangle \right\}, \quad (47)$$

where

$$a_{ij} = \epsilon_i + \epsilon_j - \epsilon_{i'} - \epsilon_{j'}, \quad (48)$$

etc. This we simplify using the identity

$$\frac{1}{a_{ij}} \frac{1}{a_{ijkl}} \left(\frac{1}{a_{ij}} + \frac{1}{a_{kl}} \right) - \frac{1}{a_{ij}^2} \frac{1}{a_{kl}} = \frac{1}{a_{ij}^2} \left(\frac{a_{ij} + a_{kl}}{a_{ijkl}} - 1 \right). \quad (49)$$

Since

$$a_{ijkl} = a_{ij} + a_{kl}, \quad (50)$$

these two terms cancel. We note that this is possible only when all of the matrix elements of the same general unlinked structure, are combined. Thus the unlinked terms (of order N^2) give zero, as we expected from the consideration of a similar cancellation in Sec. II. We shall not discuss the cancellations of the unlinked

terms of higher order; the proof has been explicitly constructed in fifth and sixth order and always involves a lengthy but trivial sequence of algebraic identities such as those of Eq. (49).

We proceed next to a more general statement of this result in the following section.

IV. LINKED PROPAGATOR

The results of the last two sections strongly suggest that the appearance of unlinked cluster terms in the energy expansion is everywhere spurious in that they are exactly canceled by similar terms appearing elsewhere in the expansion. Consequently the expansion is in each order linear in the total number of particles in that the contributions to the energy depend on N . The unlinked terms which cancel are always unphysical in that they cannot represent actual interaction energy; this situation is similar to that which exists in the S -matrix expansion of quantum field theory where similar unlinked terms can always be identified as contributing only to the phase of the wave function without giving real effects in the scattering.

In discussing these terms in Secs. II and III, we have shown that their omission results in a simple and reasonable definition of the propagator or energy denominator. The criterion of dropping unlinked terms is perhaps sufficient to apply to any given term in the energy expansion since the spurious terms can be easily recognized and dropped. It is however convenient to show that these conditions may be formulated more formally by defining what we shall call a linked propagator. We introduce a modified propagator p with the following properties. Consider a typical expression involving the nondiagonal I operators and the propagator;

$$(\phi_0, \dots \mathbf{I}_{kl, k'l'} p \mathbf{I}_{mn, m'n'} \dots \phi_0). \quad (51)$$

We define the matrix element of p in this case to be

$$p = [\sum_i (E_i - E_{i'})]^{-1}, \quad (52)$$

where the sum over i is to run over all terms linked to the left to the pair k', l' . For the energies E_i and $E_{i'}$ we shall take

$$E_i = (\phi_0, \mathbf{T}_i + \sum_j \mathbf{t}_{c, ij} \phi_0),$$

$$E_{i'} = (\phi_{i'}, \mathbf{T}_{i'} + \sum_j \mathbf{t}_{c, i'j'} \phi_{i'}). \quad (53)$$

If the energy difference is zero, then the matrix element of p is to be set equal to zero. Since the states we consider are discrete, the omission of the single term in the matrix element is unambiguous. To show that this definition leads to the desired changes; first, it is obvious that the equation for $K_{ij, ij}$ now appears directly. In addition, in a typical unlinked term of fourth order, for example, we now have the matrix element

$$(\phi_0, \mathbf{I}_{ij, i'j'} p(i'j') \mathbf{I}_{kl, k'l'} p(k'l') \times \mathbf{I}_{i'j', ij} p(ij) \mathbf{I}_{k'l', kl} \phi_0). \quad (54)$$

The propagator $p(ij)$ is for the ij and all linked pairs (in this case the $i'j'$ pair) in the ground state; hence the energy difference is zero and by the previous definition, this matrix element of the propagator is equal to zero. Thus this term is identically zero; this will be true for any such unlinked interaction.

This completes the specification of the expansion; it now has a quite simple form since it is a series in the linked or irreducible clusters alone. The great simplification of this result in comparison with the perturbation expansion, for example, is due to the immense compression of the series which is accomplished by the introduction of the K and I operators. The equation for K is nonlinear; it is the perturbation expansion of this operator which leads to the very complicated series of perturbation theory. It is worth noting, however, that the convergence of the perturbation series itself does not depend on N , each term in the series depending only on N linearly.

V. CONVERGENCE OF THE LINKED CLUSTER EXPANSION

To determine the extent to which the linked cluster expansion is sufficiently rapidly convergent to make it a useful approximation method, we shall examine the leading cluster terms. Since the convergence does not depend on N , we expect that the expansion will be characterized by other parameters of the system such as the density and the interaction strength and range. Thus we shall attempt to determine the dimensionless expansion parameter which governs the rate of convergence of the series; if this is sufficiently small, we expect that the leading term in the expansion will be an accurate representation of the many-body energy. As an explicit example we shall consider the nuclear system which has been studied elsewhere^{1,3,5-7}; the convergence of the method has been investigated in these papers but only in a semiquantative way with the intent of establishing a rough upper limit on the magnitude of the correction terms.

We consider in detail the first cluster term involving more than two particles; this is

$$\Delta E_3 = \frac{1}{8} \sum_{ijk i'j' k'} (\phi_0, \mathbf{I}_{ij, i'j'} p \mathbf{I}_{k i', k' i} p \mathbf{I}_{i'j', k j} \phi_0). \quad (55)$$

In evaluating this we shall neglect all exchange terms; these are in general much smaller and also of opposite sign to the main term so that we over-estimate ΔE_3 by making this approximation. In evaluating ΔE_3 we shall take a simple form for the nondiagonal operators I , i.e.,

$$I_{ij, kl} = \frac{2\pi V_0}{\mu\nu} [f(\mathbf{k}_i - \mathbf{k}_k) + f(\mathbf{k}_i + \mathbf{k}_k)] \times \delta(\mathbf{k}_i + \mathbf{k}_j, \mathbf{k}_k + \mathbf{k}_l), \quad (56)$$

⁵ Brueckner, Levinson, and Mahmoud, Phys. Rev. **95**, 219 (1954).

⁶ K. A. Brueckner, Phys. Rev. **96**, 908 (1954).

⁷ Brueckner, Francis, and Eden, Phys. Rev. (to be published).

where

$$f(x) = 1/(\mu^2 + x^2), \quad (57)$$

and $V_0/\mu = 0.25$, v = total normalization volume. This result is a reasonable approximation to the actual nondiagonal operators I ; it is the Born approximation matrix element given by a Yukawa well with Serber exchange mixture. In the propagator we use the approximate result for the energies

$$E_i = k_i^2/2M^*, \quad (58)$$

where $M^* = 0.54M$ is the effective mass of a nucleon moving in the velocity-dependent nuclear potential.³ Making these substitutions into Eq. (55), the energy shift ΔE_3 is

$$\begin{aligned} \Delta E_3 = & \frac{1}{8}(2M^*)^2 \left(\frac{2\pi V_0}{\mu v} \right)^3 \sum_{ijk'i'} [f(\mathbf{k}_i - \mathbf{k}_{i'}) + f(\mathbf{k}_i + \mathbf{k}_{i'})] \\ & \times (k_i^2 + k_j^2 - k_{j'}^2 - k_{i'}^2)^{-1} [f(\mathbf{k}_i - \mathbf{k}_{i'}) + f(\mathbf{k}_i + \mathbf{k}_{i'})] \\ & \times (k_j^2 + k_k^2 - k_{j'}^2 - k_{i'}^2)^{-1} \\ & \times [f(\mathbf{k}_i - \mathbf{k}_{i'}) + f(2\mathbf{k}_j + \mathbf{k}_k - \mathbf{k}_{i'})]. \quad (59) \end{aligned}$$

The sum over $ijk'i'$ is not only over the momentum states but also over spins and isotopic spins. Since the operator I does not involve spin operators, the sum over the two spin and two isotopic spin values for the states ijk introduces a factor of $(4)^3$ (the summation over the spin states of i', j', k' gives unity). The summations over momenta can be replaced by integrations, using

$$\sum_{\mathbf{k}_i} \rightarrow \frac{v}{(2\pi)^3} \int d\mathbf{k}_i. \quad (60)$$

Introducing the variable

$$\mathbf{k}_i - \mathbf{k}_{i'} = \mathbf{x}, \quad (61)$$

we then find

$$\begin{aligned} \Delta E_3 = & 8(\mu^*)^2 \left(\frac{2\pi V_0}{\mu v} \right)^3 \left[\frac{v}{(2\pi)^3} \right]^4 \int_{k_f} d\mathbf{k}_i \int_{k_f} d\mathbf{k}_j \int_{k_f} d\mathbf{k}_k \int d\mathbf{x} \\ & \times f^3(x) [x^2 - (\mathbf{k}_i + \mathbf{k}_j) \cdot \mathbf{x}]^{-1} [x^2 - (\mathbf{k}_j + \mathbf{k}_k) \cdot \mathbf{x}]^{-1}, \quad (62) \end{aligned}$$

where we have dropped some small exchange terms depending on $f(x)^2 f(\mathbf{x} + \mathbf{k}_i)$, etc., which are much smaller (for large k_i) than the main term. In this integral the restrictions arising from the exclusion principle are that

$$\begin{aligned} |\mathbf{k}_{i'}| &= |\mathbf{k}_i - \mathbf{x}| \geq k_f, \\ |\mathbf{k}_{j'}| &= |\mathbf{k}_j - \mathbf{x}| \geq k_f, \\ |\mathbf{k}_{k'}| &= |\mathbf{k}_k - \mathbf{x}| \geq k_f. \end{aligned} \quad (63)$$

Let us now examine this 12-dimensional integral to see how simplifications can be made to reduce the problem of evaluation. We note that $f(x)^3$ is a very rapidly decreasing function of x for $x > \mu$, falling off as x^{-6} . Therefore, since the rest of the integrand is also largest

for x small, it is reasonable to evaluate the integrals over $\mathbf{k}_i, \mathbf{k}_j, \mathbf{k}_k$ making the approximation that x is small. This integral is

$$\Delta = \int \frac{d\mathbf{k}_i d\mathbf{k}_j d\mathbf{k}_k}{x^2} \frac{1}{(k_i \mu_i + k_j \mu_j - x)} \frac{1}{(k_j \mu_j + k_k \mu_k - x)}, \quad (64)$$

where $\mu_i = \mathbf{k}_i \cdot \mathbf{x}/k_i x$, etc. The restriction from the exclusion principle restricts the angular range of integration; the condition is

$$\mu_i \leq (k_i^2 + x^2 - k_j^2)/2k_i x, \quad (65)$$

and similarly for the other angular variables. Keeping only the leading terms in x , the integral over k_i , for example, is

$$2\pi \int_{k_f - x}^{k_f} k_i^2 dk_i \int_{-1}^{(k_i^2 + x^2 - k_j^2)/2k_i x} \frac{d\mu_i}{k_i \mu_i + k_j \mu_j - x}. \quad (66)$$

For small x , we set $k_i = k_f$ except for the upper limit of the angular integration where we set

$$\frac{k_i^2 + x^2 - k_j^2}{2k_i x} \cong \frac{(k_i + k_j)(k_i - k_j)}{2k_i x} \cong \frac{k_i - k_j}{x}, \quad (67)$$

with these approximations, we find for the k_i integration

$$\begin{aligned} 2\pi k_f \int_{k_f - x}^{k_f} dk_i \int_{-1}^{(k_i - k_f)/x} \frac{d\mu_i}{\mu_j + \mu_i} \\ = -2\pi k_f x [1 + \mu_j \ln(1 - 1/\mu_j)]. \quad (68) \end{aligned}$$

The integral over k_k gives a similar result; the remaining integral over k_j gives

$$\begin{aligned} \Delta = & (2\pi k_f)^2 2\pi k_f^2 \\ & \times \int_{k_f - x}^{k_f} dk_j \int_{-1}^{(k_j - k_f)/x} [1 + \mu_j \ln(1 - 1/\mu_j)]^2 d\mu_j. \quad (69) \end{aligned}$$

A change in the order of integration and a change of variable in the μ_j integration leads finally to

$$\begin{aligned} \Delta = & (2\pi k_f)^3 k_f x \int_0^1 ds s \left[1 + s \ln \frac{s}{1+s} \right]^2 \\ = & (2\pi k_f)^3 k_f x (0.078). \quad (70) \end{aligned}$$

Collecting these results we find

$$\begin{aligned} \Delta E_3 = & 8(M^*)^2 \left(\frac{2\pi V_0}{\mu v} \right)^3 \left[\frac{v}{(2\pi)^3} \right]^4 (2\pi k_f)^3 \\ & \times 0.078 k_f \int_0^\infty 4\pi x^3 f^3(x) dx. \quad (71) \end{aligned}$$

Using Eq. (57) for $f(x)$, the integral over x is

$$\int f^3(x) x^3 dx = \frac{1}{4} \mu^2. \quad (72)$$

Taking $4v(4/3\pi k_f^3)/(2\pi)^3 = N$, the final result is

$$\Delta E_3 = \left(\frac{M^* V_0}{\mu} \frac{1}{\mu} \frac{1}{4\pi} \right)^3 0.468 \frac{\mu}{M^*} k_f. \quad (73)$$

For typical values of the parameters, i.e., $M^*/\mu \sim 4$, $V_0/\mu = -\frac{1}{4}$, $k_f = 1.6\mu$, this gives

$$\Delta E_3 \cong -0.0071 \text{ Mev}, \quad (74)$$

which is to be compared with the leading term in K_{ij} , ij which is about equal to 30 Mev per particle.

The very small effect of the correction term on the energy is due to the smallness of the effective expansion parameter

$$\beta = \frac{M^* V_0}{\mu} \frac{1}{4\pi} = 0.072. \quad (75)$$

It can be fairly easily shown that the four body cluster term is also given by an expression very similar to that for ΔE_3 except that it depends on β^4 ; consequently it is again the size of β which determines the next higher order correction. The smallness of the expansion parameter β is due in part to the size of $(V_0/\mu)(M^*/\mu)$ which is in this case of the order of unity. It is of course obvious that convergence cannot be expected for large values of this combination of constants. Even more important, however, is the factor of $1/(4\pi)$ which always appears paired with the mass and potential constants. This factor appears as the result of the restriction of the angular integrations by the exclusion principle; this restriction is important because the typical momenta near the Fermi limit are rather large compared with momenta for which the matrix elements of the interaction are large. We have made use of this fact in evaluating the integrals for ΔE_3 . For much stronger potentials, however, this result would no longer hold and for the same Fermi momentum the angular integrations would be less affected by the exclusion principle.

Another important and striking feature of the result for the three- (and more-) body cluster is its linear dependence on the Fermi momentum. It is easily shown that the leading term in the energy depends on k_f^3 ; thus the relative importance of the higher-order cluster terms decreases as the Fermi momentum increases or, equivalently, as the density increases. The origin of this effect is obvious; as the Fermi momentum increases, the momentum transfers required to bring particles out of the Fermi gas to excited states increase. Since the matrix elements of the interaction decrease as the momentum transfers increase, excitations involving the multiple excitation of particles are inhibited. The effect is most pronounced for the high-order clusters for this reason.

VI. METHODS OF SOLVING THE K_{ij} , ij EQUATION

We have shown that the linked cluster expansion converges very rapidly for forces of typical nuclear strength so that it is a very good approximation in this case to evaluate only the leading term in the expansion. The convergence may not be as rapid in other systems of physical interest; even in such cases, however, the leading term will give very useful semiquantitative

information about the system. The equation for the matrix elements of K_{ij} , ij as given by Eq. (23) is, in the ground state,

$$\mathbf{K}_{ij, ij} = \mathbf{v}_{ij, ij} + \mathbf{v}_{ij, i'j'} (E_i + E_j - E_{i'} - E_{j'})^{-1} \mathbf{K}_{i'j', ij}, \quad (76)$$

where

$$\begin{aligned} E_i &= (\phi_0, (\mathbf{T}_i + \sum_k \mathbf{t}_{cik}) \phi_0) \\ &= (\phi_0, [\mathbf{T}_i + \frac{1}{2} \sum_k (\mathbf{K}_{ik, ik} + \mathbf{K}_{ik, ki})] \phi_0), \end{aligned} \quad (77)$$

and

$$E_{i'} = (\phi(i'), [\mathbf{T}_{i'} + \frac{1}{2} \sum_k (\mathbf{K}_{i'k, i'k} + \mathbf{K}_{i'k, ki'})] \phi(i')). \quad (78)$$

To a good first approximation, we can suppose that the coherent potential

$$V_c(i) = \frac{1}{2} \sum_k (\mathbf{K}_{ik, ik} + \mathbf{K}_{ik, ki}) \quad (79)$$

acting on the i th particle depends only on the momentum state of the particle i and evaluate $V_c(i)$ as if particle i were always propagating on the energy shell. This is equivalent to the assumption that the potential acting on a particle is the same in virtual states as in real states. If we make this simplifying assumption, then we can write

$$\mathbf{K}_{ij, ij} = \mathbf{v}_{ij, ij} + \mathbf{v}_{ij, i'j'} [T_i + T_j - T_{i'} - T_{j'} + V_c(i) + V_c(j) - V_c(i') - V_c(j')]^{-1} \mathbf{K}_{i'j', ij}, \quad (80)$$

and

$$\begin{aligned} V_c(i) &= [\phi_0, \sum_k (\mathbf{K}_{ik, ik} + \mathbf{K}_{ik, ki}) \phi_0], \\ V_c(i') &= [\phi_0, \sum_k (\mathbf{K}_{i'k, i'k} + \mathbf{K}_{i'k, ki'}) \phi_0]. \end{aligned} \quad (81)$$

Equations (80) and (81) form a coupled system for the determination of $V_c(i)$; they are equivalent to a non-linear integral equation for $V_c(i)$. In this approximation they have been solved as a self-consistency problem in applications of the methods to the nucleus.³

A more accurate treatment of the propagator in intermediate states is possible if account is taken of the fact that propagation off the energy shell is occurring. According to our rules for defining p , in an intermediate state with $i'j'$ excited, the equation for a typical term in $V_c(i')$ is

$$\mathbf{K}'_{i'k, i'k} = \mathbf{v}_{i'k, i'k} + \mathbf{v}_{i'k, i''k'} (E_{i'} + E_j + E_k - E_{i''} - E_{j'} - E_{k'})^{-1} \mathbf{K}_{i''k', i'k}. \quad (82)$$

This differs from the equation for $K_{i'k, i'k}$ on the energy shell in that

$$\begin{aligned} \mathbf{K}'_{i'k, i'k} &= \mathbf{v}_{i'k, i'k} \\ &+ \mathbf{v}_{i'k, i''k'} (E_{i'} + E_k - E_{i''} - E_{k'})^{-1} \mathbf{K}_{i''k', i'k}. \end{aligned} \quad (83)$$

The neglect of the difference between $K_{i'k, i'k}$ and $K'_{i'k, i'k}$ leads to a high order correction term in the energy. If the energy difference in the propagator is expanded out as a perturbation, i.e., if we take

$$\begin{aligned} \mathbf{K}'_{i'k, i'k} &= \mathbf{v}_{i'k, i'k} + \mathbf{v}_{i'k, i''k'} \frac{1}{e_{ik} + \Delta E} \mathbf{K}'_{i''k', i'k} \\ &\cong \mathbf{K}_{i'k, i'k} - \mathbf{v}_{i'k, i''k'} \frac{1}{e_{ik}} \frac{1}{e_{ik}} \mathbf{K}_{i''k', i'k}, \end{aligned} \quad (84)$$

then the equation for the ground-state matrix element of $K_{ij, ij}$ is altered by the amount

$$\Delta K_{ij, ij} \cong \frac{1}{e_{i'j'}} \frac{1}{e_{ik}} \times \Delta E \frac{1}{e_{ik}} \frac{1}{e_{i'j'}} K_{i'k, i'k} K_{i'j', ij}. \quad (85)$$

Since the effect appears in such high order, the actual quantitative change is probably quite small. If greater accuracy is desired, this effect can be included at least approximately. It adds the complication that a self-consistent solution must be obtained for the effective potential V_e acting on a particle both off and on the energy shell. It is probably an excellent first approximation to neglect this effect and make the static approximation for V_e ; this has been done in the actual applications of this method.

VII. CONCLUSIONS

We have considered in detail an approximation method which has been developed to deal with the problem of determining the interaction energy of a many-body system when the particle-particle forces are not weak. The usefulness of the method is determined by the convergence of an expansion which determines the interaction energy. We have primarily interested ourselves in this paper in a study of the structure of this expansion, particularly in the dependence of the convergence on the total number of particles (which we wish to take to be very large). We have shown that the convergence of the method does not depend on N , any dependence on higher powers of N than the first being spurious and exactly canceled by similar spurious terms appearing elsewhere in the expansion. This cancellation has been shown to occur in a precisely similar manner in the expansion of conventional perturbation theory, any nonlinear dependence on the total number of particles being again spurious. As a consequence the resulting expansion is in every order proportional to the total number of particles the total number thus appearing as a common factor for the entire series. The expansion is in a series of irreducible or linked clusters which involve a succession of transitions brought about by the nondiagonal transition operators of the theory. The convergence for the nuclear system has been investigated by a detailed examination of the first irreducible cluster (involving three particles) and shown to be extremely rapid, with the exclusion principle playing an important role in determining the rate of convergence.

Finally the problem of evaluating the two-body interaction operators has been briefly discussed; it is a sum over the diagonal part of these operators which provides the leading term in the interaction energy. It is shown that in the approximation the potential acting

on a particle is the same in virtual as real states, the problem of solving for the potential reduces to the problem of self-consistency used in other applications. The accuracy of this approximation is discussed briefly and methods of improving on the approximation are described.

ACKNOWLEDGMENTS

The author is indebted to Dr. R. J. Eden for many helpful discussions.

APPENDIX

Reduction of Fourth-Order Perturbation Term

We shall show how the fourth-order perturbation term can be broken up into a variety of terms which can be identified either as modifications of the propagators of lower order clusters or as irreducible cluster terms. To make this identification possible we first expand the reaction matrix K_{ij} as given by Eq. (23) with the notation simplified for the case of distinguishable particles:

$$\langle K_{ij} \rangle = \langle v_{ij} \rangle + \left\langle \frac{1}{e_{ij}} K_{ij} \right\rangle, \quad (A1)$$

where to second order the modified propagator is

$$e_{ij} = \epsilon_i + \epsilon_j + \sum_k \left\{ \langle v_{ik} \rangle + \left\langle \frac{1}{a} v_{ik} - v_{ik} \right\rangle - v_{ik} - \frac{1}{a} v_{ik} - v_{ik} + (i \rightarrow j) \right\}. \quad (A2)$$

Expanding the equation for K_{ij} we find for the fourth-order term

$$\begin{aligned} \langle K_{ij} \rangle^{(4)} = & \sum_k \left\langle \frac{1}{a} \frac{1}{a} \frac{1}{a} v_{ij} - v_{ij} - v_{ij} - v_{ij} \right\rangle \\ & + \sum_k \left\langle \frac{1}{a} v_{ij} (v_{ik} - \langle v_{ik} \rangle) - \frac{1}{a} v_{ij} - v_{ij} \right\rangle \\ & + \sum_k \left\langle \frac{1}{a} v_{ij} - v_{ij} (v_{ik} - \langle v_{ik} \rangle) - v_{ij} \right\rangle \\ & + \sum_k \left\langle \frac{1}{a} v_{ij} \left(\frac{1}{a} v_{ik} - v_{ik} - \left\langle \frac{1}{a} v_{ik} - v_{ik} \right\rangle \right) - v_{ij} \right\rangle \\ & + \sum_k \sum_l \left\langle \frac{1}{a} v_{ij} (v_{ik} - \langle v_{ik} \rangle) - \frac{1}{a} (v_{il} - \langle v_{il} \rangle) - v_{ij} \right\rangle \\ & + \sum_k \sum_l \left\langle \frac{1}{a} v_{ij} (v_{ik} - \langle v_{ik} \rangle) - \frac{1}{a} (v_{jl} - \langle v_{jl} \rangle) - v_{ij} \right\rangle, \quad (A3) \end{aligned}$$

omitting some terms which result from interchange of ij . We shall also need for comparison the expansion of the

modified propagator for the three-body cluster which is, to fourth order,

$$\sum_{ijmk} \left\{ \left\langle \frac{1}{a} v_{ij} (v_{ik} - \langle v_{ik} \rangle) v_{jm} v_{mi} \right\rangle + \left\langle \frac{1}{a} v_{ij} v_{jm} (v_{jk} - \langle v_{jk} \rangle) v_{mi} \right\rangle \right\}. \quad (\text{A4})$$

The perturbation expansion as given in Sec. III gives for the fourth-order term in the energy

$$E_4 = \left\langle \frac{1}{a} \frac{1}{a} \frac{1}{a} \right\rangle - \left\langle \frac{1}{a^2} u \right\rangle \left\langle \frac{1}{a} u \right\rangle. \quad (\text{A5})$$

Inserting the expansion for u , this is

$$E_4 = \sum_{ijmnrstv} \left[\left\langle \frac{1}{a} u_{ij} u_{mn} u_{rs} u_{tv} \right\rangle - \left\langle \frac{1}{a^2} u_{mn} \right\rangle \left\langle \frac{1}{a} u_{rs} u_{tv} \right\rangle \right]. \quad (\text{A6})$$

The first simplification is that already noted in Sec. III, i.e.,

$$\begin{aligned} \sum_{ijmn} \left\langle \frac{1}{a} u_{ij} u_{mn} \right\rangle &= \sum_{ijmn} \left\langle \frac{1}{a} u_{ij} u_{ij} \right\rangle \delta_{im} \delta_{jn} \\ &= \sum_{ij} \left\langle \frac{1}{a} u_{ij} u_{ij} \right\rangle. \end{aligned} \quad (\text{A7})$$

Consequently the second term of Eq. (A6) is

$$\begin{aligned} \sum_{ijmn} \left\langle \frac{1}{a^2} u_{ij} u_{ij} \right\rangle \left\langle \frac{1}{a} u_{mn} u_{mn} \right\rangle \\ = \sum_{ij \neq mn} \left\langle \frac{1}{a^2} u_{ij} u_{ij} \right\rangle \left\langle \frac{1}{a} u_{mn} u_{mn} \right\rangle \\ + \sum_{ijk} \left\langle \frac{1}{a^2} u_{ij} u_{ij} \right\rangle \left\langle \frac{1}{a} u_{ik} u_{ik} \right\rangle, \end{aligned} \quad (\text{A8})$$

the first term of which is of order N^2 . Before reducing the first term of Eq. (A6), we note a simple property of the matrix element. An expression in which a term u_{mn} is not linked to any other term is zero, since in this case the expectation value of the product reduces to the product of the expectation values, and the expectation value of u_{mn} in the ground state is zero. This rule greatly

simplifies the possible terms; they can be classified according to the number of pairs of indices which are set equal. The simplest term is that with all pairs of indices equal, i.e.,

$$E_4^{(4)} = \sum_{ij} \left\langle \frac{1}{a} \frac{1}{a} \frac{1}{a} u_{ij} u_{ij} u_{ij} u_{ij} \right\rangle. \quad (\text{A9})$$

This is the fourth-order iterate of u_{ij} which appears in the expansion of K_{ij} . The next term with one u_{mn} different is

$$\begin{aligned} E_4^{(3)} = \sum_{ijk} \left[\left\langle \frac{1}{a} \frac{1}{a} \frac{1}{a} u_{ij} u_{ij} u_{ik} u_{ij} \right\rangle + \left\langle \frac{1}{a} \frac{1}{a} \frac{1}{a} u_{ij} u_{ik} u_{ij} u_{ij} \right\rangle \right], \end{aligned} \quad (\text{A10})$$

together with terms in which u_{ik} is replaced by u_{jk} . These two terms are the first-order modification in the propagator of the third-order iterate of u_{ij} in the equation for K_{ij} . The next term has two u 's the same:

$$\begin{aligned} E_4^{(2)} = \sum_{ij \neq ml} \left[\left\langle \frac{1}{a} \frac{1}{a} \frac{1}{a} u_{ij} u_{kl} u_{kl} u_{ij} \right\rangle + \left\langle \frac{1}{a} \frac{1}{a} \frac{1}{a} u_{ij} u_{kl} u_{ij} u_{kl} \right\rangle \right] \\ + \sum_{ijk} \left\langle \frac{1}{a} \frac{1}{a} \frac{1}{a} u_{ij} u_{ki} u_{ki} u_{ij} \right\rangle + \sum_{ijkl} \left\langle \frac{1}{a} \frac{1}{a} \frac{1}{a} u_{ij} u_{kl} u_{jl} u_{ij} \right\rangle. \end{aligned} \quad (\text{A11})$$

In $E_4^{(2)}$ we have not included terms which vanish because they involve matrix elements of u in the ground state or because they require intermediate transitions to the ground state. The various terms in $E_4^{(2)}$ have the following interpretation; the first two cancel with the unlinked term of Eq. (A8) as shown in Sec. III. The last two terms combine with the linked term in Eq. (A8) to modify the propagator of the second term in the expansion for K_{ij} . Finally we have the term with no u 's the same but all linked; these are for example

$$\sum_{ijmkl} \left\langle \frac{1}{a} \frac{1}{a} \frac{1}{a} u_{ij} u_{ik} u_{jm} u_{ml} \right\rangle, \quad (\text{A12})$$

which is the modification of propagator of the three-body cluster term, and

$$\sum_{ijkl} \left\langle \frac{1}{a} \frac{1}{a} \frac{1}{a} u_{ij} u_{jk} u_{kl} u_{li} \right\rangle, \quad (\text{A13})$$

which is a term typical of the fourth-order irreducible clusters.