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Pion Production in Proton-Proton Collisions*†

D. B. LICHTENBERG‡
University of Illinois, Urbana, Illinois
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The cut-off form of the Yukawa theory is applied to the reaction $p+p\rightarrow\pi^++d$. Using the Low-Wick formulation of field theory, an expression is obtained for the appropriate matrix element in terms of the wave functions of a physical deuteron and diproton. Approximations are made for these wave functions and the matrix element is evaluated. The resulting cross section agrees semiquantitatively with that part of the observed cross section due to pions emitted in a p-state.

I. INTRODUCTION

THE success of the cut-off form of the Yukawa theory[1] in correlating a variety of phenomena involving pions and nucleons has led us to apply it here to the problem of pion production in nucleon-nucleon collisions. Previous treatments[2–7] of this problem have been of two types. The first approach is to make a strictly field theoretical perturbation calculation of the matrix element for the reaction. This approach suffers from the usual difficulties of perturbation theory when applied to meson phenomena. There is an additional difficulty if the final state contains a deuteron, since present perturbation methods are not adequate to treat bound state problems. The second approach tries to avoid these troubles by treating the interaction between the two nucleons by means of a phenomenological potential. There is a practical disadvantage to this method in that the results are very sensitive to the details of the wave functions chosen for the initial and final nucleon systems. Even more important from a fundamental point of view, it has not been adequately shown under what approximations this treatment follows from field theory.

However, two recent developments have put the second approach to the pion production problem on a much sounder footing. The recent work of Low and Wick[8] in reformulating field theory in terms of physical nucleons has enabled us to show that the treatment with nuclear wave functions is indeed consistent with the prescriptions of field theory. And the work of Gartenhaus[9] in obtaining nuclear potentials from the cut-off theory has permitted us to use wave functions in the matrix element which are probably better than wave functions chosen phenomenologically.

II. FORMULATION OF THE PROBLEM

We restrict ourselves to consideration of the reaction

$$p+p\rightarrow\pi^++d, \qquad (1)$$

although the methods developed here are more generally valid. The differential cross section is given by the formula (we choose as units $\hbar=c=1$):

$$\frac{d\sigma}{d\Omega}=\frac{1}{(2\pi)^2}\frac{M}{2p}q\omega_q\sum|T_{qp}|^2 \qquad (2)$$

subject to the condition that energy be conserved. The initial protons each have momentum $\mathbf{p}$ (in the center-of-mass system), and the pion is emitted with momentum $\mathbf{q}$ and energy $\omega_q=(q^2+\mu^2)^{\frac{1}{2}}$. The quantity T_{qp} is the appropriate matrix element for the transition, $\sum$ indicates an average and sum over initial and final spin states, and M and μ are respectively the nucleon and pion mass. The recoil of the deuteron is neglected in obtaining Eq. (2).

The problem is to find an adequate estimate for the matrix element T_{qp}. To do this, we first write down an expression for the total Hamiltonian H:

$$H=T_N+T_\pi+H_1'+H_2',$$

where T_N is the kinetic energy of the two nucleons, $T_\pi=\sum_k a_k^\dagger a_k\omega_k$ is the kinetic energy of the pion field, and $H_j'(j=1,2)$ is the interaction between the pion field and the jth nucleon. The quantities a_k and $a_k^\dagger$ are the usual annihilation and creation operators for a pion of type k, where k is a generalized index referring to both the momentum and charge state of the pion. For the cut-off theory, the interaction H_j' is given by[1]

$$H_j'=\sum_k(a_kV_{jk}^0+a_k^\dagger V_{jk}^{0\dagger}),$$
$$V_{jk}^0=\left(\frac{4\pi}{2\omega_k}\right)^{\frac{1}{2}}\frac{f^0}{\mu}v(k)\boldsymbol{\sigma}_j\cdot i\mathbf{k}\tau_{jk}e^{i\mathbf{k}\cdot\mathbf{r}_j}, \qquad (3)$$

where f^0 is the unrationalized and unrenormalized coupling constant, $v(k)$ is the cutoff, $\boldsymbol{\sigma}_j$ and $\boldsymbol{\tau}_j$ are the usual spin and isotopic spin operators for the jth nucleon at position $\mathbf{r}_j$.

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‡ Now at Indiana University, Bloomington, Indiana.
1 G. F. Chew, Phys. Rev. **95**, 1669 (1954).
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9 Solomon Gartenhaus (to be published).

Following the formalism of Low and Wick, we let $\Psi_q^{D(+)}$ be an eigenfunction of the *total* Hamiltonian H with eigenvalue E_q. This function represents a physical deuteron interacting with its cloud of virtual mesons, and satisfying the boundary condition of a pion at infinity with momentum q, plus outgoing spherical waves. The eigenvalue equation satisfied by this function is

$$H\Psi_q^{D(+)}=E_q\Psi_q^{D(+)}=(E_0+\omega_q)\Psi_q^{D(+)}, \tag{4}$$

where E_0 is the energy of the deuteron when the pion is far away, and ω_q is the pion energy. We now introduce another eigenfunction Ψ_0^D of the total Hamiltonian which satisfies the equation

$$H\Psi_0^D=E_0\Psi_0^D. \tag{5}$$

This eigenfunction Ψ_0^D is the lowest state of the system, and represents the ground state of the deuteron interacting with a virtual meson cloud, but with no real pions in the field. We can produce a real pion of type q by acting on this state with the creation operator $a_q^\dagger$. The function $a_q^\dagger\Psi_0^D$ is not an eigenfunction of H, but we can write

$$\Psi_q^{D(+)}=a_q^\dagger\Psi_0^D+\Psi_s, \tag{6}$$

where Ψ_s is defined by Eq. (6) plus the outgoing wave condition.

To find an expression for Ψ_s, we substitute (6) into (4), obtaining

$$Ha_q^\dagger\Psi_0^D+H\Psi_s=(E_0+\omega_q)(a_q^\dagger\Psi_0^D+\Psi_s). \tag{7}$$

Then, noting that

$$Ha_q^\dagger\Psi_0^D=a_q^\dagger H\Psi_0^D+[H,a_q^\dagger]\Psi_0^D,$$

and

$$[H,a_q^\dagger]=a_q^\dagger\omega_q+V_{1q}^0+V_{2q}^0,$$

Eq. (7) becomes

$$(V_{1q}^0+V_{2q}^0)\Psi_0^D+H\Psi_s=(E_0+\omega_q)\Psi_s.$$

This can be written

$$\Psi_s=[1/(E_0+\omega_q-H+i\epsilon)](V_{1q}^0+V_{2q}^0)\Psi_0^D, \tag{8}$$

where the $+i\epsilon$ signifies that we have chosen the outgoing wave solution.

We now expand (8) in a complete set of eigenfunctions $\Psi_n^{(-)}$ of H which satisfy the incoming wave boundary condition:

$$\Psi_s=\sum_n \Psi_n^{(-)}\frac{(\Psi_n^{(-)},\ (V_{1q}^0+V_{2q}^0)\Psi_0^D)}{E_0+\omega_q-E_n+i\epsilon}.$$

From scattering theory, the transition matrix from the state q to the state n is given by

$$T_{nq}=(\Psi_n^{(-)},\ (V_{1q}^0+V_{2q}^0)\Psi_0^D). \tag{9}$$

In particular, if the state n consists of two protons with no real mesons (this diproton state will be denoted by $\Psi_0^{P(-)}$), we have the appropriate transition matrix for reaction (1) except that the initial and final states are reversed.

III. APPROXIMATION FOR THE MATRIX ELEMENT

For reasons of invariance,[10] if Ψ_j^A and Ψ_j^B are two different states of a *single physical* nucleon, then

$$(\Psi_j^B,V_{jq}^0\Psi_j^A)=(u_j^B,V_{jq}u_j^A), \tag{10}$$

where u_j^A and u_j^B are the corresponding states of the bare nucleon. The quantity V_{jq} has the same form as V_{jq}^0 except that the unrenormalized coupling constant f^0 is replaced by the renormalized coupling constant f.

As our first approximation, we make the assumption that Eq. (10) is valid even if the nucleon is bound. This is equivalent to saying that even in a deuteron, mesons which are emitted and absorbed by the same nucleon will contribute only to renormalization. As our second approximation, we make the assumption that the effect of the mesons which are exchanged between the two nucleons can be replaced by an effective potential between them. Gartenhaus[9] has in fact found a way to calculate approximately this effective potential from the cut-off theory.

With these two approximations, the matrix element can be written

$$\begin{aligned}T_{pq}&=(\Psi_0^{P(-)},\ (V_{1q}^0+V_{2q}^0)\Psi_0^D)\\&\simeq(\psi_P^{(-)},\ (V_{1q}+V_{2q})\psi_D),\end{aligned} \tag{11}$$

where $\psi_P^{(-)}$ and ψ_D are now the bare diproton and deuteron wave functions. These functions can be found by solving the Schroedinger equation using the effective nuclear potential.

It is possible to make a further simplification by considering the problem at an energy just above threshold for pion production. Then we can set $\exp(i\mathbf{q}\cdot\mathbf{r}_j)=1$ in the interaction, and V_{jq} becomes independent of spatial variables. With this restriction, the square of the matrix element, appropriately summed and averaged, reduces to the form previously found by Chew, Goldberger, Steinberger, and Yang[6]:

$$\begin{aligned}\sum|T_{pq}|^2=(4\pi)^2\frac{f^2}{\mu^2}\frac{v^2(q)}{\omega_q}q^2\{F_0^2+\sqrt{2}\cos(\delta_0-\delta_2)\\ \times F_0F_2(3\cos^2\theta-1)+\tfrac{1}{2}F_2^2(3\cos^2\theta+1)\},\end{aligned} \tag{12}$$

where

$$F_0=\int_0^\infty u_0(r)u_D(r)dr,\quad F_2=\int_0^\infty u_2(r)w_D(r)dr,$$

u_D and w_D being the radial parts of the deuteron S and D functions, and u_0 and u_2 being the radial diproton S and D functions. Here δ_0 and δ_2 are the S and D phase shifts for the diproton. The normalization for the deuteron and diproton wave functions is shown in Figs. 1 and 2.

[10] T. D. Lee, Phys. Rev. 95, 1329 (1954).

Gartenhaus, using his triplet potential, has computed the S and D functions for the deuteron. Using the Gartenhaus singlet potential, we have calculated the S and D diproton wave functions at the pion production threshold energy of 138 Mev in the center-of-mass system. The phase shifts turn out to be small and attractive ($\delta_0=3°$, $\delta_2=9°$).

The diproton and deuteron wave functions are not as accurate as one would hope. The reason is that the Gartenhaus potentials were calculated using the approximation that the kinetic energy of the nucleons is small compared to the pion mass. This condition is clearly not fulfilled for the diproton. The deuteron wave function also is affected, as can be seen by considering this wave function in momentum space representation. Then it is the high momentum components of the deuteron wave function which are the most inaccurate. But it is just these high momentum components which give the greatest contribution to the overlap integrals F_0 and F_2, since the diproton wave function consists chiefly of high momentum parts.

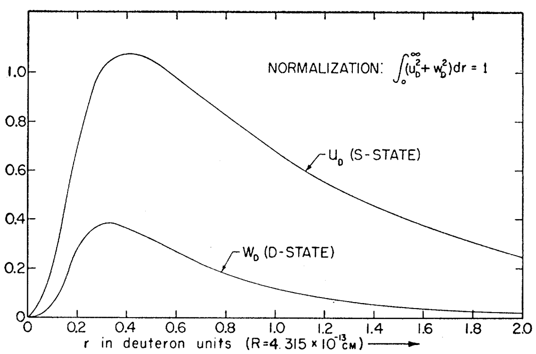

FIG. 1. Deuteron S- and D-state wave functions.

A mitigating circumstance is that the high momentum components of phenomenologically chosen wave functions are likely to be even more uncertain.

IV. RESULTS AND DISCUSSION

The cross section for reaction (1) can be expressed at low energy by the following semiempirical formula of Watson and Brueckner[4,11]:

$$4\pi(d\sigma/d\Omega)=aq+b[(c+\cos^2\theta)/(c+\tfrac{1}{3})]q^3. \tag{13}$$

The quantities a, b, and c are slowly varying functions of the energy and are usually taken as constant in the energy region under consideration. The term proportional to q is due to pions emitted in an s-state, and the term proportional to q^3 is due to p-wave pions. Crawford and Stevenson[12] compared formula (13) with experiment in the energy range from $q=0.38$ to $q=0.58$ (in units of the pion mass). Their values of the parameters a, b, and c together with our calculated values are given in Table I.

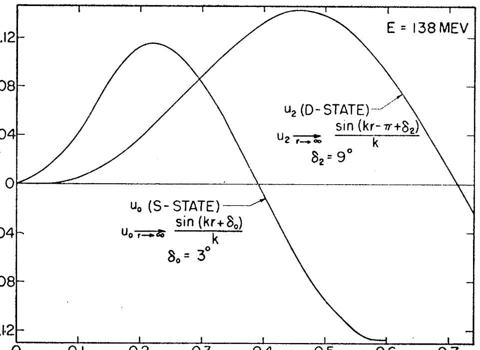

FIG. 2. Diproton S- and D-state wave functions.

The calculated constant $a=0$ because at threshold, the cutoff theory cannot lead to production of pions in the s-state unless nuclear recoil is taken into account. This has not been attempted, since it is not clear how to go about it within the framework of the cutoff theory. However, as has been pointed out by Crawford and Stevenson,[12] a simple theoretical estimate indicates that the ratio of s- to p-state production should be of the order of the ratio of the mass of the pion to the mass of the nucleon. Experiment indicates that this is indeed the case.

The calculated value of the p-wave part of the cross section depends upon the magnitude of the overlap integrals F_0 and F_2. These were found by a numerical integration using the wave functions shown in Fig. 1 and Fig. 2, with the result

$$F_0 \simeq 0, \quad F_2=0.08\mu^{-\frac{3}{2}}.$$

The value zero for the integral F_0 arises from the approximate cancellation of large positive and negative terms. In view of the approximations made in obtaining the wave functions, even the sign of F_0 is uncertain. The quantity F_2 is much more accurate, not only because there is little cancellation from negative terms, but also because the centrifugal barrier prevents the D-state wave functions from being too sensitive to the details of the potential.

Our result that the major part of the p-wave cross section can be accounted for by the contribution from

TABLE I. A comparison of experimental and calculated values of the parameters appearing in the semiempirical formula for the cross section for the reaction $p+p\rightarrow\pi^++d$.

Parameter	Crawford and Stevenson	Calculated
a (millibarns)	0.138±0.015	0
b (millibarns)	1.01 ±0.08	0.6
c	0.082±0.034	$\frac{1}{3}$

[11] This notation, slightly modified, is due to A. H. Rosenfeld, Phys. Rev. **96**, 139 (1954).
[12] F. S. Crawford and M. L. Stevenson, Phys. Rev. **97**, 1304 (1955).

F_2 differs from some previous analyses of the reaction[7] in which this contribution was neglected completely because of the deuteron's small D-state probability. The observed angular distribution was then interpreted on the basis of the strong pion-nucleon interaction in the state of isotopic spin $\frac{3}{2}$ and angular momentum $\frac{3}{2}$. The effect of the $(\frac{3}{2},\frac{3}{2})$ state interaction is not included in this calculation because we do not know how to take it into account consistently in the present formulation.

It should be noted that complete agreement with the p-wave part of the semiempirical formula would be obtained if F_2 were increased by about 20% and F_0 were made positive and about $\frac{1}{4}$ as large as F_2. In view of the approximation made in obtaining formula (12) for the matrix element, this seems as satisfactory a situation as one can expect. Further refinements of this calculation along the lines outlined in this work will depend upon better expressions for the physical deuteron and diproton wave functions.

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Total Cross Section of Hydrogen for 150- to 750-Mev Positive and Negative Pions*

S. J. LINDENBAUM AND LUKE C. L. YUAN
Brookhaven National Laboratory, Upton, New York
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The total cross section of hydrogen for positive and negative pions of 150–750 Mev has been determined.

An early preliminary survey using polyethylene-carbon differences is reported and indicated a resonance in the state of isotopic spin and angular momentum equal to $\frac{3}{2}$ (i.e., $T=J=\frac{3}{2}$).

A more recent precision determination (~3–5%) of the total hydrogen cross section for positive pions in the presumed resonance region (140–340 Mev) has been performed using liquid hydrogen transmission measurements.

A theoretical analysis of the results in conjunction with other available data strongly supports the hypothesis of a resonance. The behavior of the phase shift of the $T=J=\frac{3}{2}$ state is deduced and given. The most probable solution for the phase shift in this state is found to pass through 90° at ~175–180 Mev, although one cannot rule out the possibility of the resonance (90° phase shift) occurring up to 200 Mev.

The $T=\frac{1}{2}$ state cross section is found to be zero within the errors in the resonance region and to become appreciable only above 300 Mev.

The present experimental data and other relevant work are analyzed in terms of the recent Chew-Low theory and the Brueckner phenomenological resonance theory and is found to fit the general behavior predicted by both, which incidentally is shown to be of similar form.

I. INTRODUCTION

SINCE the original prediction by Yukawa of the meson as the particle responsible for the force field between nucleons, a considerable body of experimental evidence and theoretical speculations thereon relative to the nature of the pion-nucleon interaction have ensued. The early cosmic ray discovery of a meson[1] which is now known to be a moun favored the very weak coupling theories. Later the discovery of the pion[2] gave new impetus to the possibility of strong coupling. However, later experimental work[3] shed considerable doubt on whether the interaction was indeed strong.

The first clear picture of the nature of the pion-nucleon interaction began to emerge as a result of the study of the pion-hydrogen cross sections as a function of energy with the FM cyclotrons at Columbia and Chicago. In particular, the first really definitive work was performed by Anderson, Fermi *et al.*[4] in their measurement of the total hydrogen cross section for positive and negative pions from low energy to 135 and 200 Mev, respectively. The sharp rise of the cross sections with energy from very low values at low energies strongly suggested a gradient coupling with the P-wave predominantly responsible for the interaction.

The first successful quantitative attempt to explain the variation of the total cross sections with energy was proposed by Brueckner.[5] He employed the concept of a nucleon isobar of isotopic spin $T=\frac{3}{2}$ and angular momentum $J=\frac{3}{2}$ as first predicted for the pseudoscalar

* Work performed under the auspices of the U. S. Atomic Energy Commission.

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