Tensor Scattering Matrix for the Electromagnetic Field*

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The scattering of an arbitrary incoming electromagnetic wave by an unrestricted scattering object is described in terms of a tensor scattering matrix. General reciprocity relations and the cross-section theorem, including an interesting extension, are established using this representation. The results are related to the special case of plane-wave scattering and the scattering matrix is explicitly exhibited in terms of the plane-wave scattering amplitude for two mutually perpendicular directions of polarization.

GENERAL FORMULATION

IN the following note, we delineate those general features of the scattering of electromagnetic waves which are consequences of the asymptotic properties of Maxwell's equations but are independent of the nature of the scatterer.[1] For this purpose, it is convenient to describe the scattering by introducing a tensor scattering matrix. Using this representation, reciprocity and the cross-section theorem, including an interesting extension, are easily established as we now proceed to show.[2]

Outside the scattering region, and in particular as $r\rightarrow\infty$, the fields satisfy the free-space Maxwell's equations, which we write (in Gaussian units) in the form

$$\nabla\times\mathbf{E}=ik\mathbf{H},\quad \nabla\times\mathbf{H}=-ik\mathbf{E}, \tag{1}$$

where we have assumed harmonic time dependence $e^{-i\omega t}$ and where k is the free-space wave number. We now decompose the asymptotic field $\mathbf{E}(\mathbf{n}r)$, $r\rightarrow\infty$ into incoming and outgoing waves along $\mathbf{n}$; that is, we write

$$\mathbf{E}(\mathbf{n}r)=\mathbf{F}_1(\mathbf{n})\frac{e^{-ikr}}{r}+\mathbf{F}_2(\mathbf{n})\frac{e^{ikr}}{r}, \tag{2}$$

where, as a consequence of the divergence condition,

$$\mathbf{F}_1(\mathbf{n})\cdot\mathbf{n}=\mathbf{F}_2(\mathbf{n})\cdot\mathbf{n}=0. \tag{3}$$

In the above and throughout, we neglect terms of higher order than the first in reciprocal powers of r; we deal only with pure radiation fields.

As a consequence of the linearity of Maxwell's equations, the outgoing fields must be linearly related to the incoming fields which we assume to be arbitrary. Thus, we introduce a linear connective $\mathcal{S}(\mathbf{n},\mathbf{n}')$, called the tensor scattering matrix, by writing

$$\mathbf{F}_2(\mathbf{n})=-\int d\Omega'\mathcal{S}(\mathbf{n},\mathbf{n}')\cdot\mathbf{F}_1(-\mathbf{n}'), \tag{4}$$

where the choice of signs is such that $\mathcal{S}$ reduces to the unit operator when there is no scattering. As a consequence of Eq. (3)

$$\mathbf{n}\cdot\mathcal{S}(\mathbf{n},\mathbf{n}')=\mathcal{S}(\mathbf{n},\mathbf{n}')\cdot\mathbf{n}'=0 \tag{5}$$

so that $\mathcal{S}$ has only four independent components.[3]

We first establish reciprocity relations for the electromagnetic field starting from the fact that if $\mathbf{E}$ and $\mathbf{E}'$ are any two solutions of the source-free Maxwell's equations (but with the same harmonic time-dependence), then

$$\int_{r\rightarrow\infty} d\Omega r^2\mathbf{n}\cdot\{\mathbf{E}'(\mathbf{n}r)\times[\nabla\times\mathbf{E}(\mathbf{n}r)] -\mathbf{E}(\mathbf{n}r)\times[\nabla\times\mathbf{E}'(\mathbf{n}r)]\}=0, \tag{6}$$

provided that the dielectric, permeability, and conductivity tensors are symmetric. The proof of this relation is straightforward. Under the stated conditions it is easily established from Maxwell's equations that $\nabla\cdot(\mathbf{E}'\times\mathbf{H}-\mathbf{E}\times\mathbf{H}')$ vanishes identically. Integration over all space, followed by an application of Gauss's theorem, then yields Eq. (6) for there are no sources except at infinity and asymptotically the fields satisfy Eq. (1). (It is of interest to remark that the symmetry of the constitutive tensors seems to be a necessary condition for reciprocity.)

Decomposing $\mathbf{E}'$ as well as $\mathbf{E}$ in Eq. (6) into incoming and outgoing waves along $\mathbf{n}$, we obtain the following after some algebra:

$$\int d\Omega[\mathbf{F}_1'(\mathbf{n})\cdot\mathbf{F}_2(\mathbf{n})-\mathbf{F}_1(\mathbf{n})\cdot\mathbf{F}_2'(\mathbf{n})]=0,$$

where the prime denotes the amplitudes associated with $\mathbf{E}'$. Expressing the outgoing fields in terms of the

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[1] We assume, however, that the scattering region consists of a linear medium which contains no sources and does not extend to infinity. We do not assume that the scatterer is homogeneous or isotropic or lossless.

[2] See E. Gerjuoy and D. S. Saxon, Phys. Rev. 94, 1445 (1954), for a similar discussion of scattering without absorption for scalar fields. In the present paper, the development is generalized to the vector field, including absorption.

[3] Since $\mathbf{F}_1$ is transverse, strictly speaking $\mathcal{S}(\mathbf{n},\mathbf{n}')\cdot\mathbf{n}'$ is undefined according to Eq. (4). However, we complete the definition by taking it to be zero.

incoming by Eq. (4), we then have

$$\int d\Omega \int d\Omega' [\mathbf{F}_1'(\mathbf{n})\cdot \mathcal{S}(\mathbf{n},\mathbf{n}')\cdot \mathbf{F}_1(-\mathbf{n}') - \mathbf{F}_1(\mathbf{n})\cdot \mathcal{S}(\mathbf{n},\mathbf{n}')\cdot \mathbf{F}_1'(-\mathbf{n}')]=0.$$

Replacing $\mathbf{n}$ by $-\mathbf{n}'$ and $\mathbf{n}'$ by $-\mathbf{n}$ in the last term and transposing the product, this becomes

$$\int d\Omega \int d\Omega' \{\mathbf{F}_1'(\mathbf{n})\cdot [\mathcal{S}(\mathbf{n},\mathbf{n}') - \mathcal{S}^T(-\mathbf{n}',\, -\mathbf{n})]\cdot \mathbf{F}_1(-\mathbf{n}')\}=0,$$

where the superscript T signifies the transposed tensor:

$$(\mathcal{S}^T)_{ij}=\mathcal{S}_{ji}.$$

Since $\mathbf{F}_1$ and $\mathbf{F}_1'$ are arbitrary, we thus have

$$\mathcal{S}(\mathbf{n},\mathbf{n}')=\mathcal{S}^T(-\mathbf{n}',\, -\mathbf{n}), \tag{7}$$

and this is the reciprocity condition in its general form. Later we exhibit it in more familiar form when we apply our results to plane wave scattering.

Next consider the energy relations which hold for the field. The time-average energy flux inward through the sphere at infinity must of course be equal to the average power P absorbed by the scatterer, and hence

$$-\frac{c}{8\pi}\mathrm{Re}\left[\int_{r\to\infty} (\mathbf{E}\times\mathbf{H}^*)\cdot \mathbf{n} r^2 d\Omega\right]=P.$$

Expressing $\mathbf{H}$ in terms of $\mathbf{E}$ by the first Maxwell equation, we then find using Eq. (2) and Eq. (3)

$$\frac{c}{8\pi}\int [\mathbf{F}_2^*(\mathbf{n})\cdot \mathbf{F}_2(\mathbf{n})-\mathbf{F}_1^*(\mathbf{n})\cdot \mathbf{F}_1(\mathbf{n})]d\Omega+P=0.$$

Upon substitution of Eq. (4), we obtain, after appropriate transpositions and relabeling of the integration variables,

$$\frac{c}{8\pi}\int d\Omega' \mathbf{F}_1^*(-\mathbf{n}')\cdot\left[\int \mathcal{Q}(\mathbf{n}',\mathbf{n}'')\cdot \mathbf{F}_1(-\mathbf{n}'')d\Omega'' - \mathbf{F}_1(-\mathbf{n}')\right]+P=0, \tag{8}$$

where

$$\mathcal{Q}(\mathbf{n}',\mathbf{n}'')=\int d\Omega \mathcal{S}^{*T}(\mathbf{n},\mathbf{n}')\cdot \mathcal{S}(\mathbf{n},\mathbf{n}''). \tag{9}$$

From Eq. (8) we see that the energy loss is determined in terms of the incoming field alone and can itself be characterized by a tensor $\mathcal{K}(\mathbf{n}',\mathbf{n}'')$ defined by the relation

$$P\equiv\frac{c}{8\pi}\int\int d\Omega d\Omega' \mathbf{F}_1^*(-\mathbf{n}')\cdot \mathcal{K}(\mathbf{n}',\mathbf{n}'')\cdot \mathbf{F}_1(-\mathbf{n}''). \tag{10}$$

Actually, only the transverse parts of $\mathcal{K}$ are defined by this relation, and we complete the definition by the additional conditions:

$$\mathbf{n}'\cdot \mathcal{K}(\mathbf{n}',\mathbf{n}'')=\mathcal{K}(\mathbf{n}',\mathbf{n}'')\cdot \mathbf{n}''=0. \tag{11}$$

From the fact that P is real, it follows at once from Eq. (10) that $\mathcal{K}$ is Hermitian, $\mathcal{K}^{*T}(\mathbf{n}'',\mathbf{n}')=\mathcal{K}(\mathbf{n}',\mathbf{n}'')$. If we regard $\mathcal{K}$ as a continuous supermatrix, this can be written in matrix notation as $\mathcal{K}^\dagger=\mathcal{K}$, where the adjoint symbol means to take the complex conjugate and to transpose all indices, continuous and discrete.

Equation (8) can now be rewritten in the form

$$\int\int d\Omega' d\Omega'' \mathbf{F}_1^*(-\mathbf{n}')\cdot [\mathcal{Q}(\mathbf{n}',\mathbf{n}'')-\epsilon\delta(\mathbf{n}'-\mathbf{n}'') +\mathcal{K}(\mathbf{n}',\mathbf{n}'')]\cdot \mathbf{F}_1(-\mathbf{n}'')=0.$$

Now $\mathbf{F}_1$ is arbitrary, except that it must remain transverse, and hence we infer that the transverse part of the bracketed expression vanishes. Of course $\mathcal{Q}$ and $\mathcal{K}$ are already transverse and we need only extract the transverse part of the unit tensor. Denoting this by $\epsilon_T(\mathbf{n})$, we evidently have

$$\epsilon_T(\mathbf{n})=\epsilon-\mathbf{nn}, \tag{12}$$

since $\mathbf{n}\cdot\epsilon_T(\mathbf{n})=\epsilon_T(\mathbf{n})\cdot\mathbf{n}=0$. On the other hand, if $\mathbf{B}$ is any vector perpendicular to $\mathbf{n}$, then $\mathbf{B}\cdot\epsilon_T(\mathbf{n})=\epsilon_T(\mathbf{n})\cdot\mathbf{B}=\mathbf{B}$ so that ϵ_T maintains its character as a unit tensor with respect to such vectors. In any case, using Eq. (9), we have

$$\int d\Omega \mathcal{S}^{*T}(\mathbf{n},\mathbf{n}')\cdot \mathcal{S}(\mathbf{n},\mathbf{n}'') =\epsilon_T(\mathbf{n}')\delta(\mathbf{n}'-\mathbf{n}'')-\mathcal{K}(\mathbf{n}',\mathbf{n}''). \tag{13}$$

Finally, introducing the reciprocity relation in this last result, we find after a little manipulation

$$\int d\Omega \mathcal{S}(\mathbf{n}',\mathbf{n})\cdot \mathcal{S}^{*T}(\mathbf{n}'',\mathbf{n}) =\epsilon_T(\mathbf{n}')\delta(\mathbf{n}'-\mathbf{n}'')-\mathcal{K}'(\mathbf{n}',\mathbf{n}''), \tag{14}$$

where $\mathcal{K}'(\mathbf{n}',\mathbf{n}'')=\mathcal{K}^*(-\mathbf{n}',\, -\mathbf{n}'')=\mathcal{K}^T(-\mathbf{n}',\, -\mathbf{n}'')$. In matrix notation, these relations have the form

$$\mathcal{S}^\dagger\mathcal{S}=1-\mathcal{K},\quad \mathcal{S}\mathcal{S}^\dagger=1-\mathcal{K}'$$

and $\mathcal{S}$ is seen to be properly unitary in the lossless case when $\mathcal{K}$ and $\mathcal{K}'$ are zero.

APPLICATION TO PLANE WAVE SCATTERING

We now specialize to the case in which a plane wave is incident along the direction $\mathbf{n}_0$, so that the field at large distances from the scattering region has the form

$$\underset{r\to\infty}{E(\mathbf{n}r)}=\mathbf{q}e^{ikr\mathbf{n}_0\cdot\mathbf{n}}+\mathbf{A}_\mathbf{q}(\mathbf{n},\mathbf{n}_0)\frac{e^{ikr}}{r}. \tag{15}$$

Here $\mathbf{q}$ is a unit vector in the direction of polarization of the incident wave and $\mathbf{A}_\mathbf{q}(\mathbf{n},\mathbf{n}_0)$ is the amplitude of the wave scattered in the direction $\mathbf{n}$, the notation serving to indicate explicitly the direction of the incident wave and its polarization as well as the direction of the scattered wave. Of course $\mathbf{q}\cdot\mathbf{n}_0$ and $\mathbf{n}\cdot\mathbf{A}_\mathbf{q}(\mathbf{n},\mathbf{n}_0)$ are both zero. The decomposition of this field into its incoming and outgoing parts leads in a straightforward way[2] to the result

$$\mathbf{E}(\mathbf{n}r)_{r\to\infty}=\frac{2\pi i}{k}\mathbf{q}\delta(\mathbf{n}_0+\mathbf{n})\frac{e^{-ikr}}{r}+\left[\mathbf{A}_\mathbf{q}(\mathbf{n},\mathbf{n}_0)-\frac{2\pi i}{k}\mathbf{q}\delta(\mathbf{n}_0-\mathbf{n})\right]\frac{e^{ikr}}{r}. \quad (16)$$

Hence, regarding this as a special form of Eq. (12), we have at once from the definition of the scattering matrix,

$$\mathbf{A}_\mathbf{q}(\mathbf{n},\mathbf{n}_0)=\frac{2\pi i}{k}\delta(\mathbf{n}_0-\mathbf{n})\mathbf{q}-\frac{2\pi i}{k}\mathcal{S}(\mathbf{n},\mathbf{n}_0)\cdot\mathbf{q}. \quad (17)$$

Equivalently, we can write

$$\mathbf{A}_\mathbf{q}(\mathbf{n},\mathbf{n}_0)=\mathcal{A}(\mathbf{n},\mathbf{n}_0)\cdot\mathbf{q}, \quad (18)$$

where $\mathcal{A}(\mathbf{n},\mathbf{n}_0)$, which we shall call the tensor scattering amplitude,[4] is given by

$$\mathcal{A}(\mathbf{n},\mathbf{n}_0)=\frac{2\pi i}{k}[\epsilon_T(\mathbf{n}_0)\delta(\mathbf{n}_0-\mathbf{n})-\mathcal{S}(\mathbf{n},\mathbf{n}_0)]. \quad (19)$$

Regarding (18) as the definition of $\mathcal{A}$, and supposing that $\mathbf{q}_1$ and $\mathbf{q}_2$ are two mutually perpendicular polarization directions and $\mathbf{A}_{\mathbf{q}1}$ and $\mathbf{A}_{\mathbf{q}2}$ are the corresponding scattering amplitudes, we have simply

$$\mathcal{A}(\mathbf{n},\mathbf{n}_0)=\mathbf{A}_{\mathbf{q}1}(\mathbf{n},\mathbf{n}_0)\mathbf{q}_1+\mathbf{A}_{\mathbf{q}2}(\mathbf{n},\mathbf{n}_0)\mathbf{q}_2. \quad (20)$$

Accordingly, by (19),

$$\begin{aligned}\mathcal{S}(\mathbf{n},\mathbf{n}_0)&=\epsilon_T(\mathbf{n}_0)\delta(\mathbf{n}_0-\mathbf{n})-\frac{k}{2\pi i}\mathcal{A}(\mathbf{n},\mathbf{n}_0)\\&=\epsilon_T(\mathbf{n}_0)\delta(\mathbf{n}_0-\mathbf{n})\\&\quad-\frac{k}{2\pi i}[\mathbf{A}_{\mathbf{q}1}(\mathbf{n},\mathbf{n}_0)\mathbf{q}_1+\mathbf{A}_{\mathbf{q}2}(\mathbf{n},\mathbf{n}_0)\mathbf{q}_2], \quad (21)\end{aligned}$$

which is thus an explicit construction of $\mathcal{S}$ in terms of the plane wave scattering amplitude for two mutually perpendicular direction of polarization.

We are more interested however in examining the properties of $\mathcal{A}$ corresponding to the previously derived general features of $\mathcal{S}$. Thus the reciprocity relation Eq. (7) yields at once, with the aid of the first form of Eq. (21)

$$\mathcal{A}(\mathbf{n},\mathbf{n}_0)=\mathcal{A}^T(-\mathbf{n}_0,-\mathbf{n}), \quad (22)$$

or equivalently, using Eq. (18)

$$\mathbf{q}'\cdot\mathbf{A}_\mathbf{q}(\mathbf{n},\mathbf{n}_0)=\mathbf{q}\cdot\mathbf{A}_{\mathbf{q}'}(-\mathbf{n}_0,-\mathbf{n}). \quad (23)$$

For the special case in which $\mathbf{q}=\mathbf{q}'$, this last relation states that if we interchange the direction of the incident wave and the direction of observation, the component of the scattered wave in the direction of polarization is unchanged; and this is the more customary (and more restricted) statement of reciprocity.

Next we examine the consequences of energy conservation as expressed by Eq. (13). Upon substitution of the first form of Eq. (21), we obtain

$$\int d\Omega\left[\epsilon_T(\mathbf{n}_0)\delta(\mathbf{n}_0-\mathbf{n})+\frac{k}{2\pi i}\mathcal{A}^{T*}(\mathbf{n},\mathbf{n}_0)\right]\cdot\left[\epsilon_T(\mathbf{n}_0')\delta(\mathbf{n}_0'-\mathbf{n})-\frac{k}{2\pi i}\mathcal{A}(\mathbf{n},\mathbf{n}_0')\right]=\epsilon_T(\mathbf{n}_0')\delta(\mathbf{n}_0'-\mathbf{n}_0'')-\mathcal{K}(\mathbf{n}_0',\mathbf{n}_0''). \quad (24)$$

Noting that $\epsilon_T(\mathbf{n}_0)\cdot\mathcal{A}(\mathbf{n}_0,\mathbf{n}_0')=\mathcal{A}(\mathbf{n}_0,\mathbf{n}_0')$, since $\mathcal{A}$ is transverse, we then find, after the δ function terms are integrated,

$$\int\mathcal{A}^{T*}(\mathbf{n},\mathbf{n}_0)\cdot\mathcal{A}(\mathbf{n},\mathbf{n}_0')d\Omega+\frac{4\pi^2}{k^2}\mathcal{K}(\mathbf{n}_0,\mathbf{n}_0')=\frac{2\pi}{ik}[\mathcal{A}(\mathbf{n}_0,\mathbf{n}_0')-\mathcal{A}^{T*}(\mathbf{n}_0',\mathbf{n}_0)]; \quad (25)$$

and this is a rather interesting extension of the cross-section theorem. It reduces to its conventional form upon setting $\mathbf{n}_0'=\mathbf{n}_0$ and taking scalar products from left and right with $\mathbf{q}$, for we then have, using Eq. (18),

$$\int\mathbf{A}_\mathbf{q}^*(\mathbf{n},\mathbf{n}_0)\cdot\mathbf{A}_\mathbf{q}(\mathbf{n},\mathbf{n}_0)d\Omega+\frac{4\pi^2}{k^2}\mathbf{q}\cdot\mathcal{K}(\mathbf{n}_0,\mathbf{n}_0)\cdot\mathbf{q}=(4\pi/k)\,\mathrm{Im}[\mathbf{q}\cdot\mathbf{A}_\mathbf{q}(\mathbf{n}_0,\mathbf{n}_0)]. \quad (26)$$

Now from Eq. (10), and recalling that $\mathbf{F}_1(\mathbf{n})=(2\pi i/k)\mathbf{q}\delta(\mathbf{n}_0+\mathbf{n})$ for a plane wave incident along $\mathbf{n}_0$, the rate of energy absorption P is

$$P=\frac{c}{8\pi}\cdot\frac{4\pi^2}{k^2}\mathbf{q}\cdot\mathcal{K}(\mathbf{n}_0,\mathbf{n}_0)\cdot\mathbf{q},$$

while the energy flux in the incident plane wave is $c/8\pi$. Hence the second term on the left of Eq. (26) is simply the absorption cross section σ_a while the first term is, of course, the total scattering cross section σ_s and thus, as expected,

$$\sigma_s+\sigma_a=(4\pi/k)\,\mathrm{Im}[\mathbf{q}\cdot\mathbf{A}_\mathbf{q}(\mathbf{n}_0,\mathbf{n}_0)]. \quad (27)$$

It is worth emphasizing that this result, as well as the more general relation Eq. (25), both of which are statements of conservation of energy, involve inter-

[4] *Note added in proof.*—Morse and Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., 1953). See pp. 1897–1898 where a tensor scattering amplitude is introduced to describe scattering from a sphere.

ference between the scattered and incident fields. To make this more explicit, recall that in our treatment the incident wave is represented by δ functions. Thus the left side of Eq. (24), representing the total outgoing energy, contains the scattered energy, the outgoing energy in the incident wave and interference terms between the scattered and incident fields. The right side represents the incoming energy (arising only from the incident wave, of course) from which is subtracted the energy absorbed in the scatterer. The purely incident wave terms cancel, as they must, and the scattered plus absorbed energy is accounted for by destructive interference between the incident wave and the forward scattered wave. This is the physical content of Eq. (27).

The structure of these relations perhaps becomes more transparent if we repeat our derivation using matrix notation. Thus, corresponding to Eq. (21) we have

$$\mathcal{S}=1-k\mathcal{A}/2\pi i,$$

and hence

$$\mathcal{S}^\dagger\mathcal{S}=1-k(\mathcal{A}-\mathcal{A}^\dagger)/2\pi i+k^2\mathcal{A}^\dagger\mathcal{A}/4\pi.$$

Recalling that from energy conservation $\mathcal{S}^\dagger\mathcal{S}=1-\mathcal{K}$, we thus obtain, corresponding to Eq. (25)

$$\mathcal{A}^\dagger\mathcal{A}+4\pi\mathcal{K}/k^2=2\pi(\mathcal{A}-\mathcal{A}^\dagger)/ik. \qquad (28)$$

The diagonal elements of this result give directly the cross-section theorem (for specific directions of incidence and polarization), the usefulness of which is so well known. However, perhaps because of their complexity, little consideration appears to have been given to the off-diagonal elements.[5] It is our belief, as we now attempt to demonstrate, that extension of the cross-section theorem to off-diagonal elements also has its utility.

One obvious application is simply as an algebraic check on specific calculations of scattering amplitudes. As an example of the kind of relations which can be derived, consider a lossless scatterer, $\mathcal{K}=0$, which is axially symmetric and has a symmetry plane perpendicular to its symmetry axis; e.g., a spheroid or a dumbell shaped scatterer. Consider now plane waves incident along the axis of symmetry in both directions; i.e., in Eq. (25) take $\mathbf{n}_0$ along the axis of symmetry and take $\mathbf{n}_0'=-\mathbf{n}_0$. Further, let each incident wave have the same polarization $\mathbf{q}$. Using Eq. (18), we then obtain

$$\int \mathbf{A}_q^*(\mathbf{n},\mathbf{n}_0)\cdot\mathbf{A}_q(\mathbf{n},-\mathbf{n}_0)d\Omega =\frac{2\pi}{ik}\mathbf{q}\cdot[\mathbf{A}_q(\mathbf{n}_0,-\mathbf{n}_0)-\mathbf{A}_q^*(-\mathbf{n}_0,\mathbf{n}_0)].$$

Because of the symmetry, $\mathbf{A}_q(\mathbf{n},-\mathbf{n}_0)=\mathbf{A}_q(-\mathbf{n},\mathbf{n}_0)$ while the two backward waves are equal and have the same polarization as the incident waves and can thus be written as $\mathbf{q}A_b$. Referring $\mathbf{n}$ to $\mathbf{n}_0$ as polar axis, we thus obtain the simple result

$$\int \mathbf{A}_q^*(\theta,\varphi)\cdot\mathbf{A}_q(\pi-\theta,\pi+\varphi)d\Omega=\frac{4\pi}{k}\,\mathrm{Im}[A_b].$$

This relation is easily verified for a spherically symmetric scatterer directly from the (vector) spherical harmonic expansions. Evidently similar but more complicated relations can be derived for other symmetries.

Finally we mention a rather different kind of application in which one uses off-diagonal elements to learn something about the large-angle scattering. For this purpose, consider a lossless, rather soft, spherically symmetrical scatterer in the short-wave limit. Here, the scattering is predominantly in the forward direction and the integral in Eq. (25) can be approximately evaluated as follows. Assuming the effective radius of the scatterer to be a, the angular width $\delta\theta$ of the main diffraction peak is roughly $(1/ka)$ and the main contributions to the integral evidently occur when $\mathbf{n}$ is near $\mathbf{n}_0$ and when it is near $\mathbf{n}_0'$, i.e., when one of the factors in the integrand is the forward scattering amplitude. For simplicity, let us take both incident waves to have the same polarization $\mathbf{q}$ (which means that we restrict our attention to the situation in which $\mathbf{n}$ and $\mathbf{n}_0$ lie in the plane perpendicular to $\mathbf{q}$) and let us assume that the large-angle scattering amplitude does not fluctuate violently over $\delta\theta$. We then obtain,

$$\frac{\pi}{(ka)^2}\{\mathbf{A}_q^*(\mathbf{n}_0,\mathbf{n}_0)\cdot\mathbf{A}_q(\mathbf{n}_0,\mathbf{n}_0')+\mathbf{A}_q^*(\mathbf{n}_0,\mathbf{n}_0')\cdot\mathbf{A}_q(\mathbf{n}_0',\mathbf{n}_0')\} \simeq\frac{2\pi}{ik}\mathbf{q}\cdot[\mathbf{A}_q(\mathbf{n}_0,\mathbf{n}_0')-\mathbf{A}_q^*(\mathbf{n}_0',\mathbf{n}_0)].$$

Using the obvious relations between the scattering amplitudes which follow from the spherical symmetry, this can be reduced to the relation

$$\sin\gamma\simeq\frac{1}{2ka^2}|A_f|\cos(\gamma_0-\gamma), \qquad (29)$$

where we have written

$$\mathbf{A}_q(\mathbf{n}_0,\mathbf{n}_0)=\mathbf{q}|A_f|e^{i\gamma_0},$$

and

$$\mathbf{q}\cdot\mathbf{A}_q(\mathbf{n}_0,\mathbf{n}_0')=|\mathbf{q}\cdot\mathbf{A}|e^{i\gamma},$$

so that γ is the phase angle of the component of the complex scattering amplitude along $\mathbf{q}$ and γ_0 denotes this phase angle for forward scattering. Using similar evaluation techniques for the total cross section,[6] we have

$$\sigma_{\mathrm{scatt}}\simeq\pi|A_f|^2/(ka)^2, \qquad (30)$$

[5] *Note added in proof.*—However, see Glauber and Schomaker [Phys. Rev. **89**, 667 (1953)] for a discussion in the scalar field case without dissipation.

[6] E. Gerjuoy and D. S. Saxon, Phys. Rev. **94**, 478 (1954), Eq. (70) ff.

and hence, according to the cross-section theorem for lossless scatterers,

$$\sin\gamma_0 \simeq |A_f|/4ka^2. \quad (31)$$

Eliminating $|A_f|$ between Eq. (29) and Eq. (31), we see that $\gamma=2\gamma_0$ to the accuracy of these relations. In words, under the stated assumptions, the phase of the complex scattering amplitude (more precisely, of its component along the direction of polarization of the incident wave) increases from its value γ_0 for forward scattering to roughly twice this value outside the main diffraction peak and, surprisingly, then becomes independent of scattering angle. Further, using Eq. (30), we note that in the usual electromagnetic scattering problem, where the total cross section tends to a constant value in the short-wave limit, the phasee γ_0 and γ also tend to definite limiting values. On ths other hand, if the total cross section tends to zero, as for example in the scattering by a free electron cloud, then the phases tend to zero as $\sigma^{\frac{1}{2}}$. This latter case, which is infrequently encountered in electromagnetic scattering, is the usual situation in quantum mechanics. In this connection see reference 6, in which some discussion of the Born and variational approximations is given on this basis. Presumably, relations such as those above can be generally useful in evaluating the consistency and convergence of approximation methods, but we shall not elaborate further.

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Scattering by a Symmetric Potential

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Scattering by a symmetric structure, such as a regular molecule or crystal, is discussed in terms of a variational principle. The appropriate partial waves are labelled by the irreducible representations of the crystallographic groups and may be generated by a set of projection operators, one for each irreducible representation. The trial functions are chosen to lie in a space spanned by the crystallographic partial waves. Methods are indicated for generating either the infinite set of partial waves needed for an exact solution or a finite set more useful in an approximate calculation.

INTRODUCTION

THE analysis of term splitting in symmetric potentials[1] is one of the well-known physical applications of group theory, but no systematic remarks on the corresponding scattering problem appear to have been published. In this connection, one might consider the scattering of particles by regular molecules or crystals, or macroscopically the scattering of sound by regular polyhedra. As long as these problems are done in Born approximation they are sufficiently simple so that there is no appreciable advantage in discussing further simplifications associated with the symmetry. But there are, of course, situations in which the Born approximation may not be adequate, such as electron diffraction by molecules containing potentials of very different strengths.[2] In this case more refined methods are needed, and then it may be helpful to handle the symmetry conditions in a systematic way. In the following, this will be done in connection with a variational principle proposed by Schwinger.[3,4]

[1] H. A. Bethe, Ann. Physik **3**, 133 (1929).
[2] The example of uranium hexafluoride has been discussed by R. Glauber and V. Schomaker, Phys. Rev. **89**, 667 (1953).
[3] J. Schwinger, *Lectures on Nuclear Physics*, Harvard University, 1947 (unpublished).
[4] E. Gerjuoy and D. Saxon, Phys. Rev. **94**, 478 (1954). Gerjuoy and Saxon have used this principle to discuss scattering from a spherically symmetric potential and have shown that even in this case it is convenient to utilize a cyclic group. The present note was suggested by their work.

PARTIAL WAVES

Consider scattering by a potential u having the symmetry of a rotation group G. Let the incoming wave be ψ_c and the outgoing wave be ψ_d. They are connected by S, the scattering operator:

$$\psi_d = S\psi_c \equiv e^{i\eta}\psi_c, \quad (1)$$

where S and η commute with all the elements R of G:

$$(S,R) = (\eta,R) = 0. \quad (2)$$

Equation (2) implies that η and S are multiples of the unit matrix, when the representation of G is irreducible, and that they have no elements connecting different irreducible representations. When G is the complete rotation group, one obtains the irreducible representations by choosing the basis functions to be the spherical harmonics Y_{lm}, and then

$$\eta(i,j) = \eta(lm;l'm') = \eta_l\delta(lm,l'm'). \quad (3)$$

The eigenvalues η_l determine the phase shifts and there is one for each irreducible representation.

In the case of the crystallographic groups, G contains only a finite number of elements and has only a small number of irreducible representations (Γ_γ). Every representation of the complete rotation group D_l may be decomposed into the Γ_γ:

$$D_l = \sum a_{l\gamma}\Gamma_\gamma.$$