TABLE A-3-a. ($E \geq 10^{-4} E_0$).

t \ N_e	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0.5	650	130	750	206	470	289	268	105	168	158	132	65	51	17	16	9	6	4	3	2	1	0
0.4	1152	288	520	236	501	221	204	60	211	22	53	7	16	4	3	0	2	0	0	0	0	0
0.3	1463	308	823	326	252	75	194	10	41	2	6	0	0	0	0	0	0	0	0	0	0	0
0.2	1896	237	1033	60	243	3	27	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0.1	2871	0	603	0	26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE A-3-b. ($E \geq 10^{-3} E_0$).

t \ N_e	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0.5	910	260	768	433	464	189	231	106	66	49	11	6	5	2	0
0.4	1440	72	852	401	391	117	155	32	33	1	6	0	0	0	0
0.3	1848	0	1096	178	285	48	38	5	2	0	0	0	0	0	0
0.2	2212	158	955	40	128	2	5	0	0	0	0	0	0	0	0
0.1	2871	0	603	0	26	0	0	0	0	0	0	0	0	0	0

TABLE A-3-c. ($E \geq 10^{-2} E_0$).

t \ N_e	0	1	2	3	4	5	6	7	8	9
0.5	1625	195	1010	295	244	78	44	7	2	0
0.4	1872	216	1010	173	180	36	11	2	0	0
0.3	2310	77	968	22	118	1	4	0	0	0
0.2	2528	158	719	40	52	2	1	0	0	0
0.1	2871	87	516	13	13	0	0	0	0	0

radiation length and taking the point of conversion as the origin. The letters a, b, c here again refer to $\alpha = 10^{-4}$, 10^{-3}, and 10^{-2} respectively where now E_0 = initial pair energy. Here N_e is the number of additional electrons formed in the distance t radiation lengths from the first pair origin and the numbers in the rows opposite t represent the number of cases.

Tables A-3-a, A-3-b, and A-3-c represent the development for the initial condition of one electron pair and one photon of equal energy, E_0, incident at the origin and were made by a superposition of Tables A-1 and A-2. Here again a, b, c, refer respectively to $\alpha = 10^{-4}$, 10^{-3}, and 10^{-2} and N_e is the number of additional electrons made in the distance t radiation lengths from the origin and the numbers in the rows opposite t represent the number of cases.

With these results we clearly see the large fluctuations in electron numbers in the cascade development. As a matter of fact, these fluctuations can cover almost all anomalies in electron numbers in cascade showers thus far observed if the initial energy is sufficiently high.

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Causality in the Pion-Proton Scattering*

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Dispersion relations applicable to particles with mass and charge have been used for analyzing pion-proton scattering data. In these relations, experimental values of the total cross sections for π^+ and π^- from 0 to 1.9 Bev were used to calculate the real part of the forward scattering amplitudes, and these were compared with the results of phase-shift analyses. With suitable choice of the pion-nucleon coupling constant, good agreement can be obtained for the phase-shift solutions with a resonant behavior for α_{33}.

INTRODUCTION

A RELATION between the real and imaginary parts of the forward scattering amplitude for the scattering of light on atoms has been known for some time. The relation, known as the Kramers-Kronig dispersion relation[1,2] is derived from the condition that the scattered wave should have zero amplitude until the incident wave reaches the scatterer. Following a suggestion by Kronig[3] that such a causality condition might be extended to apply to the scattering of particles

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[3] R. Kronig, Physica **12**, 543 (1946).

with mass, a generalized dispersion relation for Bose particles with finite rest mass has now been obtained by Goldberger.[4] Goldberger's derivation uses the formalism of quantum field theory. The demand that waves do not propagate faster than the velocity of light is expressed by requiring that measurements of two observable quantities made at space like points, should not interfere. Thus, the causality condition is imposed by setting the commutator of two Heisenberg operators for the boson field equal to zero when these are taken at space like points.

Application of such a dispersion relation to the pion-proton scattering was first made by Karplus and Ruderman[5] who were able to show that the causality condition required a positive sign for the real part of the forward scattering amplitude of the low-energy pion-proton scattering. This was in agreement with the result of the study of the effect of the Coulomb interference.[6,7]

Our interest was to make use of the causality condition in the phase-shift analysis of the pion-proton scattering, following a suggestion by Wigner.[8] Wigner had found a lower limit for the derivative of the phase shifts with momentum. This made clear the relevancy of causality for the phase-shift analysis, but seemed less useful than a dispersion relation. The dispersion relation obtained by Karplus and Ruderman applies to the scattering of π^0 by protons. The formula obtained by Goldberger, Miyazawa, and Oehme[9] applies explicitly to the scattering of charged pions by protons and was used in the present work.

DISPERSION RELATION

In a dispersion relation, the real part of the forward scattering amplitude is calculated from a knowledge of the imaginary part over the entire energy range, from $-\infty$ to $+\infty$. For any given scattering process, it is of course possible to measure the total cross section and hence the imaginary part of the forward scattering amplitude, only at energies above μ, the rest energy. According to Goldberger, the usual assumptions made in field theory have the consequence that the cross sections for the charge conjugate particle may be used for energies more negative than $-\mu$. For this reason, both the cross sections $\sigma_+(\omega)$ for π^+ as well as $\sigma_-(\omega)$ for π^- contribute in the dispersion formulas.

There is an additional contribution which occurs at energy $\omega=\mu^2/2M$ for π^+ or $\omega=-\mu^2/2M$ for π^- scattering, where M is the nucleon rest energy. This is the only contribution in the energy range between $-\mu$ and $+\mu$ and arises from the fact that the neutron can be a possible intermediate state in the scattering. This contribution may be expressed in terms of the small coupling constant characteristic of the pseudovector interaction, and is numerically very important. An important object of an experimental check of the dispersion relations is to test the correctness of these ideas.

We reproduce below the formulas given by Goldberger, Miyazawa, and Oehme[9] for the scattering of positive and negative pions by protons.

$$D_+(k)-\tfrac{1}{2}\left(1+\frac{\omega}{\mu}\right)D_+(0)-\tfrac{1}{2}\left(1-\frac{\omega}{\mu}\right)D_-(0) =\frac{k^2}{4\pi^2}\int_\mu^\infty \frac{d\omega'}{k'}\left[\frac{\sigma_+(\omega')}{\omega'-\omega}+\frac{\sigma_-(\omega')}{\omega'+\omega}\right]+\frac{2f^2}{\mu^2}\frac{k^2}{\omega-\frac{\mu^2}{2M}}, \quad (1)$$

$$D_-(k)-\tfrac{1}{2}\left(1+\frac{\omega}{\mu}\right)D_-(0)-\tfrac{1}{2}\left(1-\frac{\omega}{\mu}\right)D_+(0) =\frac{k^2}{4\pi^2}\int_\mu^\infty \frac{d\omega'}{k'}\left[\frac{\sigma_-(\omega')}{\omega'-\omega}+\frac{\sigma_+(\omega')}{\omega'+\omega}\right]-\frac{2f^2}{\mu^2}\frac{k^2}{\omega+\frac{\mu^2}{2M}}, \quad (2)$$

where k is the wave number of the pion and ω its total energy, both in the laboratory system.

The formulas have been written to give the difference between the real part of the forward scattering amplitude $D_+(k)$ for positive pions of wave number k, or $D_-(k)$ for negative pions and the values for these quantities taken at $k=0$. Both integrals contain the total cross sections for π^+ and π^- and it is the principal part of the integral which is to be evaluated. The final term in both expressions gives the contribution of the bound state.

It is well known that in the pion-nucleon scattering, the state of isotopic spin $T=\frac{3}{2}$ is dominant. For this reason, we preferred to calculate the quantity $D_3(k)$ corresponding to this state. This is identical to $D_+(k)$ given in (1).

$$D_3(k)=D_+(k). \quad (3)$$

However, the corresponding quantity for the state $T=\frac{1}{2}$ is given by the linear combination

$$D_1(k)=\tfrac{3}{2}D_-(k)-\tfrac{1}{2}D_+(k). \quad (4)$$

These quantities are readily expressed in terms of the phase shifts which are determined from the angular distribution measurements. We have up to terms including p waves,

$$D_3(k)=(k/2k_b^2)(\sin 2\alpha_3+2\sin 2\alpha_{33}+\sin 2\alpha_{31}+\cdots), \quad (5)$$

and

$$D_1(k)=(k/2k_b^2)(\sin 2\alpha_1+2\sin 2\alpha_{13}+\sin 2\alpha_{11}+\cdots). \quad (6)$$

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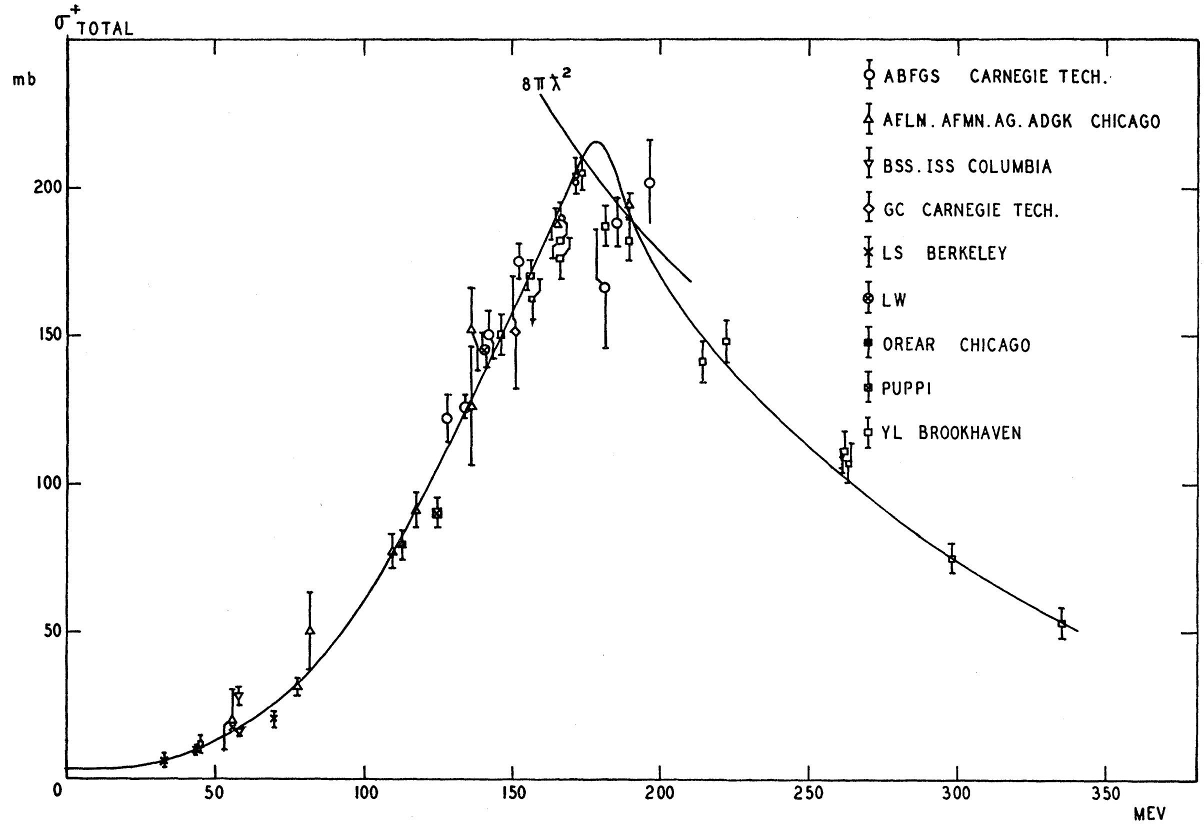

FIG. 1. Total cross section for positive pions on protons *vs* pion kinetic energy in the laboratory system; energy range from 0-350 Mev. Sources for data are listed in references 11–27.

Here k_b is the pion wave number in the center-of-mass system. The transformation from the center of mass to the laboratory system is carried out by recognizing that the ratio $D(k)/k$ is an invariant.

The connection between k_b and k is

$$k_b=k\left(1+\frac{2\omega}{M}+\frac{\mu^2}{M^2}\right)^{-\frac{1}{2}}. \tag{7}$$

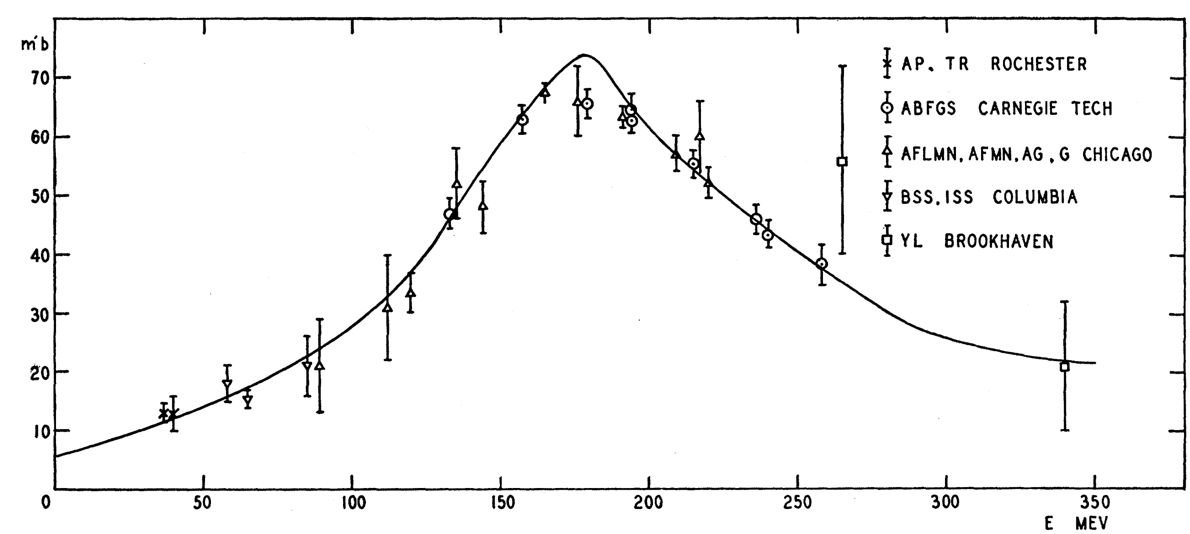

FIG. 2. Total cross section for negative pions on protons *vs* pion kinetic energy in the laboratory system; energy range from 0-350 Mev. Sources for data are listed in references 11–27.

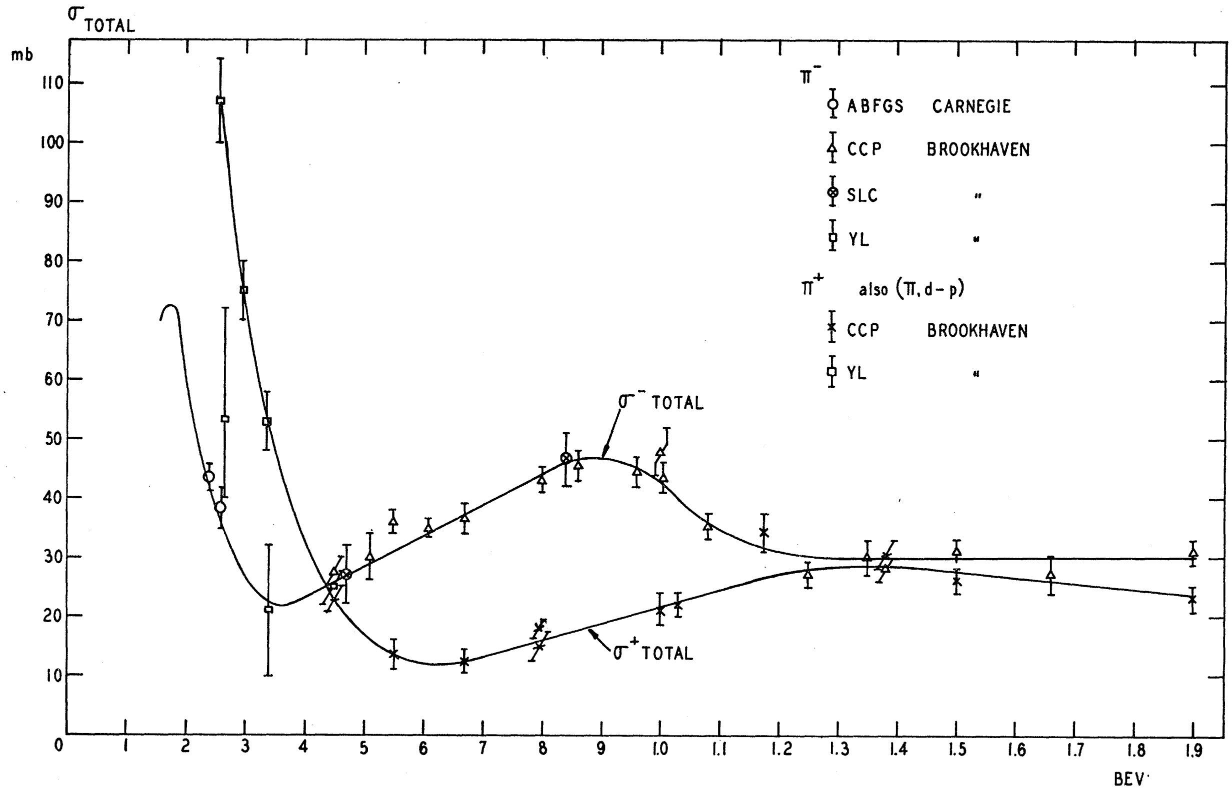

FIG. 3. Total cross section for positive and negative pions *vs* pion kinetic energy in the laboratory system; energy range from 0.25 to 1.9 Bev. Sources for data are listed in references 11–27.

The quantities $D_+(0)$ and $D_-(0)$ may be evaluated from the low-energy behavior of the phase shifts. As

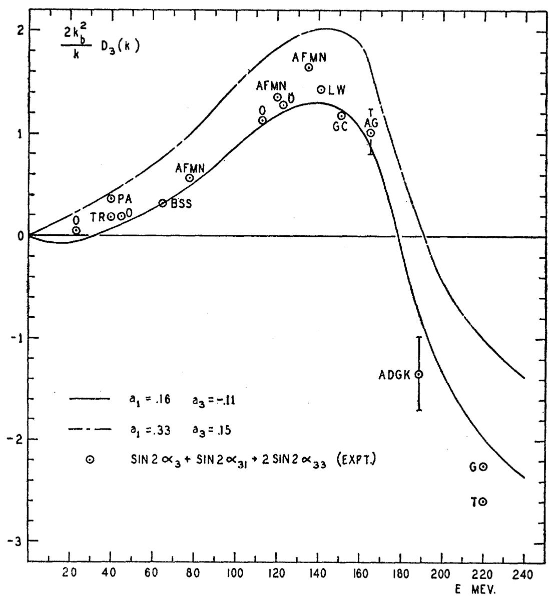

FIG. 4. Comparison of $(2k_b^2/k)D_3(k)$ calculated from causality conditions with $\sin 2\alpha_3 + \sin 2\alpha_{31} + 2\sin 2\alpha_{33}$ obtained from phase-shift analysis. The abscissa represents the laboratory kinetic energy in Mev. Sources for data are listed in references 11–27.

is well known, the phase shift corresponding to the state of angular momentum l varies asymptotically as k_b^{2l+1}. Thus, in the limit as $k_b \rightarrow 0$, only the s wave phase shifts contribute. These are written:

$$\alpha_3 = a_3 \eta, \tag{8}$$

$$\alpha_1 = a_1 \eta, \tag{9}$$

where $\eta = k_b \lambda\!\!\!^{-}_c$ with $\lambda\!\!\!^{-}_c$ the pion Compton wavelength. Thus, from (5) and (6) using (3), (4), and (7), we write

$$D_+(0) = \lambda\!\!\!^{-}_c \left(1 + \frac{\mu}{M}\right) a_3, \tag{10}$$

$$D_-(0) = \lambda\!\!\!^{-}_c \left(1 + \frac{\mu}{M}\right) \left(\tfrac{2}{3} a_1 + \tfrac{1}{3} a_3\right). \tag{11}$$

The quantities a_3 and a_1 may be obtained from an analysis of low-energy pion scattering data such as has been made by Orear.[10] The coupling constant f remains as an adjustable parameter.

COMPARISON WITH EXPERIMENT

The cross sections for the scattering of both π^- and π^+ by protons are now known up to 1.9 Bev.[11–27] We

[10] J. Orear, Phys. Rev. **96**, 176 (1954).

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have plotted all this data as shown in Figs. 1–3. At low energies we have drawn the curve according to Orear's[10] prescriptions for the phase shifts. At high energies we supposed the cross sections to continue as constant and equal to their values at 1.9 Bev. These extrapolations have only a minor effect on the results.

For a more direct comparison with the phase-shift analysis of the scattering experiments we have plotted the quantity $(2k_b^2/k)D_3(k)$ in Fig. 4 and the quantity $(2k_b^2/k)D_1(k)$ in Fig. 5. In the calculations we have taken $M/\mu=6.73$ and $\lambda_c=1.42\times10^{-13}$ cm. For $2f^2$ we used the value 0.161 obtained by Chew from the analysis of pion-nucleon scattering and photomeson production.[28] The quantities $a_3=-0.11$ and $a_1=+0.16$ as taken from Orear[10] gave the lower curves. For comparison we also calculated using $a_3=0.15$ and $a_1=0.33$ according to Noyes and Woodruff.[29]

The experimental points are the quantities $\sin2\alpha_3+2\sin2\alpha_{33}+\sin2\alpha_{31}$ for Fig. 4 and $\sin2\alpha_1+2\sin2\alpha_{13}+\sin2\alpha_{11}$ for Fig. 5. In the case of the points at 165 Mev and 189 Mev an error analysis was available so that it was possible to give some idea of the experimental error. Presumably, errors comparable in magnitude should be associated with the other experimental points.

The outstanding feature of the curve for $T=\frac{3}{2}$ is the abrupt change in sign near 180 Mev. The positive sign of $D_+(k)$ below 180 Mev is in agreement with the result of measurements of the Coulomb interference effect at 113 Mev[6] and at 126 Mev.[7] The change in sign of $D_+(k)$ above 180 Mev implies that the forward scattering should be augmented rather than diminished by the Coulomb interference. This effect has now been observed by Taft[30] with positive pions at 220 Mev.

The abrupt change in sign for $D_3(k)$ has made possible a rather unambiguous choice of one out of several possible sets of phase shifts which fit the pion-proton angular distribution of the solutions recently found at 189 Mev;[12] the one in which $\alpha_{33}=98.5°$, $\alpha_3=-11.3°$, and $\alpha_{31}=-11.6°$ is in agreement with this behavior. Alternative solutions with $\alpha_{33}=81.5°$, $\alpha_3=+11.3°$, and $\alpha_{31}=+11.6°$ or one with $\alpha_3=-44.5°$ give the wrong sign for $D_3(k)$. A fourth solution at 189 Mev has $\alpha_3=-11.6°$, $\alpha_{33}=126°$, and $\alpha_{31}=56°$ or 236°. This gives the proper behavior for $D_3(k)$ but must be compared with solutions at 165 Mev[15] in which $\alpha_3=-20°$, $\alpha_{33}=34°$, and $\alpha_{31}=94°$ or $\alpha_3=-20°$, $\alpha_{33}=+63°$, and $\alpha_{31}=4°$. This seems unlikely in view of the large jumps for two phase shifts which are called for. The choice coincides with that selected by de Hoffmann, Metropolis, Alei, and Bethe[31] on other grounds. This is the solution in which α_{33} passes through 90°. A non-resonant behavior for α_{33} seems excluded by causality.

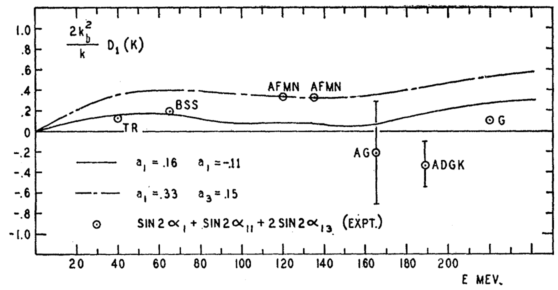

FIG. 5. Comparison of $(2k_b^2/k)D_1(k)$ calculated from causality conditions with $\sin2\alpha_1+\sin2\alpha_{11}+2\sin2\alpha_{13}$ obtained from phase shift analysis. The abscissa represents the laboratory kinetic energy in Mev. Sources for data are listed in references 11–27.

The general fit of the experimental data is surprisingly good in view of the rather considerable experimental errors. Neglect of the presence of d waves in the analysis of the scattering may be responsible for part of the discrepancy which occurs at the higher energies. The agreement with Orear's prescription for a_3 and a_1 is noticeably better than that using the values of Noyes and Woodruff. Adjustment of the curves by altering the value of f^2 can serve to bring down the curve for $T=\frac{3}{2}$ but this raises the $T=\frac{1}{2}$ curve by twice the amount.

The importance of the neutron term is of special significance. Neglect of this term would have the quantity $(2k_b^2/k)D_1(k)$ rise rather substantially at the high-energy end. Its inclusion serves to keep the value of this quantity near zero in conformity with the experiments. For this reason, the dispersion relations can serve as a means to evaluate the strength of the pion-nucleon coupling. The role of the neutron as an intermediate state of the pion-proton system is made more evident because of these results.

ACKNOWLEDGMENTS

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