Let us assume the Schrödinger equation for the spin function to be

$$i\hbar\partial\Phi/\partial t=[H_0(t)+H_1(t,\alpha)]\Phi, \tag{C1}$$

where $H_0(t)$ is independent of the orbital parameters α and H_1 is small.

If

$$\Phi(t)=U(t)\Phi(0), \tag{C2}$$

we obtain

$$U=Q(t)W(t,\alpha), \tag{C3}$$

where

$$i\hbar\partial Q/\partial t=H_0Q, \tag{C4}$$

and

$$i\hbar\partial W/\partial t=Q^{-1}H_1QW. \tag{C5}$$

Assuming $Q(t)$ known for Eq. (C4), $W(t,\alpha)$ is, correct to first order in H_1,

$$W(t,\alpha)=1+W_1(t,\alpha), \tag{C6}$$

where

$$W_1(t,\alpha)=\frac{1}{i\hbar}\int_0^t Q^{-1}(t')H_1(t',\alpha)Q(t')dt'. \tag{C7}$$

Using (13), the definition (12), and the approximate formula for U we obtain

$$\text{D.P.}=\frac{-2\,\text{trace}\{[\rho(0)]^2[\langle W_1^2(t)\rangle-\langle W_1(t)\rangle^2]\}}{\text{trace}\rho^2(0)-\frac{1}{2}}+\frac{2\,\text{trace}\{\langle\rho(0)W_1Q^{-1}\rho(0)QW\rangle-\langle\rho(0)W_1Q^{-1}\rangle\langle\rho(0)QW_1\rangle\}}{\text{trace}\rho^2(0)-\frac{1}{2}}. \tag{C8}$$

$\rho(0)$ is the initial spin density matrix. The angular braces denote averages with respect to α. Equation (C8) is correct to second order in H_1. (This follows somewhat indirectly from an assumed Hermiticity of H_1.) It may be remarked, that since the limitations on experiments are obtained by requiring the depolarization to be small, these could be derived directly using (C8) and

$$H_0=f(t)\boldsymbol{\sigma}\cdot\mathbf{h}, \tag{C9}$$

$$H_1(t,\alpha)=\mathbf{b}(t,\alpha)\cdot\boldsymbol{\sigma}, \tag{C10}$$

with $\mathbf{h}$ a constant unit vector, $f(t)$ a given function of time, and b a small, variable vector.

PHYSICAL REVIEW VOLUME 100, NUMBER 6 DECEMBER 15, 1955

Diffusion of Like Particles Across a Magnetic Field

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(Received July 25, 1955)

It is shown that the diffusion rate across a magnetic field due to collision of like charged particles is derivable from the macroscopic equations of the plasma. However, it is necessary to include the off-diagonal terms in the stress tensor. The resultant diffusion rate does not obey Fick's law and is proportional to the inverse fourth power of magnetic field strength. This diffusion rate is usually smaller than that due to unlike particle collisions, but may sometimes dominate.

I. INTRODUCTION

MANY of the gross properties of a plasma may be obtained from a consideration of the hydrodynamical (or macroscopic) equations of the plasma.[1] Thus, for example, for a gas consisting of ions and electrons and assuming an isotropic stress tensor, one has the following momentum equation in the steady state

$$\nabla P=\mathbf{j}\times\mathbf{H}+\epsilon\mathbf{E}. \tag{1}$$

Here P is the gas pressure, H and E are the magnetic and electric field strengths respectively, j is the current and ϵ the charge density in the plasma. Note that a nonlinear term in the velocity is neglected. In addition to this equation, we have another expression representing the generalized Ohm's law:

$$\mathbf{E}+(\mathbf{v}\times\mathbf{H})/c=\mathbf{j}/\sigma+\nabla P_i/en. \tag{2}$$

Again, steady state has been assumed and nonlinear terms neglected. The mass velocity of the plasma is denoted by $\mathbf{v}$, the ion partial pressure by P_i and the conductivity of the plasma by σ. The conductivity is defined as

$$\sigma\cong ne^2/mc\nu, \tag{3}$$

where n is the density of electrons, m the electron mass, and ν is the collision frequency for electron-ion impact.

As Spitzer has shown,[1] an expression for the diffusion rate across a magnetic field may be readily derived from Eqs. (1) and (2). Assuming that the density gradient and the electron field are in a single direction (say x) and the magnetic field is in the z-direction, one may eliminate $\mathbf{j}$ between Eqs. (1) and (2). The result is

$$v_x=-(c/\sigma H^2)(\nabla P-\epsilon E). \tag{4}$$

This has the usual form for diffusion in that Fick's law

[1] Lyman Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, New York, to be published).

is obeyed ($v\sim\nabla n$). The effective diffusion coefficient is

$$D=cnkT/\sigma H^2, \tag{5}$$

which is proportional to H^{-2} and to the cross section for electron-ion collisions. The mobility u has the usual form

$$u=eD/kT. \tag{6}$$

It is the purpose of this note to point out that the assumption of an isotropic stress tensor in Eq. (1) omits the effects due to like particle collisions and leads to a contradiction for the case of a gas consisting of one type of particle only (simple gas). Inclusion of the off-diagonal terms in the stress tensor removes the contradiction and yields a diffusion rate for collisions of like particles which is of an unusual form in that Fick's law is not obeyed and the diffusion coefficient is proportional to H^{-4}.

II. DIFFUSION IN A SIMPLE GAS

Consider a simple gas of charged particles. In this case, Eq. (2) does not exist. Equation (1) is still valid; however $\mathbf{v}$ and $\mathbf{j}$ are no longer independent and are related by

$$\mathbf{j}=\epsilon\mathbf{v}/c=ne\mathbf{v}/c. \tag{7}$$

As a result, Eq. (1) predicts that no mass flow occurs in the direction of a density gradient which is perpendicular to the magnetic field and hence that the diffusion rate is zero. It is also clear from the resulting form of the diffusion coefficient in Eq. (5) that like-particle collisions are not included in this expression; and hence the paradoxical result for a simple gas, which was just obtained, is not unexpected.

A very different result may now be obtained by including the off-diagonal terms of the stress tensor T_{ij} in Eq. (1) for a simple gas. Assume that the magnetic field is in the z-direction, that all quantities vary only in the x-direction and that the mass velocity in the direction of the magnetic field is constant everywhere. The resultant equation has the following three components:

$$\partial T_{xx}/\partial x=(nev_y/c)H+\epsilon E, \tag{8}$$

$$\partial T_{xy}/\partial x=-(nev_x/c)H, \tag{9}$$

$$\partial T_{xz}/\partial x=0. \tag{10}$$

Chapman and Cowling[2] have derived the proper expressions for the stress tensor of a simple gas in a magnetic field. The relevant expressions are

$$T_{xx}=P-\frac{2\mu}{1+(16/9)\omega^2\tau^2}\left[\tilde{e}_{xx}+\tfrac{1}{2}(\tilde{e}_{xx}+\tilde{e}_{yy})\cdot\frac{16}{9}\omega^2\tau^2+\tilde{e}_{xy}\cdot\frac{4}{3}\omega\tau\right], \tag{11}$$

$$T_{xy}=-\frac{2\mu}{1+(16/9)\omega^2\tau^2}\left[\tilde{e}_{xy}+\frac{1}{2}(\tilde{e}_{yy}-\tilde{e}_{xx})\cdot\frac{4}{3}\omega\tau\right], \tag{12}$$

$$T_{xz}=-\frac{2\mu}{1+(4/9)\omega^2\tau^2}\{\tilde{e}_{zx}+\tfrac{2}{3}\omega\tau\tilde{e}_{zy}\}, \tag{13}$$

where μ is the usual coefficient of viscosity in the absence of a magnetic field. Thus

$$\mu\cong\tfrac{2}{3}P\tau. \tag{14}$$

The mean free time between collisions is denoted by $\tau(=\lambda/v)$ and $\omega=eH/mc$. The symbol $\tilde{e}_{ij}$ represents the symmetrical and traceless tensor $\nabla\mathbf{v}$. The usual case of interest is where $\omega\tau\gg1$, which corresponds to a particle making very many gyrations between collisions. Clearly the opposite limit $\omega\tau<1$ is of no interest since the magnetic field is then unimportant.

By our assumptions $\tilde{e}_{zx}=\tilde{e}_{zy}=0$ and hence $T_{xz}\equiv0$. Equation (10) is thus satisfied identically. It will be seen from the form of the final result that $\mu\tilde{e}_{xy}\ll\omega\tau P$ and that $\tilde{e}_{xy}/\tilde{e}_{xx}\gg\omega\tau$. If one assumes these inequalities, the remaining tensor components can be written as

$$T_{xx}\cong P, \tag{15}$$

$$T_{xy}\cong-\frac{9}{16}\frac{\mu}{(\omega\tau)^2}\frac{dv_y}{dx}. \tag{16}$$

Solving Eq. (8) for v_y and substituting in Eq. (9), one obtains

$$v_x=\frac{c}{neH}\frac{d}{dx}\left\{\frac{9}{16}\frac{\mu}{(\omega\tau)^2}\frac{d}{dx}\left[\frac{c}{neH}\left(\frac{dP}{dx}-\epsilon E\right)\right]\right\}. \tag{17}$$

Assume a constant temperature in the gas and ignore the electric field. By virtue of the fact that $\tau\sim n^{-1}$ and hence that μ is independent of n, Eq. (17) can be rewritten as

$$v_x=\frac{3}{8}\frac{v^4}{\omega^4\tau}\frac{1}{n^2}\frac{d}{dx}\left\{n^2\frac{d}{dx}\left(\frac{1}{n}\frac{dn}{dx}\right)\right\}$$

or

$$v_x=\frac{3}{8}\frac{r_0^4}{\tau}\cdot\frac{1}{n}\left\{\frac{d^3n}{dx^3}-\frac{1}{n}\frac{d^2n}{dx^2}\frac{dn}{dx}\right\}, \tag{18}$$

where $r_0=mvc/eH$ is the Larmor radius. Equation (18) is the desired result. The diffusion velocity due to like particle collisions is clearly proportional to H^{-4} and does not obey Fick's law. Instead, the diffusion rate depends on the second and third space derivatives of the particle density. An alternative form for Eq. (18) is

$$v_x=\frac{3}{8}\frac{r_0^4}{\tau}\frac{d}{dx}\left[\frac{1}{n}\frac{d^2n}{dx^2}\right]. \tag{19}$$

Equations (18) and (19) can be expected to give the proper form for the diffusion due to like particle collisions even in a nonsimple gas. An expression which

[2] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, Cambridge, 1952), p. 338.

is identical to that given in Eq. (19), except for the factor $\frac{3}{8}$, has been obtained by Kruskal by the use of a model based on individual particle orbits.[3]

It remains now to justify the approximation made in deriving the results. All order of magnitude estimates of the various terms which have been used above rest on the following assertion. It is extremely unlikely that the ion density n, in any practical problem, can fall off faster than with an e-folding length equal to the Larmor radius r_0. In fact, the actual e-folding length should usually be considerably larger than this. Hence, order-of-magnitude estimates may be made by replacing each space derivative of n by the quantity $(\kappa r_0)^{-1}$, where κ is substantially larger than unity.

By combining this assertion with the result of Eq. (19), it is seen that

$$v_x = O[r_0/\tau\kappa^3], \tag{20}$$

where the symbol O denotes "of the order of."

Similarly, by Eqs. (8) and (15),

$$v_y = O\left[\frac{ckT}{eH}\frac{1}{\kappa r_0}\right] = O\left[\frac{r_0}{\tau}\frac{\omega\tau}{\kappa}\right]. \tag{21}$$

Hence

$$v_y/v_x = O[\omega\tau\kappa^2] \gg \omega\tau.$$

As a result, then,

$$\frac{\tilde{e}_{xy}}{\tilde{e}_{xx}} \cong \frac{\partial v_y/\partial x}{\partial v_x/\partial x} \gg \omega\tau,$$

as was asserted earlier. In addition,

$$\mu\tilde{e}_{xy} \cong P\tau\partial v_y/\partial x = P\cdot O[\omega\tau/\kappa^2].$$

Hence $\mu\tilde{e}_{xy} \ll \omega\tau P$ as was also asserted earlier.

III. COMPARISON OF DIFFUSION RATES

It is of interest to compare the rate of diffusion of ions due to ion-ion collisions with that due to electron-ion collisions in a plasma consisting of both types of particles (assumed in approximate space-charge neutrality). The rate due to ion-ion collisions is obtained from Eq. (19) and is of the order of magnitude

$$v = O[r_{0i}/\tau_i\kappa^3],$$

where r_{0i} is the Larmor radius of the ions and τ_i is the mean free time between ion-ion collisions. The ion diffusion rate due to electron-ion collisions is obtained from Eqs. (4) and (3), (assuming $E=0$) and is of the order of magnitude

$$v' = O\left[\frac{r_{0e}^2}{\tau_e\kappa r_{0i}}\right],$$

where r_{0e} is the electron Larmor radius and τ_e is the mean free time between electron-ion collisions. Hence,

$$\frac{v}{v'} = O\left[\frac{r_{0i}^2\tau_e}{r_{0e}^2\tau_i\kappa^2}\right] = O\left[\left(\frac{M}{m}\right)^{\frac{1}{2}}\frac{\lambda_e}{\lambda_i}\frac{1}{\kappa^2}\right],$$

where equal temperatures have been assumed for the electron and ion gas, and where M denotes the ion mass. For the case of equal temperatures, $\lambda_e \cong \lambda_i$ and

$$\frac{v}{v'} = O\left[\left(\frac{M}{m}\right)^{\frac{1}{2}}\frac{1}{\kappa^2}\right]. \tag{22}$$

Hence it is possible for the diffusion rate due to ion-ion collisions to be larger than that due to electron-ion collisions, since $(M/m)^{\frac{1}{2}}$ is of the order of 10^2. However, in most cases one might expect $\kappa \gg 1$ and then the self-diffusion rate will become comparable to or smaller than that due to electron-ion collisions.

A curious point which is worth noting is that if one were dealing with a truly simple gas with density varying only in the x-direction then the electric field E is related to the density by Poisson's law $dE/dx = 4\pi ne$. Upon substitution of this relation in Eq. (17), one now finds that a term proportional to dn/dx contributes to v_x. Hence, to this extent, Fick's law is obeyed. The coefficient of this term is still proportional to H^{-4}. Moreover, in a nonsimple gas, which is all that may be obtained in practice, the electric field gradient is no longer simply related to the density of one species of charged particle.

ACKNOWLEDGMENTS

The author wishes to take this opportunity to thank Dr. Lyman Spitzer for a prepublication copy of his forthcoming tract on *The Physics of Fully Ionized Gases*.

[3] Martin Kruskal (private communication).