Relativistic Theory of Discrete Momentum Space and Discrete Space-Time

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An examination is made of the result of restricting translations and Lorentz transformations in space-time to those with rational coefficients. This removes the major defect of Schild's model of discrete space-time by elimination of the lower bound on the relative velocities of reference systems. The theory is formulated in two stages. In the first the energy and momentum components of a particle are restricted to a countable set satisfying the relativistic energy-momentum relation while the space-time variables are continuous. This gives a theory of discrete energy-momentum space in which wave functions are almost periodic functions. In the second stage the space-time variables are restricted to rational values. This leads to the theory of discrete space-time.

1. INTRODUCTION

It has been conjectured frequently that a treatment of space and time as having a discrete structure would be of interest as a test of the character of physical theories.[1] Heisenberg's suggestion of the existence of a minimal space length has aroused particular interest among physicists.[2] A number of trials have been made at the reformulation of quantum mechanical theory by assumptions depending on the nature of the space and time variables considered as operators.[3]

One of the major difficulties encountered arises from the supposed necessity of preserving invariance of the theory under the full continuous group of Lorentz transformations, L. Some years ago Schild examined the consequences of assuming a lattice structure of space-time, requiring only the discrete group of Lorentz transformations under which the lattice as a whole was invariant.[4] For the case of a cubic lattice he found the smallest permissible velocity parameter to be $v/c=\frac{1}{2}\sqrt{3}$, which is impossibly large to make the model of use for physical purposes. There is little doubt that other space-time lattices suffer from the same defect.

We assume the character of a space to be determined by its allowed group of symmetry transformations in the same sense as was used in Schild's model. The property of discreteness can be achieved by other methods of selecting sets of transformations from L. The principal requirements which it seems necessary to impose in order to make any such set a reasonable starting point for a physical theory of space-time are the following: (a) the set must form a group in order to have closure under successive transformations and symmetry with respect to the senses of backward and forward, and (b) it must be dense in the full group L. To form a discrete subgroup of L is easy. One has only to take any finite or countably infinite complex K from L which contains the inverse of every element in it, and then form all possible products from the elements in K. The resulting extended complex will be a subgroup of L which will be at most countably infinite. However, when the complex K is chosen in an arbitrary manner it will be generally quite difficult to test whether its extension will be dense in L.

The present work starts with the selection of transformations from L which have *rational* coefficients. In this case the necessary analysis reduces to certain problems in the theory of numbers. The general properties of the resulting infinite discrete group, to which we shall refer as the rational proper Lorentz group, L_r, are developed briefly in the Appendix. The group L_r is dense in L and so is sufficiently extensive to remove doubt that it can be made the basis of a physical theory. It thus avoids the principal weakness of Schild's model in preserving the property of discreteness without setting lower bounds on the parameters of its transformations.

The theory is developed along two lines. In the first the space-time variables are allowed to be continuous but the energy and momentum variables of a particle are restricted to a certain countable set which is described later (Sec. 2). We refer to this as the theory of discrete energy-momentum space. The major result of quantum mechanical interest is that wave functions here become a special case of almost periodic functions. This interpretation provides an approach to the handling of continuous spectrum problems which, in principle at least, is more in consonance with current quantum mechanical theory in which strict normalization of the wave functions is not required than is the Hilbert space theory of von Neumann.[5] However, it does not appear that it will lead to any great improvement in the handling of convergence problems in perturbation theory without a revision in the nature of interaction operators.

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[1] An interesting discussion is given by B. Russell, *The Analysis of Matter* (Dover Publications, New York, 1954), especially Chaps. 11 and 29.

[2] W. Heisenberg, Z. Physik **120**, 513 (1943); W. Heisenberg, *Festschrift der Akad. Wissen. Göttingen* (Springer-Verlag, Berlin, 1951), p. 50.

[3] H. S. Snyder, Phys. Rev. **71**, 38 (1947); **72**, 68 (1947); V. Rojansky, Phys. Rev. **97**, 507 (1955).

[4] A. Schild, Can. J. Math. **1**, 29 (1949). The problem was under independent investigation in this Laboratory by C. N. Kelber at the time of the appearance of Schild's paper. The writer is indebted to Dr. Kelber for discussions of his work.

[5] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955).

In the second form of the theory the property of discreteness is extended to space-time. One has the choice of treating the energy-momentum variables as discrete or continuous. This leads to alternative interpretations of wave functions of which the implications have not been analyzed fully.

The major part of the discussion will be devoted to the formulation of the models of discrete energy-momentum space and discrete space-time. The treatment of wave functions will be incomplete owing to the fact that a suitable reformulation of the quantum mechanical operator theory has not yet been carried through.

2. MODEL OF DISCRETE ENERGY-MOMENTUM SPACE

The energy and momentum of a particle are invariant under all translations of the reference system in space-time. The translations are therefore equivalent to the identity for the discussion of this section and so can be omitted.

We take as the coordinates of the energy-momentum space of a particle the four quantities

$$\pi_0 = E/m_0c^2, \quad \pi_i = p_i/m_0c. \quad (i=1, 2, 3) \tag{1}$$

Under all transformations of the full continuous group of proper Lorentz transformations, L, these quantities transform like the components of a real four-vector with the invariant (energy-momentum equation of the particle)

$$\pi_0^2 - \pi_1^2 - \pi_2^2 - \pi_3^2 = 1. \tag{2}$$

More particularly, under the discrete group of rational Lorentz transformations, L_r, the set of points in energy-momentum space having *rational* coordinates which satisfy Eq. (2) will be invariant. These points constitute our discrete energy-momentum space.

The possible values of the energy and momentum of a particle which satisfy these restrictions can be characterized in the following manner. We consider only positive values of the energy for definiteness. Let

$$\pi_0 = m/n, \quad \pi_i = m_i/n_i. \quad (i=1, 2, 3) \tag{3}$$

These rational fractions are supposed to be reduced to their lowest terms so that[6]

$$(m,n)=1, \quad (m_i,n_i)=1. \tag{4}$$

It is clear that $m \geqslant n$. Since the choice $m=n$ leads to the obviously acceptable energy value $E=m_0c^2$, it will be convenient in the following discussion to make the restriction $m>n$.

Equation (2) hence takes the form

$$(m/n)^2 - (m_1/n_1)^2 - (m_2/n_2)^2 - (m_3/n_3)^2 = 1. \tag{5}$$

If $\{n, n_1, n_2, n_3\} = sn = s_1n_1 = s_2n_2 = s_3n_3$ is the least common multiple of the n's, we can write (5) in the form

$$s^2(m^2-n^2) = (s_1m_1)^2 + (s_2m_2)^2 + (s_3m_3)^2. \tag{6}$$

Every solution of this Diophantine equation in integers leads to a solution of Eq. (2) in rationals, and conversely.

Equation (6) shows that the positive integer $s^2(m^2-n^2)$ is expressible as the sum of three squares of integers. It is known that an integer is expressible in this manner if and only if it is not of the form $4^a(8b+7)$, where a and b are non-negative integers.[7] To make use of this result in connection with Eq. (6) we employ the following lemma.[8]

Lemma.—The positive integer $s^2(m^2-n^2)$ with $(m,n)=1$ will be of the form $4^a(8b+7)$ if and only if $(m^2-n^2) \equiv 7 \pmod 8$.

It follows immediately that the rational values $E/m_0c^2 = m/n$, with $(m,n)=1$, will be allowed energy values if and only if $(m^2-n^2) \not\equiv 7 \pmod 8$; that is, if and only if (m^2-n^2) is not an integer of the form $8b+7$. For example, the value 4/3 is not allowed since $4^2-3^2=7$.

One can show easily that the allowed values of energy are dense in the range $E/m_0c^2 \geqslant 1$. For since the rational numbers are dense in this range we have only to show that the allowed values of E/m_0c^2, all of which are rational, are dense among the rationals. Let x/y be any positive rational number such that $x>y$ and $(x,y)=1$. Then if r is any positive integral multiple of 8, so that $r \equiv 0 \pmod 8$, the rational number $(rx+3)/(ry+2)$ will be an allowed value of E/m_0c^2, since $[(rx+3)^2-(ry+2)^2] \equiv 5 \pmod 8$.[9] By taking a sufficiently large value of r this number can be made to approximate x/y with arbitrary accuracy. Similarly, the numbers $(rx+4)/(ry+3)$ will be unallowed values, from which we can conclude that the rational numbers which are unallowed for E/m_0c^2 are also dense in the open interval $(1, \infty)$.

For any allowed value of E/m_0c^2 other than unity there will be infinitely many rational solutions of Eq. (2) for the components of the momentum vector, $\mathbf{p}/m_0c$.[10] The allowed momentum vectors will form an infinite countable set which is dense in direction and magnitude in space. No convenient parametrization has been found for this set. In the following discussion the allowed energy values will be designated as E_α and the allowed momentum vectors by $\mathbf{p}_{\alpha\beta}$ with $\alpha, \beta = 1, 2, 3 \cdots$.

[6] We use the notation given in the text by G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Oxford University Press, Oxford, 1945), second edition. The greatest common divisor of two integers m and n is indicated by (m,n) and the least common multiple by $\{m,n\}$.

[7] See reference 6, Sec. 20.10.

[8] Write s in the form $s=2^p r$, where r is odd. Since the square of any odd integer is of the form $8t+1$, one has $s^2=4^p(8t+1)$. Since $(m,n)=1$, m^2-n^2 can at most be of the form $8q+7$. Suppose $m^2-n^2=8q+7$, then it is apparent at once that $s^2(m^2-n^2) = 4^p(8b+7)$ with $b=8tq+7t+q$, which proves the first part of the lemma. Conversely, if $s^2(m^2-n^2)=4^p(8t+1)(m^2-n^2)=4^a(8b+7)$ then we must have $a=p$, so that $(8t+1)(m^2-n^2)=8b+7$. But in any event $(m^2-n^2)=8r+g$, where $0 \leqslant g \leqslant 7$. It is apparent by inspection that the only possible choice is $g=7$, which proves the second part of the lemma.

[9] No integer of the form $4^a(8b+7)$ is congruent to 5 (mod 8).

[10] These solutions are to be obtained from Eq. (6) by specifying the integers m and n, with $(m,n)=1$, and then taking all cases $s=1, 2, 3, \cdots$. Almost all of the solutions will arise from very large values of s.

The usual representation of the wave function of a particle in terms of a Fourier integral must now be replaced by a sum over the allowed states.

$$\psi(x,y,z,t)=\sum_\alpha \sum_\beta A_{\alpha\beta} \exp[i(\mathbf{p}_{\alpha\beta}\cdot\mathbf{r}-E_\alpha t)/\hbar]$$
$$=\sum_\alpha \sum_\beta A_{\alpha\beta} \exp[i(m_0c/\hbar)(\boldsymbol{\pi}\cdot\mathbf{r}-\pi_0 ct)]. \quad (7)$$

The space-time coordinates will be considered to be continuous variables and no restriction will be placed on the numerical value of the Compton wavelength.

Functions of the form (7), with appropriate convergence requirements, belong to the special class of almost periodic functions known as limit periodic functions.[11] They are the natural generalization of the very special periodic functions which appear in the method of "quantization in a box," or that of periodic boundary conditions. They are very different from the functions which are representable by Fourier integrals. This is shown by the fact that no almost periodic function can vanish at infinity without being identically zero. The sums in (7) are free from the measure restrictions associated with Fourier integrals, but of course they are not normalizable in the usual sense.

The coefficients in the expansion (7) can be evaluated by the method of mean values, in the sense of an initial value problem. If $M\{F(x,y,z)\}$ is the mean value of the function F taken over all space, and if $u(x,y,z)$ is the form of the wave function at $t=0$, then[12]

$$A_{\alpha\beta}=M\{u(x,y,z)\exp[-i(\mathbf{p}_{\alpha\beta}\cdot\mathbf{r})/\hbar]\}. \quad (8)$$

The wave functions (7) will be solutions of the Klein-Gordon equation, of course, but an exactly similar procedure can be used for the construction of free particle solutions of the Dirac equation and the other equations of quantum mechanics.

The manner in which these wave functions are to be used for the calculation of interactions between particles and fields presents some questions of principle which have not been solved. It does not appear likely that convergence difficulties will be lessened appreciably so long as the usual differential interaction operators are allowed. The situation may be improved if it is found possible to revise the operators in such a manner that they are given, or at least are approximated, by matrices having nonvanishing elements only connecting the allowed states in the momentum representation. The results which have been obtained to date are too scanty to justify their further discussion here.

3. MODEL OF DISCRETE SPACE-TIME

Under the group L_r the set of points of space-time (coordinates $x^0=ct$, $x^1=x$, $x^2=y$, $x^3=z$) having rational coordinates will be invariant. This property will be preserved under translations provided the components of the translation vectors are restricted to rational values; these will be referred to as rational translations.

[11] A. S. Besicovitch, *Almost Periodic Functions* (Dover Publications, New York, 1954), p. 32. The theory can be extended to functions of any number of variables.

[12] See reference 11, p. 12.

This allows us to define the rational translation group T_r in space-time as the family of all rational translations. Clearly there is in it no smallest step in any of the variables.

The group combination laws for successive applications of transformations of L_r and T_r will be valid and any transformation generated in this manner will leave the set of points in space-time having rational coordinates invariant. This provides our model of discrete space-time.

The construction of wave functions in this case depends on the interpretation assigned to energy-momentum space. If the energy-momentum variables are allowed to be continuous then Fourier integrals can be formed in the usual manner, but the symmetry between energy-momentum and space-time wave functions will be lost. If both spaces are taken to be discrete the situation becomes more extreme. The mathematical implications of these methods have not been examined in sufficient detail to permit their application to physical problems.

4. CONCLUSION

The foregoing theory has been designed mainly to show the possibility of constructing a self-consistent theory of discrete energy-momentum space and discrete space-time without doing utter violence to the invariance under Lorentz transformations. The extensiveness of the group of allowed transformations is so great as to leave no room for direct experimental disproof of the theory. However, this does not mean that there may not be some consequences which will be subject to experimental verification.

Since the group L_r is embedded in the full continuous group L and is dense in it, L_r will have most of the properties of L which are used in physical theory. In particular, the matrix representations of L which are now used will still be applicable. However, the discrete group L_r may well allow the existence of representations which cannot be extended to the full group. This suggests the following point which might ultimately become of some importance as a possible means of experimental test. The restriction of the quantized values of angular momentum to those employed in the usual vector model is dependent on the assumption of continuity of the matrix representations of the 3-dimensional rotation group. These representations will still be available for L_r, with a slight reinterpretation to take account of the discreteness of L_r. But if L_r admits of other representations for its subgroup of rational rotations then the vector coupling model may no longer be applicable. This seems to be a very delicate question which the writer has not succeeded in working out as yet.

APPENDIX A. RATIONAL HOMOGENEOUS PROPER LORENTZ GROUP, L_r

The continuous group of proper homogeneous Lorentz transformations, L, is given by the set of real 4×4 matrices $l=[l_j{}^i]$

$(i,j=0,1,2,3)$ subject to the conditions[13]

$$l_0^0 \geqslant 1, \quad \det(l)=+1, \quad l^{-1}=\eta l^T \eta, \tag{A-1}$$

where l^T is the transpose of l and

$$\eta=\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{A-2}$$

It is evident from an inspection of these conditions that they will all be satisfied, as will the law of group multiplication, if we restrict the matrices to those having rational numbers as elements. This set of matrices forms a subgroup of L to which we shall refer as the *rotational Lorentz group*, L_r, as stated in the text, and which forms the basis of the theory given in this paper. The separation of the transformations of L into rotations and special Lorentz transformations holds as it does for L simply because L_r is embedded in L.

The writer has been unable to give a direct algebraic proof that L_r is dense in L. To avoid this we show in the next three parts of the Appendix that L_r contains a subgroup which is dense in L. While this is sufficient to show that L_r itself is dense in L, it leaves open the question whether the Euler factorization process is valid for L_r.

APPENDIX B. RATIONAL ROTATION SUBGROUP OF L_r

If $R(\mathbf{n};\lambda)$ is a rotation through an angle λ about an axis in space specified by the unit vector $\mathbf{n}$, the associated coordinate transformations on the space-time variables are

$$\mathbf{r}'=\mathbf{r}\cos\lambda+\mathbf{n}(\mathbf{n}\cdot\mathbf{r})2\sin^2\lambda/2+(\mathbf{n}\times\mathbf{r})\sin\lambda, \quad t'=t. \tag{B-1}$$

Examination of the coefficients in these equations shows that a necessary and sufficient condition that they be rational is that the quantities $\cos\lambda$ and $n_i \sin\lambda$ $(i=1,2,3)$ be rational. The search for the set of rational rotations can be reduced to the solution of Eq. (6). For, if we set

$$\cos\lambda=p/q, \quad n_i \sin\lambda=p_i/q_i, \tag{B-2}$$

where these rational fractions are reduced to their lowest terms, we must have

$$(p/q)^2+(p_1/q_1)^2+(p_2/q_2)^2+(p_3/q_3)^2=1. \tag{B-3}$$

With $\{q,q_1,q_2,q_3\}=rq=r_1q_1=r_2q_2=r_3q_3$ as the least common multiple of the q's, we can write

$$r^2(q^2-p^2)=(r_1p_1)^2+(r_2p_2)^2+(r_3p_3)^2, \tag{B-4}$$

which is equivalent to Eq. (6) of the text. There is thus a simple one-to-one correspondence between the allowed points of the discrete energy-momentum space and the rational rotation subgroup of L_r.

For the special case of a rotation through an angle θ about one of the coordinate axes $\cos\theta$ and $\sin\theta$ must be rational. In this case the allowed rotation angles are easily determined. Since $\cos^2\theta+\sin^2\theta=1$, we must solve this Diophantine equation in rational numbers. The general solution is[14]

$$|\cos\theta(t)|=\frac{1-t^2}{1+t^2}, \quad |\sin\theta(t)|=\frac{2t}{1+t^2}, \tag{B-5}$$

where t is an arbitrary rational number such that $0\leqslant t\leqslant 1$. An alternative formulation is $|\tan(\theta/2)|=t$.

Since the values of t are dense in $[0,1]$ and since relations $(B-5)$ are continuous in this interval, the allowed rotation angles will be dense in the interval $[0,2\pi]$. We can make the usual identification $R(\mathbf{n};\pi+\theta)=R(-\mathbf{n};\pi-\theta)$.

APPENDIX C. RATIONAL SPECIAL LORENTZ TRANSFORMATIONS OF L_r

The special Lorentz transformations to moving reference systems, the direction and speed of the relative motion being arbitrary, can be written in the form

$$\mathbf{r}'=\mathbf{r}-\frac{\mathbf{v}}{c}\left\{\frac{\mathbf{r}\cdot\mathbf{v}/c}{v^2/c^2}\left[1-\frac{1}{(1-v^2/c^2)^{\frac{1}{2}}}\right]+\frac{ct}{(1-v^2/c^2)^{\frac{1}{2}}}\right\}$$
$$ct'=\frac{ct-\mathbf{r}\cdot\mathbf{v}/c}{(1-v^2/c^2)^{\frac{1}{2}}}. \tag{C-1}$$

These transformations do not form a group by themselves, of course, but by combination with the three-dimensional rotations the full group L can be generated.

Inspection of the coefficients shows that a necessary and sufficient condition that they be rational is that $(1-v^2/c^2)^{\frac{1}{2}}$ and v_i/c $(i=1,2,3)$ be rational. From this one can reduce the analysis of the rational transformations of type (C-1) to that of Eq. (6).

Again restricting attention to motion along one of the coordinate axes it is easy to find the allowed values of the relative velocity. With the notation

$$a=\frac{1}{(1-v^2/c^2)^{\frac{1}{2}}}, \quad b=\frac{v/c}{(1-v^2/c^2)^{\frac{1}{2}}}, \tag{C-2}$$

these quantities must be rational. Since $a^2-b^2=1$ we need the general solution in rationals of this Diophantine equation. It is[14]

$$a=(r^2+1)/2r, \quad |b|=(r^2-1)/2r, \tag{C-3}$$

where r is any rational number $r\geqslant 1$. Alternatively, the allowed values of v/c are given by $|v/c|=(r^2-1)/(r^2+1)$. These values are dense in the open interval $(-1, 1)$.

APPENDIX D. DENSENESS OF L_r IN L

Let L_x', L_y', L_z' represent special Lorentz transformations of type (C-1) along the coordinate axes, and R_x', R_y', R_z' be rotations about the same axes. The Euler factorization process for the full group L shows that any transformation in it can be factorized as a product of the form $R_z'R_y'R_x'L_z'L_y'L_x'$.[15] Furthermore, the resultant transformation is a continuous function of each of the transformations in the product.

It is not known to the writer whether the group of rational transformations L_r admits this same type of factorization. In any event, starting with the allowed rational transformations of the indicated special forms as a complex of L_r we can form all possible products. In this way we obtain a countably infinite subgroup of L_r, which may or may not be a proper subgroup but which we argue is dense in L. For taking any transformation of L and writing it as this type of product we can approximate each of the separate transformations by a corresponding rational transformation to any desired degree of accuracy. The continuity of the product in each of the multiplicands assures us that the final product transformation (which is in L) can be approximated arbitrarily closely by a product of transformations from L_r. Hence L_r must be dense in L.

[13] F. D. Murnaghan, *The Theory of Group Representations* (Johns Hopkins University Press, Baltimore, 1938), Chap. 12.

[14] Hardy and Wright (reference 6, Sec. 13.2) give the solution of the equation $a^2+b^2=c^2$ in integers from which our result can be found. The solution in rationals is given directly by O. Ore, *Number Theory and its History* (McGraw-Hill Book Company, Inc., New York, 1948), p. 169.

[15] See reference 13, p. 357.