# Momentum Dependence of Phase Shifts

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A lower limit is found for the momentum derivative of the scattering phase shifts of a relativistic neutral two-particle system when the interaction is of finite range.

SOME restrictions on the momentum dependence of the phase shifts for a nonrelativistic scattering problem have been discussed recently by Wigner.[1] In this note we derive a somewhat stronger restriction for the phase shifts for the relativistic scattering of neutral particles confined to a single channel. The latter condition essentially means that we are dealing with a first-quantized field.

The proof is based upon the unitarity of the partial $S$-matrix referring to a single phase shift and to the completeness, outside some radius $a_0$, of the asymptotic wave functions describing the scatterer and scattered particle in the center-of-mass system in a given state of angular momentum. We shall restrict our discussion to $S$-waves; analogous theorems are valid for other angular momenta.

For nonrelativistic $S$-wave scattering, unitarity and completeness have been shown[2,3] to lead to the conclusion that $S(k)e^{2iak}[a>a_0]$, if it has an analytic continuation,[4] is a regular function of $k$ for $\text{Im}\, k>0$ as long as no bound states are present. If there are no branch points, Ning Hu[3] has shown that one can therefore write

$$S(k)=e^{2i\delta(k)}=e^{-2iak}\prod_s\frac{k-k_s^*}{k-k_s}\frac{k+k_s}{k+k_s^*},\qquad(1)$$

$$a>a_0,\quad \text{Im}k_s<0.$$

The $k_s$ for different $s$ are not assumed to be necessarily distinct. The form of Eq. (1) is dictated by the absence of poles in the upper half plane and by the condition

$$S(k)=e^{2i\delta(k)}=e^{-2i\delta(-k)}=S^*(-k).\qquad(2)$$

For a relativistic particle the energy is a double-valued function, $E=\pm(k^2+\mu^2)^{\frac{1}{2}}$; in general the scattering amplitude depends both on $k$ and $E$ so that cuts must be introduced in the $k$-plane along the imaginary $k$ axis from $i\mu$ to $i\infty$ and $-i\mu$ to $-i\infty$. Following Hu,[3] we can write

$$S(k,E)=e^{-2iak}\prod_s\frac{(k_s+k_s^*)(E-E_s^*)-(E_s^*-E_s)(k-k_s^*)}{(k_s+k_s^*)(E-E_s)-(E_s-E_s^*)(k-k_s)}.\qquad(3)$$

$S(k)$ has a pole at $k=k_s$, $E=E_s=(k_s^2+\mu^2)^{\frac{1}{2}}$ but need not have a pole at $k=k_s$, $E=-E_s$; we have $S^*(k,E)=S(-k,E)$ in accord with unitarity but there is no simple relation between $S(k,E)$ and $S(k,-E)$. But for a real field (neutral particles only) $S(k,E)=S(k,-E)$, so that for every factor in Eq. (3) contributing a pole at $k_s$, $E_s$ there is one contributing a pole at $k_s$, $-E_s$. When the pair of factors with $E_s$ are multiplied, Eq. (3) reduces to the form of Eq. (1).

We may solve Eq. (1) for the phase shift and obtain

$$\delta(k)=-ak+\textstyle\sum_s \tan^{-1}\alpha_s(k)=-ak+\sum_s\theta_s(k),\qquad(4)$$

where we have introduced the angle

$$\theta_s(k)=\tan^{-1}\alpha_s(k),\qquad(5)$$

and the function $\alpha_s(k)$,

$$\alpha_s(k)=2k\,\text{Im}k_s/(k^2-|k_s|^2).\qquad(5')$$

The properties of the derivative of $S(k)$ with respect to $k$, denoted by a dot, are related to those of the derivatives of $\theta$ and $\alpha$,

$$\dot\delta=-a+\sum_s\dot\theta_s=-a+\sum_s\frac{\dot\alpha_s}{1+\alpha_s^2}.\qquad(6)$$

Because the poles of $S$ are all in the lower half-plane, however, $\dot\alpha$ is always positive,

$$\dot\alpha_s=(-2\,\text{Im}k_s)\frac{k^2+|k_s|^2}{(k^2-|k_s|^2)^2}>0,\qquad(7)$$

whence

$$\dot\delta>-a.\qquad(8)$$

Actually, the properties of $\alpha_s$ permit a still stronger statement on the lower limit of $\dot\delta$. By comparing Eqs. (5′) and (7), we may conclude that

$$\dot\alpha_s>|\alpha_s|/k.\qquad(9)$$

But this implies that $\theta_s$ has the lower limit

$$\dot\theta_s>\frac{1}{k}\frac{|\alpha_s|}{1+\alpha_s^2}=\frac{1}{2k}|\sin2\theta_s|\qquad(10)$$

and that, in turn, $\dot\delta$ has to be greater than $-a$ by a positive definite amount

$$\dot\delta>-a+\frac{1}{2k}\sum_s|\sin2\theta_s|\geq-a+\frac{1}{2k}|\sin2(\delta+ka)|.\qquad(11)$$

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[1] E. P. Wigner, Phys. Rev. **98**, 145 (1955).
[2] W. Heisenberg, Z. Naturforsch. **1**, 608 (1946).
[3] Ning Hu, Phys. Rev. **74**, 131 (1948).
[4] N. G. van Kampen, Phys. Rev. **89**, 1072 (1953) and Phys. Rev. **91**, 1267 (1953).

The last part of the inequality can be easily deduced by induction, starting with a sum over two terms,

$$|\sin(\theta_1+\theta_2)|=|\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2| \leq |\sin\theta_1||\cos\theta_2|+|\sin\theta_2||\cos\theta_1| \leq |\sin\theta_1|+|\sin\theta_2|. \quad (12)$$

Equation (11) above corresponds to Eq. (5a) of reference 1, but differs in that the oscillating term always makes a positive contribution.

When bound states are present $S(k)$ of Eq. (1) has simple poles in the upper half-plane at $k=ik_\lambda$ such that the energy of the bound state is $(\mu^2-k_\lambda^2)^{\frac{1}{2}}$. Then the *rhs* of Eq. (1) will have a factor

$$\prod_\lambda \frac{k+ik_\lambda}{k-ik_\lambda} \quad (13)$$

and Eq. (11) becomes

$$\dot{\delta}>-a+\frac{1}{2k}|\sin(2\delta+2ka)|-\sum_\lambda \frac{k_\lambda}{k^2+k_\lambda^2}. \quad (14)$$

Another inequality which can be stronger than Eq. (11) and is valid for real fields even if inelastic processes enter at higher $k$, follows from an application of Cauchy's theorem to $\exp[2i\delta(k)+2ika]$. Since the integrand has no poles in the finite half-plane if there are no bound states, Cauchy's theorem yields

$$e^{2i[\delta(k)+ka]}=\frac{P}{\pi i}\int_{-\infty}^{\infty}\frac{dt}{t-k}e^{2i[\delta(t)+ta]}. \quad (15)$$

By differentiating Eq. (15) logarithmically and using $\delta(k)=-\delta(-k)$, we obtain

$$\dot{\delta}(k)=-a+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dt\,\sin^2[\delta(t)-\delta(k)+at-ak]}{(t-k)^2} > -a. \quad (16)$$

The integrand of the right-hand side is always positive. Therefore a knowledge of $\delta(k)$, even over a limited region of $k$, contributes a lower limit to the entire integral in Eq. (16).[5]

For the partial $S$-matrix of Eq. (3), which describes the scattering of a charged relativistic particle, even the inequality (8) will not hold in general; it does not seem possible, therefore, to apply the inequalities (8), (12), and (16) to the scattering of charged mesons. Furthermore the possibility of charge exchange makes even Eq. (3) inadequate for the $S$-wave scattering of $\pi^0$ mesons by nucleons. Under the transformation $E\rightarrow -E$ the partial $S$-matrix $e^{2i\delta_3}$ for $\pi^++p\rightarrow\pi^++p$ becomes that for $\pi^-+p\rightarrow\pi^-+p$.[6] In a charge-independent theory, for instance, $\pi^-+p\rightarrow\pi^0+n$ can take place, $e^{2i\delta_3(-E)}$ cannot be unitary, and $\delta_3(-E)$ cannot be real.

Nevertheless some information about the range of interaction can be inferred for the meson-nucleon system. Completeness and unitarity give the relation

$$\int_{-\infty}^{\infty}dke^{ikr}S[k,+(k^2+\mu^2)^{\frac{1}{2}}]+\int_{-\infty}^{\infty}dke^{ikr}S[k,-(k^2+\mu^2)^{\frac{1}{2}}]=0,$$

where $S[k,+(k^2+\mu^2)^{\frac{1}{2}}]$ may be the partial $S$-matrix for $\pi^\pm+p\rightarrow\pi^\pm+p$ or $\pi^-+p\rightarrow\pi^0+n$. Charge independence relates the charge exchange amplitude to the two elastic amplitudes, and the $\pi^+$ amplitude becomes the complex conjugate of that for $\pi^-$ when the sign of the energy is changed. Completeness, unitarity, and charge independence lead to the single relation

$$2e^{2i\phi_3(k)}+e^{2i\phi_1(k)}=\frac{P}{\pi i}\int_{-\infty}^{\infty}\frac{2e^{2i\phi_3(t)}+e^{2i\phi_1(t)}}{t-k}dt, \quad (17)$$

where

$$\phi_3(k)=\delta_3(k)+ka \quad \text{and} \quad \phi_1(k)=\delta_1(k)+ka.$$

Differentiating both sides of Eq. (17) with respect to $k$ and then setting $k=0$, we obtain

$$2\dot{\delta}_3(0)+\dot{\delta}(0)+3a=\pi\int_{-\infty}^{\infty}\frac{2\sin^2\phi_3(t)+\sin^2\phi_1(t)}{t^2}dt. \quad (18)$$

With Orear's extrapolated phase shifts,

$$\delta_3=-0.11\frac{k}{\mu c} \quad \text{and} \quad \delta_1=+0.16\frac{k}{\mu c}, \quad (19)$$

the complete neglect of the positive definite integral yields $a\geq 0.02(\hbar/\mu c)$.

A stronger limit on $a$ is easily obtained if we use a linear approximation to the $S$-wave phase shifts.[7] The integration in Eq. (18) can be carried out, with the result

$$2\phi_3(0)+\phi_1(0)=2|\phi_3(0)|+|\phi_1(0)|. \quad (20)$$

From this we can infer

$$a\geq 0.11(\hbar/\mu c). \quad (21)$$

[5] If multiple processes occur at momenta greater than $k$, Eq (16) still holds with the $\sin^2(\ )$ of the integrand replaced by $e^{-2\delta_2}\sin^2(\ )+e^{-\delta_2}\sinh(\delta_2)$, where $\delta_2$ is the imaginary part of $\delta$.

[6] M. Goldberger, Phys. Rev. **99**, 979 (1955). We are indebted to Professor Goldberger for furnishing us a copy of his paper before publication.

[7] J. Orear, Phys. Rev. **96**, 176 (1954).