Scattering of Mesons by a Fixed Scatterer

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The meson-nucleon scattering equations of Chew and Low are generalized to the case of mesons and scatterers having arbitrary angular momenta. In consequence, the algebraic structure of the equations is made clearer. Two coupling schemes for the angular momenta are studied, the J-scheme in which momenta of meson and scatterer in the initial state are coupled, and the N-scheme in which the meson momenta in initial and final states are coupled. The condition of unitarity of the S-matrix is simple only in the J-scheme, the condition of causality is simple only in the N-scheme. The interlock between the two schemes gives rise to the peculiar linking of different J-values in the Chew-Low equations. The linkage coefficients are shown to be ordinary Racah coefficients.

I. INTRODUCTION

CHEW[1] has introduced the so-called "fixed-source model" as an approximate model for the meson-nucleon interaction, and has found remarkably good agreement between this model and the experimental data on photo-production and meson-proton scattering. Low[2] has discovered a new type of scattering equation which greatly simplifies the mathematical analysis of the Chew model. Low has also proved[3] a theorem which states that his scattering equation is in a certain sense equivalent to the requirement that the meson-nucleon interaction satisfy the two conditions of causality and unitarity of the S-matrix. This means that the consequences of the Low scattering equation are to some extent independent of the details of the model, and therefore to be taken more seriously than the model itself.

The present paper contains a simple generalization of the Chew-Low analysis to the scattering of mesons by a fixed scatterer, the mesons and the scatterer having arbitrary angular momenta. The treatment is general enough so that scattering matrix elements which are mixtures of different multipole orders can be included. It is found that the essential features of the Chew-Low theory are entirely unchanged by the generalization. In particular, the Low theorem about unitarity and causality still holds.

The purpose of this generalization of the theory was to understand the algebraic structure of the scattering equation [Eq. (3.11) of reference 2], particularly the linkage between scattering amplitudes of different total angular momentum. The generalized equations make the structure clear. The essential results thus obtained are two.

(i) The linkage coefficients are Racah coefficients, which enter here for the same reason that they occur in the theory of angular correlations of successive nuclear radiations.[4]

(ii) The unitarity and causality conditions can be simply formulated in terms of two different coupling schemes for the angular momenta, and it is only in the transformation between the two coupling schemes that the Racah coefficients make an appearance.

The discussion in this paper is intentionally confined to the algebraic properties of the scattering equations. Concerning the physical meaning of the equations, we have nothing to add to what Chew and Low have said.

II. THE SCATTERING EQUATION

We shall consider the interaction of mesons with a heavy scatterer to be a particular case of the following general situation. There is a scatterer whose states are represented by $|S\sigma\rangle$, $|T\tau\rangle$, where σ or τ is the z-component of the angular momentum, and S or T denotes the total angular momentum together with any other variables needed to specify the state completely. There is a neutral spin-zero meson field $\phi(r)$. States of the scatterer plus one meson are denoted by $|S\sigma L\lambda x\rangle$, where L is the orbital angular momentum of the meson, λ is the z-component of L, and x is the magnitude of the meson momentum at infinite distance from the scatterer. The states are normalized by the convention

$$\langle T\tau M\mu y|S\sigma L\lambda x\rangle=\delta_{TS}\delta_{\tau\sigma}\delta_{LM}\delta_{\lambda\mu}\delta(x-y). \tag{1}$$

The Hamiltonian of the whole system is

$$H=H_0+H'+H_m, \tag{2}$$

where H_0 is the energy of the scatterer alone, H_m is that of the free meson field, and H' is the interaction

$$H'=\int U(r)\phi(r)d_3r, \tag{3}$$

where $U(r)$ is a Hermitian matrix operating on the states of the scatterer.

The Fourier transform of the operator $U(r)$ may be expressed as a sum of multipoles:

$$U(k)=\int U(r)e^{ik\cdot r}d_3r=\sum_{L\lambda}(-1)^\lambda Y_{-\lambda}{}^L(k)U_\lambda{}^L(k),$$

[1] G. F. Chew, Phys. Rev. **89**, 591 (1953); **94**, 1748 and 1755 (1954); **95**, 1669 (1954).
[2] F. Low, Phys. Rev. **97**, 1392 (1955).
[3] G. F. Chew (unpublished communication).
[4] G. Racah, Phys. Rev. **84**, 910 (1951).

where $Y_{-\lambda}{}^L(k)$ is a normalized surface harmonic, and $U_\lambda{}^L(k)$ is an operator depending on the magnitude of k only. The basic hypothesis of the fixed-scatterer model is that the $U_\lambda{}^L(k)$ be factorizable. That is to say, for each multipole separately we have

$$U_\lambda{}^L(k)=u_L(k)O_\lambda{}^L, \tag{4}$$

where $u_L(k)$ is a c-number function of $|k|$, and $O_\lambda{}^L$ is an operator independent of k. The $O_\lambda{}^L$ are irreducible tensor operators in the sense of Racah.[5] According to Eq. (29) of reference 5, the matrix elements of $O_\lambda{}^L$ are given by

$$\langle R\rho|O_\lambda{}^L|S\sigma\rangle=(-1)^{R+\rho}V(RSL,\,-\rho\sigma\lambda)\langle R\|O^L\|S\rangle, \tag{5}$$

where V is a Wigner coefficient, and the reduced matrix elements $\langle R\|O^L\|S\rangle$ express the physical properties of the scatterer.

The interaction operator is subject to the further conditions that it be Hermitian and invariant under time reversal. The first condition gives

$$(R\|O^L\|S)^*u_L{}^*(k)=(-1)^{S-R+L}(S\|O^L\|R)u_L(k). \tag{6}$$

The second condition, using the conventions of Coester[6] to define the time-reversal operator, gives

$$(R\|O^L\|S)=\eta(-1)^{R-S+L}(S\|O^L\|R), \tag{7}$$

where $\eta=\pm1$ expresses the intrinsic parity of the meson field under time-reversal. According to (6), the phase of $u_L(k)$ is independent of k. We may therefore choose the phase of $O_\lambda{}^L$ in (4) so that $u_L(k)$ is real for $\eta=+1$ and pure imaginary for $\eta=-1$. Then (6) and (7) imply that every $(R\|O^L\|S)$ is real.

Consider the part of the S-matrix for which the initial state $|S\sigma L\lambda x\rangle$ is a state of the scatterer with one meson of momentum x, while the final state $|R\rho\rangle$ is any state of the complete system of scatterer plus meson field. For the final state, R represents the total angular momentum and any other variables needed to specify the state, while ρ is the z-component of angular momentum. Let E_R be the energy of the final state, E_S that of the scatterer in the initial state, and $\Delta_R=E_R-E_S$. Let $\omega_x=(\mu^2+x^2)^{\frac{1}{2}}$ be the energy of the meson with momentum x. According to Low [Eq. (3.6) of reference 2], this part of the S-matrix may be written

$$S=I+2i(x/\omega_x)\delta(\omega_x-\Delta_R)F(x), \tag{8}$$

where $F(x)$ is an operator whose relevant matrix elements are given by

$$\langle R\rho|F(x)|S\sigma L\lambda\rangle=-\tfrac{1}{4}(\omega_x/\pi)^{\frac{1}{2}}u_L(x)\langle R\rho|O_\lambda{}^L|S\sigma\rangle. \tag{9}$$

This $F(x)$ is the scattering amplitude operator for a meson of momentum x. In a scattering state which is an eigenstate of $F(x)$, the eigenvalue will be $e^{i\delta}\sin\delta$, where δ is the phase shift.

[5] G. Racah, Phys. Rev. **62**, 438 (1942).
[6] F. Coester, Phys. Rev. **89**, 619 (1953).

The scattering equation of Low [Eq. (3.7) of reference 2], for the case of elastic scattering of a meson with momentum x, gives

$$\langle T\tau M\mu|F(x)|S\sigma L\lambda\rangle=(-1)^{M+\mu}\frac{xu_L(x)u_M(x)}{16\pi^2}$$
$$\times\sum_{R\rho}\left[\frac{1}{\Delta_R-\omega_x-i\epsilon}\langle T\tau|O_{-\mu}{}^M|R\rho\rangle\langle R\rho|O_\lambda{}^L|S\sigma\rangle\right.$$
$$\left.+\frac{1}{\Delta_R+\omega_x}\langle T\tau|O_\lambda{}^L|R\rho\rangle\langle R\rho|O_{-\mu}{}^M|S\sigma\rangle\right]. \tag{10}$$

Here it is supposed that the initial and final states of the scatterer have equal energy E_S. Equation (10) is exact and depends only on the hypothesis of factorizability of the interaction. The approximation of neglecting states $|R\rho\rangle$ containing more than one meson will not be made anywhere in this paper.

III. THE TWO COUPLING SCHEMES

The usual method of eliminating the magnetic quantum numbers τ, μ, σ, λ from Eq. (10) is to use the fact that the total angular momentum J of the system, and its z-component j, are constants of the motion. Thus we may write

$$\langle T\tau M\mu|F(x)|S\sigma L\lambda\rangle=\textstyle\sum_{Jj}(2J+1)V(TMJ,\,\tau\mu-j)$$
$$\times V(SLJ,\,\sigma\lambda-j)\langle TM|F_J(x)|SL\rangle. \tag{11}$$

Substituting Eqs. (5) and (11) into (10) gives the Low equation in the J-coupling scheme

$$\langle TM|F_J(x)|SL\rangle=\frac{xu_L(x)u_M(x)}{16\pi^2}\sum_R\left[\frac{(-1)^{M+T-R}}{\Delta_R-\omega_x-i\epsilon}\right.$$
$$\times\langle T\|O^M\|R\rangle\langle R\|O^L\|S\rangle\frac{1}{2J+1}\delta_{JR}+\frac{(-1)^{M+S-R}}{\Delta_R+\omega_x}$$
$$\left.\times\langle T\|O^L\|R\rangle\langle R\|O^M\|S\rangle W(LSTM,JR)\right]. \tag{12}$$

Here W is the Racah coefficient defined in reference 5. This Eq. (12) is the direct generalization of Low's equation (3.11) in reference 2, and it shows how the coupling between different J-values in Low's equation comes about. The numerical factors 8/3, 2/9, etc., in Low's equation are products of Racah coefficients, as shown explicitly in Eq. (33) below.

An alternative way of eliminating the magnetic quantum numbers from Eq. (10) is to couple together the initial and final angular momenta of the meson; thus we couple together (LM,ST) instead of (TM,SL). We write

$$\langle T\tau M\mu|F(x)|S\sigma L\lambda\rangle=\textstyle\sum_{N\nu}(2N+1)V(LMN,\,-\lambda\mu\nu)$$
$$\times V(STN,\,\sigma-\tau\nu)(-1)^{\sigma+\mu-S-N}\langle TM|G_N(x)|SL\rangle. \tag{13}$$

Substituting Eqs. (5) and (13) into (10) gives the Low equation in the N-coupling scheme

$$\langle TM|G_N(x)|SL\rangle=\frac{xu_L(x)u_M(x)}{16\pi^2}\sum_R\bigg[\frac{(-1)^L}{\Delta_R-\omega_x-i\epsilon}\times\langle T\|O^M\|R\rangle\langle R\|O^L\|S\rangle W(TMSL,RN)+\frac{(-1)^{M+N}}{\Delta_R+\omega_x}\times\langle T\|O^L\|R\rangle\langle R\|O^M\|S\rangle W(TLSM,RN)\bigg]. \quad (14)$$

For given L, M, S, T, the number of possible values of N is always equal to the number of values of J. From the symmetry of the Racah coefficients, Eqs. (12) and (14) imply

$$\begin{aligned}\langle TM|F_J(x)|SL\rangle&=\langle SL|F_J(x)|TM\rangle,\\ \langle TM|G_N(x)|SL\rangle&=(-1)^{S-T}\langle SL|G_N(x)|TM\rangle.\end{aligned} \quad (15)$$

Thus the S-matrix is symmetric in both J and N coupling schemes, in accordance with the general theorem of Coester.[6]

Equation (14) has in addition a property of symmetry[7] with respect to the interchange of L and M. Thus for $\alpha=0$, 1, we consider the sum or difference defined by

$$\langle TM|G_{N\alpha}(x)|SL\rangle=[\langle TM|G_N(x)|SL\rangle+(-1)^\alpha\langle TL|G_N(x)|SM\rangle]. \quad (16)$$

For this quantity, Eq. (14) gives

$$\langle TM|G_{N\alpha}(x)|SL\rangle=\frac{xu_L(x)u_M(x)}{16\pi^2}\sum_R\bigg[\frac{1}{\Delta_R-\omega_x-i\epsilon}+\frac{(-1)^{N+\alpha}}{\Delta_R+\omega_x}\bigg]\times\{(-1)^L\langle T\|O^M\|R\rangle\langle R\|O^L\|S\rangle W(TMSL,RN)+(-1)^{M+\alpha}\langle T\|O^L\|R\rangle\langle R\|O^M\|S\rangle\times W(TLSM,RN)\}. \quad (17)$$

The F_J and G_N are linear combinations of each other according to Eqs. (11) and (13),

$$\begin{aligned}F_J&=\textstyle\sum_N(2N+1)(-1)^{M+T-L-J}W(TMSL,JN)G_N,\\ G_N&=\textstyle\sum_J(2J+1)(-1)^{M+T-L-J}W(TMSL,JN)F_J.\end{aligned} \quad (18)$$

Substituting Eq. (18) into (14) reproduces Eq. (12) by virtue of the Racah sum-rule [Eq. (43) of reference 5]

$$\textstyle\sum_N(2N+1)(-1)^{L+M+S+T+J+R+N}W(TMSL,JN)\times W(TLSM,RN)=W(LSTM,JR). \quad (19)$$

The physical meaning of the amplitudes $\langle TM|G_N\times(x)|SL\rangle$ is not very clear. They are the "tensor parameters" of the S-matrix, according to the definition of Fano.[8] The following seems to be their most concrete interpretation. For any direction p, let $|L0_p\rangle$ be the state of the meson having total angular momentum L and zero component of angular momentum about p. Then Eq. (13) implies

$$\langle T\tau M0_p|F(x)|S\sigma L0_p\rangle=\textstyle\sum_N[4\pi(2N+1)]^{\frac{1}{2}}V(LMN,000)V(STN,\sigma-\tau\tau-\sigma)\times(-1)^{\sigma-S+L+M}\langle TM|G_N(x)|SL\rangle Y_{\sigma-\tau}{}^N(p). \quad (20)$$

This exhibits the G_N as coefficients of Legendre polynomials in the angular distribution of a scattering amplitude.

IV. UNITARITY AND CASUALITY

Every quantity appearing on the right of Eqs. (10), (12), (14), and (17) is real, except for the infinitesimal $(-i\epsilon)$ in the denominators. Therefore the imaginary part on the right side of these equations comes only from states $|R\rho\rangle$ with $\Delta_R=\omega_x$. Using (9), the imaginary part of (10) becomes

$$\mathrm{Im}\langle T\tau M\mu|F(x)|S\sigma L\lambda\rangle=(x/\omega_x)\textstyle\sum_{R\rho}\langle T\tau M\mu|F^*(x)|R\rho\rangle\times\delta(\Delta_R-\omega_x)\langle R\rho|F(x)|S\sigma L\lambda\rangle, \quad (21)$$

which is simply a statement that the S-matrix (8) is unitary. Since Eqs. (12), (14), and (17) are all equivalent to (10), the imaginary parts of these equations are also equivalent to the unitarity of the S-matrix. For example, the imaginary part of (12) is

$$\mathrm{Im}\langle TM|F_J(x)|SL\rangle=\frac{xu_L(x)u_M(x)}{(2J+1)16\pi}(-1)^{M+T-J}\times\textstyle\sum_R\delta_{RJ}\delta(\Delta_R-\omega_x)\langle T\|O^M\|R\rangle\langle R\|O^L\|S\rangle, \quad (22)$$

and this is another statement of the unitarity condition.

We may now use Eq. (22) in order to simplify (12), following the ideas of Low.[3] The contribution to the right side of (12) from all states R with $\Delta_R=\omega_z>\mu$ may be expressed by (22) in terms of $\mathrm{Im}F_R(z)$. Thus (12) becomes

$$\langle TM|F_J(x)|SL\rangle=\frac{xu_L(x)u_M(x)}{16\pi^2}\sum_R{}'\bigg[\frac{(-1)^{M+T-R}}{\Delta_R-\omega_x}\times\langle T\|O^M\|R\rangle\langle R\|O^L\|S\rangle\frac{1}{2J+1}\delta_{JR}+\frac{(-1)^{M+S-R}}{\Delta_R+\omega_x}\times\langle T\|O^L\|R\rangle\langle R\|O^M\|S\rangle W(LSTM,JR)\bigg]+\frac{1}{\pi}\int_0^\infty\frac{dz}{\omega_z}\frac{xu_L(x)u_M(x)}{u_L(z)u_M(z)}\bigg[\frac{\mathrm{Im}\langle TM|F_J(z)|SL\rangle}{\omega_z-\omega_x-i\epsilon}+\sum_R\frac{\mathrm{Im}\langle TL|F_R(z)|SM\rangle}{\omega_z+\omega_x}(-1)^{M+S-L-T}\times(2R+1)W(LSTM,JR)\bigg]. \quad (23)$$

[7] The extra symmetry is related to the Gell-Mann and Goldberger "crossing theorem" [the theorem is mentioned by H. W. Wyld, Phys. Rev. **96**, 1661 (1954), end of Sec. III, but is otherwise unpublished]. The purpose of introducing the quantum-number N is to make this symmetry evident in the equations.

[8] U. Fano, Phys. Rev. **90**, 577 (1953).

Here the first sum extends only over the states R of the system for which $\Delta_R<\mu$, and is in most cases a discrete sum over a finite number of states. The integral has the appearance of a dispersion formula, except that the amplitudes for different angular momenta R are mixed together in the last term.

To obtain a true dispersion formula, we must use the N-coupling scheme and apply the same argument to the real and imaginary parts of Eq. (17). Let $Q_{N\alpha}(R)$ be the quantity in curly brackets on the right of (17). Let the function $g_{N\alpha}(\omega)$ be defined by

$$g_{N\alpha}(\omega_x)=[16\pi^2/xu_L(x)u_M(x)]\langle TM|G_{N\alpha}(x)|SL\rangle. \quad (24)$$

Then using the unitarity of the S-matrix to simplify the right side, Eq. (17) becomes

$$g_{N\alpha}(\omega)=\sum_R{}' Q_{N\alpha}(R)\left[\frac{1}{\Delta_R-\omega}+\frac{(-1)^{N+\alpha}}{\Delta_R+\omega}\right]$$
$$+\frac{1}{\pi}\int_\mu^\infty[\mathrm{Im}g_{N\alpha}(\omega')]d\omega'\left[\frac{1}{\omega'-\omega-i\epsilon}+\frac{(-1)^{N+\alpha}}{\omega'+\omega}\right]. \quad (25)$$

This is a dispersion formula of the customary kind, expressing the fact that $g_{N\alpha}(\omega)$ is an analytic function of ω for complex ω in the upper half-plane, and is an even or odd function according as $(N+\alpha)$ is even or odd. We may call Eq. (25) a causality condition, since it is formally similar to the dispersion formulas which have been deduced from requirements of causality.[9] The fact that the linear transformation (18), (16) from F_J to $G_{N\alpha}$ diagonalizes the Eqs. (23) is due directly to the Racah sum-rule (19).

These results may be summed up in the statement that the Low scattering equation is equivalent to a unitarity condition (22) which is simple in the J-coupling scheme, together with the causality condition (25) which is simple in the N-coupling scheme. The interlock between the two conditions is given by the transformation formulas (18).

When the meson energy is below the threshold for inelastic processes, the unitarity condition is particularly simple. In this case the only states $|R\rho\rangle$ which contribute to (21) are one-meson states $|P\pi K\kappa x\rangle$, and Eq. (22) reduces to

$$\mathrm{Im}\langle TM|F_J(x)|SL\rangle$$
$$=\sum_{PK}\langle TM|F_J^*(x)|PK\rangle\langle PK|F_J(x)|SL\rangle,$$

or simply

$$\mathrm{Im}F_J(x)=F_J^*(x)F_J(x), \quad (26)$$

where $F_J(x)$ is considered as a matrix in the indices $TMSL$.

The "one-meson approximation" of Low[2] means that inelastic processes are neglected and so (26) is assumed to hold for all energies. We shall not make this approximation, but we shall specialize the system slightly in order to bring the dispersion formula (25) into a very simple form. We suppose that the scatterer has a ground state with spin S, and no other state with energy below $E_S+\mu$. We consider the amplitude $\langle SM|G_N|SL\rangle$ for elastic scattering which leaves the scatterer in the ground state, not assuming that competing inelastic processes are absent or unimportant. In this case the causality condition can be deduced directly from (14) instead of from (17), and the sum over states R in (25) reduces to the single term $R=S$ with $\Delta_R=0$. If we write

$$\langle SM|G_N(x)|SL\rangle=[Q_N/16\pi^2]xu_L(x)u_M(x)g_N(\omega_x),$$
$$Q_N=(-1)^L\langle S\|O^M\|S\rangle\langle S\|O^L\|S\rangle W(SMSL,SN), \quad (27)$$

the unitary condition applied to Eq. (14) gives the result

$$g_N(\omega)=[(-1)^N-1]\frac{1}{\omega}$$
$$+\frac{1}{\pi}\int_\mu^\infty[\mathrm{Im}g_N(\omega')]d\omega'\left[\frac{1}{\omega'-\omega-i\epsilon}+\frac{(-1)^N}{\omega'+\omega}\right]. \quad (28)$$

This equation is precisely equivalent to the following four statements:

(a) $g_N(\omega)$ is an analytic function of ω in the whole complex plane with cuts from μ to $+\infty$ and from $-\mu$ to $-\infty$.
(b) $g_N(\omega)$ is real for real ω in the range $-\mu<\omega<\mu$.
(c) $g_N(\omega)$ is even or odd according as N is even or odd.
(d) $g_N(\omega)$ has only a simple pole with residue -2 at $\omega=0$ when N is odd, and it has no pole when N is even.

The fact that these conditions, together with the unitarity condition (21), are equivalent to the scattering Eq. (10), is the theorem of Low which was mentioned in Sec. I.

V. EXTENSION TO CHARGED MESONS. THEORY OF MESON-NUCLEON SCATTERING

The theory which has been developed for neutral mesons can be extended immediately to mesons which possess a total isotopic spin L' and a charge quantum-number λ', scattered by a scatterer possessing total isotopic spin S' and charge quantum-number σ', provided the interaction is invariant under rotations in isotopic space. Particles having several values L', M' of total isotopic spin may also be included. The resulting equations will be the same as for the neutral case, except that each Racah coefficient is replaced by a product of two Racah coefficients operating independently on the angular momentum and isotopic spin quantum numbers. For example in Eq. (12), $W(LSTM,JR)$ is replaced by $W(LSTM,JR)W(L'S'T'M',J'R')$.

The Chew model of p-wave meson-nucleon scattering is the special case of this theory in which always

[9] Gell-Mann, Goldberger, and Thirring, Phys. Rev. **95**, 1612 (1954); M. L. Goldberger, Phys. Rev. **97**, 508 (1955); M. L. Goldberger, Phys. Rev. **99**, 979 (1955); Goldberger, Miyazawa, and Oehme, Phys. Rev. **99**, 986 (1955).

$L=L'=M=M'=1$, $S=S'=T=T'=\frac{1}{2}$. The possible values of J, J', R, R' are $\frac{1}{2}$, $\frac{3}{2}$, and the possible values of N, N' are 0, 1. We write

$$\begin{aligned} W_{JR}&=W(1\tfrac{1}{2}\tfrac{1}{2}1,JR),\\ V_{RN}&=W(\tfrac{1}{2}1\tfrac{1}{2}1,RN), \end{aligned} \tag{29}$$

so that W and V are numerical 2×2 matrices

$$W_{JR}=\frac{1}{12}\begin{bmatrix}-2 & 4\\ 4 & 1\end{bmatrix},\quad V_{RN}\doteq\frac{1}{6}\begin{bmatrix}-\sqrt{6} & 2\\ +\sqrt{6} & 1\end{bmatrix}. \tag{30}$$

The physical properties of the scatterer are expressed only in the form-function $u_1(x)$ and in the single reduced matrix element

$$f=\langle\tfrac{1}{2}\tfrac{1}{2}\|O^{11}\|\tfrac{1}{2}\tfrac{1}{2}\rangle, \tag{31}$$

which fixes the strength of the interaction. We may identify f with the renormalized coupling constant of the Chew theory. The scattering amplitude

$$F_{JJ'}(x)=\langle\tfrac{1}{2}\tfrac{1}{2}11\,|\,F_{JJ'}(x)\,|\,\tfrac{1}{2}\tfrac{1}{2}11\rangle \tag{32}$$

is equal to $e^{i\delta}\sin\delta$, where δ is the phase-shift for a meson of momentum x in the state (J,J') of the system meson-plus-scatterer. In general δ will be complex; it is real below the threshold for inelastic processes, in the range where (26) holds.

The scattering Eq. (23) for the Chew model becomes

$$\begin{aligned} F_{JJ'}(x)=&-(f^2/16\pi^2)\{x[u_1(x)]^2/\omega_x\}\\ &\times[\tfrac{1}{4}\delta_{J\frac{1}{2}}\delta_{J'\frac{1}{2}}-W_{J\frac{1}{2}}W_{J'\frac{1}{2}}]\\ &+\frac{1}{\pi}\int_0^\infty\frac{dz}{\omega_z}\frac{x[u_1(x)]^2}{[u_1(z)]^2}\biggl[\frac{\operatorname{Im}F_{JJ'}(z)}{\omega_z-\omega_x-i\epsilon}\\ &+\sum_{RR'}\frac{\operatorname{Im}F_{RR'}(z)}{\omega_z+\omega_x}(2R+1)(2R'+1)W_{JR}W_{J'R'}\biggr]. \end{aligned} \tag{33}$$

This is identical with Low's equation (3.11) of reference 2 if we choose[10]

$$u_1(x)=(48\pi)^{\frac{1}{2}}(x/\mu), \tag{34}$$

corresponding to the conventional point-source charge-symmetric pseudovector interaction, and assume (26) to hold for all x (one-meson approximation).

If, however, we make the substitution

$$F_{JJ'}(x)=(f^2/16\pi^2)x[u_1(x)]^2\sum_{NN'}Z_{JN}Z_{J'N'}g_{NN'}(\omega_x), \tag{35}$$

$$Z_{JN}=(-1)^{J-\frac{1}{2}}(2N+1)V_{\frac{1}{2}N}V_{JN}, \tag{36}$$

which is the analog of (18), (27) for the Chew model, then the functions $g_{NN'}(\omega)$ satisfy the simple dispersion formula

[10] According to (6) and (7) the time parity of the meson field must be odd, and the $u_L(k)$ pure imaginary, for neutral pseudoscalar mesons. In the charge-symmetric theory the meson field also has odd parity under inversion in isotopic space. This makes the $u_L(k)$ real when the charge variables are inserted in (6) and (7).

$$\begin{aligned} g_{NN'}(\omega)=&[\pm1-1]\frac{1}{\omega}\\ &+\frac{1}{\pi}\int_\mu^\infty[\operatorname{Im}g_{NN'}(\omega')]d\omega'\biggl[\frac{1}{\omega'-\omega-i\epsilon}\pm\frac{1}{\omega'+\omega}\biggr]. \end{aligned} \tag{37}$$

The ambiguous sign is plus for $N=N'$ and minus for $N\neq N'$. By (30), the numerical values of the coefficients Z_{JN} in (35) are

$$Z_{JN}=\frac{1}{6}\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}. \tag{38}$$

The relations (35) can be inverted to give

$$(f^2/16\pi^2)x[u_1(x)]^2g_{NN'}(\omega_x)=\sum_{JJ'}X_{NJ}X_{N'J'}F_{JJ'}(x), \tag{39}$$

with

$$X_{NJ}=2\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}. \tag{40}$$

It is the $g_{NN'}$, and not the usual scattering amplitudes $F_{JJ'}$, which obey causality relations of the form (37). It happens that the $g_{NN'}$ for the Chew model have a direct interpretation in terms of observable processes. Let $a^{\pm}(0)$ be the amplitude for elastic scattering without charge exchange of a positive or negative meson by a proton, without changing the proton spin state. Let $a^{\pm}(1)$ be the amplitude for scattering a positive or negative meson, with change of the proton spin state. Then g_{N0} is proportional to

$$[a^-(N)+a^+(N)],$$

and g_{N1} is proportional to

$$[a^-(N)-a^+(N)].$$

The fact that it is these combinations of the amplitudes which have a simple causal behavior has been already pointed out by Low and by Goldberger.[9]

The causality relations of Goldberger refer always to forward scattering amplitudes, which are expressions of the form (20). The coefficient $V(LMN,000)$ is zero when $(L+M+N)$ is odd, therefore in the case of the Chew model the right side of (20) brings in only the amplitudes with $N=0$ and not those with $N=1$. For this reason the argument of Goldberger demonstrates a causal behavior only for the no-spin-flip amplitudes, which are those with $N=0$. The causal behavior of the spin-flip $N=1$ amplitudes is thus a prediction of the Chew model which is not required (so far as we know) by general principles of field-theory, and may therefore serve as a test of the model.

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