Three-Body Contributions to the Triton Binding Energy

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A variational calculation shows that the central part of the Lévy potential is not sufficient to bind the triton. One finds a bound S-state wave function only if the coupling constant is increased appreciably. Using wave functions determined in this way, the author has computed the binding energy contributed by five three-body terms, derived from the $ps-ps$ theory. Results of this calculation indicate that the so-called "leading" three-body term is not the dominant term, even if damping is taken into account. The total contribution of the potentials considered here is attractive, not repulsive.

I. INTRODUCTION

CALCULATIONS of the triton binding energy have, in the past, always neglected the effect of many-body forces.[1–3] However, the existence of many-body forces follows from the $ps-ps$ meson theory. It has, in fact, been suggested by Wentzel[4] that many-body potentials are largely responsible for the saturation of nuclear forces. Recent calculations by Drell and Huang[5] seem to support this contention. But the approximations in the work of Drell and Huang are only valid for heavy nuclei. In view of the importance attributed to many-body forces in heavy nuclei, we propose to investigate the contribution of three-body forces to the binding energy of the triton. In this case, also, three-body forces may be of considerable importance.

Despite difficulties which have been pointed out by Klein[6] and others, we adopt here the point of view of the Lévy theory.[7] Because of the uncertainties inherent in the $ps-ps$ theory in general, and in the Lévy theory in particular, an accurate evaluation of the three-body contribution is impossible. Even the treatment of the two-body problem is not free of ambiguity. Only the order of magnitude of the three-body effects can be determined.

We shall select those terms in the three-body potential which are of lowest order in the coupling constant, G, and in the mass ratio, μ/M_{H}. It will be assumed, in the spirit of the Lévy theory, that these are the most important three-body terms, though our results will show that the validity of this assumption is questionable. In fact it is not at all clear that the series of three-body potentials converges.

It seems reasonable to assume, here, that the two-body potential is mainly responsible for the binding of the triton. Since the accuracy of our results will necessarily be limited by theoretical difficulties, the two-body potential alone may be used to determine the triton wave function. This function then gives the expectation values of the "major" terms in the three-body potential. Specifically, the two-body Lévy potential will be used to determine the wave function. We shall adopt, also, Lévy's value of the core radius and of the coupling constant, though larger values of the coupling constant will also be considered. The effect of the hard core on the wave function must, of course, be taken into account.

One of the three-body terms is of sixth order in G, and zeroth order in the mass ratio. Though smaller powers of G do occur, these are always associated with higher powers of the mass ratio. Therefore this particular term is usually referred to as the "leading" term. Of the three-body terms considered by Drell and Huang, the leading term is by far the largest in absolute value. Now, the leading term depends only on the distances between nucleons. For this reason one may argue that it is primarily the S-state wave function which determines the magnitude of the three-body contribution. An attempt has been made to find an appropriate S-state function by means of a variational calculation. For this purpose, the central part of the Lévy potential was taken as the two-body interaction. The form of the trial function (with no D-state admixture) and the details of the calculations will be discussed below. With Lévy's value of G, it was found that none of the trial functions give any binding energy at all, though functions with as many as five variable parameters were considered. It does not necessarily follow that this particular hard core theory is unsatisfactory. Quite possibly the binding energy of the triton is supplied by the tensor forces.

Our procedure, then, does not give the form of the wave function. One can, however, establish a reasonable range of values for the parameters in the trial function, by a procedure to be described in Sec. III. Contributions of various three-body terms have been computed for this range of values. Within this range, the ratios of the computed expectation values do not depend critically on the wave function parameters. It is possible, therefore, to determine which terms are large and which are small, and to arrive at some conclusions which will be independent of the exact form of the function. We shall also

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[1] E. Gerjuouy and J. Schwinger, Phys. Rev. **61**, 138 (1942).
[2] T.-M. Hu and K.-N. Hsu, Phys. Rev. **78**, 633 (1950).
[3] R. L. Pease and H. Feshbach, Phys. Rev. **88**, 945 (1952).
[4] G. Wentzel, Helv. Phys. Acta, No. 5, 569 (1952).
[5] S. D. Drell and K. Huang, Phys. Rev. **91**, 1527 (1953).
[6] A. Klein, Phys. Rev. **89**, 1158 (1953) and Phys. Rev. **90**, 1101 (1953).
[7] M. Lévy, Phys. Rev. **88**, 725 (1952).

attempt, in various ways, to compare the two- and three-body contributions to the binding energy.

The effect of some of the higher order terms has been discussed by Brueckner, Gell-Mann, and Goldberger.[8] This so-called "damping" effect modifies the Lévy potential. If one takes damping into account, the effective value of the coupling constant in the G^4-term of the two-body potential is diminished. Using a different approach, Wentzel has arrived at essentially the same conclusion.[9] A potential, modified by the damping effect, has been introduced by Jastrow.[10] We shall discuss here, briefly, the modified Lévy potential which Jastrow uses, and the corresponding effect of damping on the three-body potentials.

II. S-STATE WAVE FUNCTION

A triton wave function, satisfying the Pauli principle, may be written in the individual particle form:

$$\psi(123)=(1/\sqrt{3})[\Phi(12,3)N(1)N(2)P(3)+\Phi(23,1)N(2)N(3)P(1)+\Phi(31,2)N(3)N(1)P(2)].$$

Here $\Phi(ij,k)$ is a function of spin and space variables, antisymmetric with respect to the nucleon pair (i,j). It can be shown that $\psi(123)$, so defined, represents one component of a pure charge doublet state if

$$\Phi(12,3)+\Phi(23,1)+\Phi(31,2)=0. \tag{1}$$

We take, for the function $\Phi(ij,k)$, the product of a space-dependent factor φ, and a spin-dependent factor, S. Further it will be assumed that φ is completely symmetric, and represents an S-state. The spin function, then, must be antisymmetric with respect to an exchange of neutrons, particles i and j, in this case). We must write:

$$S(ij,k)=(1/\sqrt{2})[\alpha(i)\beta(j)-\beta(i)\alpha(j)]\alpha(k),$$

or

$$S(ij,k)=(1/\sqrt{2})[\alpha(i)\beta(j)-\beta(i)\alpha(j)]\beta(k).$$

No other eigenfunctions of the z-component of spin are antisymmetric with respect to the neutron pair, and invariant with respect to space rotations.[11] It is clear that our wave function satisfies Eq. (1).

There is actually another term, symmetric in the neutron spins, which may be included in the S-state wave function. However, Feshbach and Rarita[12] find that this second term adds only 0.4 percent to their computed binding energy. Furthermore, the two S-state functions are coupled through the agency of the tensor forces alone. Since we are neglecting tensor forces, we shall also neglect the spin-symmetric term.

As a trial function we take, first:

$$\varphi(123)=\mu^3(r_{12}-r_0)(r_{23}-r_0)(r_{31}-r_0)\times\exp[-\tfrac{1}{2}\omega\mu(r_{12}+r_{23}+r_{31})]=\varphi(12)\varphi(23)\varphi(31),$$

for $r_{12}, r_{23}, r_{31}>r_0$, and $\varphi(123)=0$, otherwise. Here r_0 is the core radius, ω the variable parameter. The function, as written above, is not normalized. Normalization constants will be computed in the next section. It will be noted that the wave function vanishes for small internucleon distances, in conformity with the requirements of the hard core theory.

When a better approximation to the true wave function is desired, linear combinations of such trial functions, with different values of ω, will be used in the variational calculation. One may object that the wave functions, as we have written them, do not have the correct asymptotic form.[13] Such trial functions have, however, given fair results in earlier calculations.[2,3] Furthermore, we shall be interested, primarily, in relations between expectation values of the various three-body potentials. Relations between these expectation values should not depend critically on the asymptotic form, because of the short range of the three-body potentials.

III. EXPECTATION VALUES, SINGLE TERM WAVE FUNCTIONS

It will often be necessary to evaluate integrals of the form:

$$I=\int_{x_0}^{\infty}dx_{31}\int_{x_0}^{\infty}dx_{23}\int_{L}^{x_{23}+x_{31}}f(x_{12},x_{23},x_{31})dx_{12},$$

where

$$x_0=\mu r_0,\quad x_{ij}=\mu r_{ij}.$$

Here L is the larger of the quantities x_0, $|x_{31}-x_{23}|$. For convenience we shall abbreviate the limits in such integrals:

$$I=\int_V f(x_{12},x_{23},x_{31})dx_{12}dx_{23}dx_{31}. \tag{2}$$

The integral may be split into four terms:

$$I=I_0-I_1-I_2-I_3, \tag{3}$$

where

$$I_0=\int_{x_0}^{\infty}dx_{31}\int_{x_0}^{\infty}dx_{23}\int_{x_0}^{\infty}dx_{12}f(x_{12},x_{23},x_{31}), \tag{4}$$

$$I_1=\int_{x_0}^{\infty}dx_{31}\int_{x_0}^{\infty}dx_{12}\int_{x_{12}+x_{31}}^{\infty}f(x_{12},x_{23},x_{31})dx_{23}, \tag{5}$$

$$I_2=\int_{x_0}^{\infty}dx_{12}\int_{x_0}^{\infty}dx_{23}\int_{x_{12}+x_{23}}^{\infty}f(x_{12},x_{23},x_{31})dx_{31}, \tag{6}$$

$$I_3=\int_{x_0}^{\infty}dx_{31}\int_{x_0}^{\infty}dx_{23}\int_{x_{31}+x_{23}}^{\infty}f(x_{12},x_{23},x_{31})dx_{12}. \tag{7}$$

[8] Brueckner, Gell-Mann, and Goldberger, Phys. Rev. **90**, 476 (1953).
[9] G. Wentzel, Helv. Phys. Acta **15**, 111 (1942).
[10] R. Jastrow, Phys. Rev. **91**, 749 (1953).
[11] Gerjuouy and Schwinger[1] list the various triton spin functions.
[12] H. Feshbach and W. Rarita, Phys. Rev. **75**, 1384 (1949).
[13] G. Morpurgo, Nuovo cimento **9**, 461 (1952).

In evaluating I_0, we treat the variables of integration as independent variables. Because of the nature of the limits in the integral I, these variables must, in fact represent the lengths of sides of a triangle. Regions where $x_{12}+x_{23}-x_{31}<0$, or $x_{12}-x_{23}+x_{31}<0$, or $-x_{12}+x_{23}+x_{31}<0$ do not contribute to the integral I. Therefore contributions from these regions (the integrals I_1, I_2, and I_3) must be subtracted from I_0. Note that no two of the preceding inequalities can be satisfied simultaneously.

We turn our attention, now, to the normalization integral. We shall take, first, a wave function containing only one term and, therefore, only one variable parameter ω. After evaluating spin and charge matrix elements we get, for the normalization integral[3]

$$N=\frac{C}{\mu^6}\int_V[\varphi(12)\varphi(23)\varphi(31)]^2x_{12}x_{23}x_{31}dx_{12}dx_{23}dx_{31}.$$

The value of the constant, C, need not concern us here. We see that the normalization integral has the form (2). It may be split into four terms, as above, and evaluated exactly.

For the potential, we take the central part of the Lévy potential:

$$V_{ij}=(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)U(x_{ij})-W(x_{ij}),$$

$$U(x_{ij})=\frac{\mu}{3}\left(\frac{G^2}{4\pi}\right)\left(\frac{\mu}{2M}\right)^2\left(\frac{e^{-x_{ij}}}{x_{ij}}\right),$$

$$W(x_{ij})=3\mu\left(\frac{G^2}{4\pi}\right)^2\left(\frac{\mu}{2M}\right)^2\times\left\{\frac{2}{\pi}K_1(2x_{ij})+\frac{\mu}{2M}\left[\frac{2}{\pi}K_1(x_{ij})\right]^2\right\}\Big/x_{ij}^2.$$

As we have already pointed out, many unsatisfactory features remain in the Lévy treatment of the two-body problem. In particular, the form of the two-body potential has not been clearly established. But it is the essential properties of the Lévy potential which we are interested in, namely, the presence of a very short-ranged component, and the hard core. Since the parameters in the original Lévy potential have been fixed through previous calculations,[7] we shall use the potential in its original form here, and throughout most of the work that follows.

Again, after computing spin and charge matrix elements we find for the expectation value of the two-body potential:

$$\langle V\rangle=-\frac{C}{N}\left(\frac{3}{\mu^6}\right)\int_V[\varphi(12)\varphi(23)\varphi(31)]^2\times[3U(x_{12})+W(x_{12})]x_{12}x_{23}x_{31}dx_{12}dx_{23}dx_{31},$$

where N is the normalization integral. The above expression represents the contribution of all three nucleon pairs to the potential energy.

While the potential integral cannot be evaluated exactly, it can be approximated with good accuracy by means of a simple device. We plot the function $x_{12}W(x_{12})$ on semilog paper for $x_0\leq x_{12}\leq 2.5$. Within this range the function is analyzed, graphically, into a sum of three exponentials. We shall not describe this analysis in detail. The same procedure is sometimes used to determine the half-lives of elements in a mixture of radioactive materials.[14]

The function $W(x_{12})$, then, is represented as a series of Yukawa potentials. Since $U(x_{12})$ is already a Yukawa potential, all integrations may now be carried out exactly.

Our analysis of $W(x_{12})$ gives:

$$x_{12}W(x_{12})\cong 1.533e^{-2.875x_{12}}+13.68e^{-6.216x_{12}}+72.97e^{-13.37x_{12}}.$$

If one uses this approximating function, the error in the potential $3U(x_{12})+W(x_{12})$ is no greater than 2 percent for $x_{12}\leq 2.5$. Over most of this range, the error is considerably smaller, but it increases for large distances. At $x_{12}=3$, for instance, the error reaches 3 percent, but the potential has dropped to one ten-thousandth of its maximum value (i.e., its value at $x_{12}=x_0$). Since the wave function should also be small for such large distances, the contribution to the expectation value from regions where $x_{12}>2.5$ should be negligible. The exact and approximate potentials appear in Table I.

TABLE I. Two-body potentials, V_E (exact) and V_A (approximate) as functions of x.[a]

x	V_E	V_A	% Error
0.38	154.4	153.6	−0.48
0.40	127.9	127.6	−0.19
0.45	82.63	82.80	0.21
0.50	55.61	55.75	0.25
0.55	38.54	38.70	0.42
0.60	27.46	27.56	0.38
0.65	20.00	20.06	0.30
0.70	14.84	14.86	0.17
0.75	11.19	11.19	0.067
0.80	8.555	8.552	−0.034
0.85	6.626	6.619	−0.098
0.90	5.190	5.183	−0.12
0.95	4.107	4.108	0.021
1.00	3.280	3.280	0.010
1.05	2.642	2.646	0.14
1.10	2.145	2.151	0.31
1.30	0.9980	1.011	1.3
1.50	0.5096	0.5196	2.0
1.75	0.2449	0.2494	1.8
2.00	0.1298	0.1307	0.69
2.50	0.04560	0.04469	−2.0
3.00	0.01952	0.01894	−3.0

[a] $V_E=\frac{e^{-x}}{x}+(9.7)\frac{1}{x^2}\left\{\frac{2}{\pi}K_1(2x)+\frac{\mu}{2M}\left[\frac{2}{\pi}K_1(x)\right]^2\right\}\propto V_2{}^c$,

$V_A=\frac{e^{-x}}{x}+(9.7)\frac{1}{x}\{1.533e^{-2.875x}+12.68e^{-6.216x}+72.97e^{-13.37x}\}$.

V_E and V_A are proportional to the exact and approximate two-body potentials, respectively.

[14] See, for instance, D. Halliday, *Introductory Nuclear Physics* (John Wiley and Sons, Inc., New York, 1950), p. 33.

Again the integral to be evaluated splits into four terms [see Eq. (3)]. One of these has the form:

$$I_3=\int_{x_0}^{\infty}dx_{31}\int_{x_0}^{\infty}dx_{23}\int_{x_{23}+x_{31}}^{\infty}[3U(x_{12})+W(x_{12})]\times[\varphi(12)\varphi(23)\varphi(31)]^2x_{12}x_{23}x_{31}dx_{12}$$

[see Eq. (7)]. Since the potential is small for large values of x_{12}, this term should be small compared to the other three. It has, therefore, been neglected. A rough estimate shows that the resulting error is less than 3 percent of the expectation value of the potential. The term we have discarded is positive. Consequently the above approximation gives a potential energy which is a little too large in absolute value. The error in the approximating function, $f(x_{12})$, which we have inserted in the integrand has just the opposite effect for large values of x_{12}.

The kinetic energy may be written in the form:

$$\text{K.E.}=-(1/2M)[(\nabla_{12}-\nabla_{23})^2+(\nabla_{23}-\nabla_{31})^2+(\nabla_{31}-\nabla_{12})^2],$$

since $\hbar=1$ in our unit system. For the expectation value we have

$$\langle\text{K.E.}\rangle=-\frac{C}{N}\frac{3}{M}\int_V r_{12}r_{23}r_{31}\psi(123)\times[\nabla_{12}^2-\nabla_{12}\cdot\nabla_{23}]\psi(123)dr_{12}dr_{23}dr_{31},$$

if one notes the symmetry of the wave function. Here

$$\nabla_{12}^2=\frac{\mu^2}{x_{12}^2}\frac{\partial}{\partial x_{12}}x_{12}^2\frac{\partial}{\partial x_{12}},$$

$$\nabla_{12}\cdot\nabla_{23}=\mu^2\frac{\mathbf{x}_{12}\cdot\mathbf{x}_{23}}{x_{12}x_{23}}\frac{\partial}{\partial x_{12}}\frac{\partial}{\partial x_{23}}=\frac{\mu^2}{2}\left[\frac{x_{13}^2-x_{12}^2-x_{23}^2}{x_{12}x_{23}}\right]\frac{\partial}{\partial x_{12}}\frac{\partial}{\partial x_{23}}.$$

Hence:

$$\langle\text{K.E.}\rangle=-K_1+K_2-K_3, \tag{8}$$

where

$$K_1=\frac{C}{N}\frac{3}{M\mu^4}\int_V x_{23}x_{31}[\varphi(12)\varphi(23)\varphi(31)]\times\frac{\partial^2}{\partial x_{12}^2}x_{12}[\varphi(12)\varphi(23)\varphi(31)]dx_{12}dx_{23}dx_{31}, \tag{9}$$

$$K_2=\frac{C}{N}\frac{3}{2M\mu^4}\int_V x_{31}^3[\varphi(12)\varphi(23)\varphi(31)]\times\frac{\partial}{\partial x_{12}}\frac{\partial}{\partial x_{23}}[\varphi(12)\varphi(23)\varphi(31)]dx_{12}dx_{23}dx_{31}, \tag{10}$$

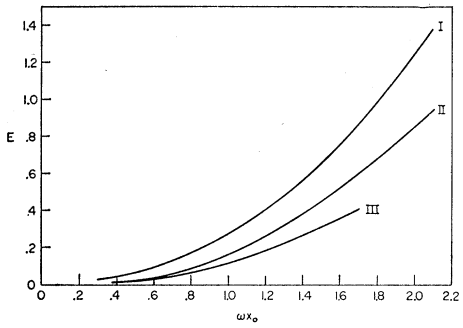

FIG. 1. Potential and kinetic energies as a function of ωx_0. See text for fuller explanation.

$$K_3=\frac{C}{N}\frac{3}{M\mu^4}\int_V x_{31}x_{12}^2[\varphi(12)\varphi(23)\varphi(31)]\times\frac{\partial}{\partial x_{12}}\frac{\partial}{\partial x_{23}}[\varphi(12)\varphi(23)\varphi(31)]\partial x_{12}\partial x_{23}\partial x_{31}. \tag{11}$$

All these integrals may be evaluated exactly.

Figure 1 shows the results of the above calculation for single term trial functions. Curve I represents the kinetic energy, plotted as a function of ωx_0. Curve II is a plot of the absolute value of the potential. It will be seen that the trial function gives no binding energy for any value of the parameter. If one plots the difference between curves I and II, the resulting curve shows neither a minimum, nor even an inflection point.

Postponing the discussion of refined trial functions, we consider next a modified two-body potential, introduced by Jastrow.[10] In accordance with recent criticisms of the Lévy paper,[6] Jastrow omits, from the two-body potential, the term in $[K_1(x_{12})]^2$. Two central potential terms remain:

$$V^{(2c)}(x_{12})=\mu\left(\frac{\mu}{2M}\right)^2\left(\frac{G^2}{4\pi}\right)\frac{(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)}{3}\frac{e^{-x_{12}}}{x_{12}},$$

$$-V^{(4c)}(x_{12})=-3\mu\left(\frac{\mu}{2M}\right)^2\left(\frac{G^2}{4\pi}\right)^2\frac{2}{\pi}\frac{K_1(2x_{12})}{x_{12}^2},\quad\begin{matrix}(x_{12}>x_0)\\(x_{12}>x_0).\end{matrix}$$

The damping effect, discussed in the introduction, should diminish the relative importance of the second term.[8,9] For this reason, a factor $\alpha<1$ is inserted before the potential $V^{(4c)}$. The central part of the two-body potential then becomes:

$$V^{(c)}(x_{12})=V^{(2c)}(x_{12})-\alpha V^{(4c)}(x_{12}).$$

Parameters in the above expressions are chosen to fit various two-body data:

$$\alpha=\tfrac{1}{4},\quad G^2/4\pi=16,\quad x_{ct}=0.61,\quad x_{cs}=0.40.$$

Here x_{ct} and x_{cs} are core radii in the triplet and singlet states, respectively.

TABLE II. Kinetic, potential, and binding energies for various coupling constants and trial functions. All energies are given in terms of the measured binding energy of the triton, 8.49 Mev. The trial function is given by the expression $\psi=f(\omega_0x_0)+a_1f(\omega_1x_0)+a_2f(\omega_2x_0)$.

$G^2/4\pi$	K.E.	P.E.	B.E.	ω_0	ω_1	ω_2	$a_1\times10^3$	$a_2\times10^5$
11.8	14.07	15.00	0.93	1.7	0.7	$\sim$	0.8	0
11.5	11.68	11.95	0.27	1.7	0.7	$\sim$	1.2	0
11.3	9.17	9.10	−0.07	1.7	0.7	0.3	1.6	0.2

Using a two-body interaction of this form, we have calculated the potential energy for various values of ω. For α and G, we have taken Jastrow's values. However, in order to make use of our previous results, we have assumed the same core radius, $x_0=0.38$, for both triplet and singlet states. Again, the term $V^{(4c)}(x_{12})$ was represented as a sum of Yukawa potentials. Curve III, in Fig. 1, is a plot of the magnitude of the potential energy against ωx_0. Here, also, one finds no binding for any value of ω. In fact curve III lies below curve II over the entire range for which integrals have been computed. For larger core radii the absolute value of the potential energy should be even smaller.

IV. REFINED TRIAL FUNCTIONS

So far we have considered only simple trial functions, containing only one variable parameter. The single term trial function gives no binding energy. We wish to determine whether it is possible to construct a linear combination of such functions (with various values of ω) which does give binding energy. For this purpose we have adopted the following procedure.

The coupling constant is increased so that $G^2/4\pi=12.2$. Now the variational calculation described above is repeated with this larger value of the coupling constant. For the optimum value, ω_0, of the trial parameter, the binding energy so calculated is greater than the measured binding energy of the triton. The single term trial function containing this parameter will be called $f(\omega_0x_0)$.

Terms are then added to the trial function one by one. As a next approximation we take

$$F_1=f(\omega_0x_0)+a_1f(\omega_1x_0),$$

and vary a_1 and ω_1 so as to maximize the binding energy. Finally,

$$F_2=f(\omega_0x_0)+a_1f(\omega_1x_0)+a_2f(\omega_2x_0).$$

Keeping ω_0, a_1 and ω_1 fixed at their predetermined values, we vary a_2 and ω_2 separately. A small readjustment of all parameters will then improve the binding energy slightly. As a check, the entire procedure is repeated with a slightly higher, and a slightly lower, initial value ω_0. In either case the binding energy diminishes.

Now the coupling constant is gradually reduced, while the wave function is modified to keep the binding energy at a maximum.

Results of the calculation are summarized in Table II. For $G^2/4\pi=11.8$ and $G^2/4\pi=11.5$, we list trial functions with only two terms, since the addition of a third term does not increase the binding energy appreciably. The table shows that all binding energy disappears for $G^2/4\pi=11.3$. The wave function listed for $G^2/4\pi=11.3$ minimizes the excess of kinetic energy. We find, then, that no linear combination gives binding energy for this value of the coupling constant, even if the (presumably) repulsive three-body forces are ignored. Possibly it is necessary to raise the coupling constant from $G^2/4\pi=9.7$ to $G^2/4\pi=11.8$ simply in order to compensate for the neglect of tensor forces.

It will be seen that the quantities a_1 and a_2 are very small. The energy and normalization integrals, however, increase very rapidly as ω decreases. For this reason, all the terms in the trial function are significant.

V. LEADING THREE-BODY TERM

The leading term in the three-body potential has the form[5,15]

$$V_3=\frac{12\lambda^3\mu}{x_{12}x_{23}x_{31}}\frac{2}{\pi}-K_1[x_{12}+x_{23}+x_{31}].$$

Here $\lambda=(G^2/4\pi)(\mu/2M)$ if the damping effect is neglected. At this point we wish to calculate the expectation value of V_3, using single-term trial functions, for various values of ω. After computing spin and charge matrix elements, we find:

$$\langle V_3\rangle=\frac{12\lambda^3}{\mu^5}\int_V[\varphi(12)\varphi(23)\varphi(31)]^2\times\frac{2}{\pi}-K_1[x_{12}+x_{23}+x_{31}]dx_{12}dx_{23}dx_{31}.$$

In order to evaluate this integral, we express the Hankel function as a sum of exponentials:

$$(2/\pi)K_1[x_{12}+x_{23}+x_{31}]\equiv(2/\pi)K_1[S]\cong0.925e^{-1.198S}+2.25e^{-3.072S}+10.0e^{-10.37S}.$$

The error in the approximating function is less than 1 percent for $S\leq3.5$. We insert this approximating function into the integrand. The resulting integral may now be split into four terms, as in the previous calculations. These four terms have been evaluated exactly.

We have calculated the contribution of the major three-body term, using single-term wave functions, for several values of ω. The results (for $G^2/4\pi=11.8$) appear in the first column of Table III. Now, the refined trial functions all contain the term $f(1.7)$. No higher value of ωx_0 occurs in any of the functions, for any value of G. For this reason we confine our attention to the range $\omega x_0\leq1.7$. It will be seen that the contribution of the leading term is significant when $\omega x_0\geq1$.

[15] G. Wentzel, Phys. Rev. **91**, 1573 (1953).

One may also use the refined wave functions to calculate the three-body contribution. Since the leading term is repulsive, it will reduce the magnitude of the potential energy. For $G^2/4\pi=11.8$, the potential energy is diminished by 16.9 percent of the two-body contribution, to 12.5 times the measured triton binding energy. The wave function listed in Table I then gives no binding. In fact, if energies are written as multiples of the triton binding energy (8.49 Mev), the kinetic energy exceeds the absolute value of the potential by 1.6.

VI. THREE-BODY TERMS V_{3a}, V_{3b}, AND V_{3c}

We consider, now, the minor three-body terms discussed by Drell and Huang. Following the notation of Drell and Huang, we call these terms V_{3a}, V_{3b}, and V_{3c}.[16]

$$V_{3a}=\lambda^2\frac{\mu^2}{M}\left[\frac{(\tau_2\cdot\tau_3)(\sigma_2\cdot\mathbf{r}_{12})(\sigma_3\cdot\mathbf{r}_{13})}{r_{12}r_{13}}\left(1+\frac{1}{x_{12}}\right)\left(1+\frac{1}{x_{31}}\right)\times\frac{e^{-(x_{12}+x_{31})}}{x_{12}x_{31}}+\text{symmetric terms}\right].$$

Averaging over all orientations of the three-nucleon triangle, we get, for the "central" part of the interaction:

$$V_{3a}{}^c=\lambda^2\frac{\mu^2}{M}\left[\frac{(\tau_2\cdot\tau_3)(\sigma_2\cdot\sigma_3)(\mathbf{r}_{12}\cdot\mathbf{r}_{13})}{3}\left(1+\frac{1}{x_{12}}\right)\left(1+\frac{1}{x_{31}}\right)\times\frac{e^{-(x_{12}+x_{31})}}{x_{12}x_{31}}+\text{symmetric terms}\right].$$

Now $\mathbf{r}_{12}\cdot\mathbf{r}_{13}=(x_{12}^2+x_{31}^2-x_{23}^2)/2x_{12}x_{31}$. Because of the symmetry of the wave function, the terms $x_{12}^2/2x_{12}x_{31}$ and $x_{31}^2/2x_{12}x_{31}$ contribute equally to the expectation value of $V_{3a}{}^c$, which is given by the expression:

$$\langle V_{3a}{}^c\rangle=-\frac{C}{N}\frac{3\lambda^2}{M\mu^4}\int_V[\varphi(12)\varphi(23)\varphi(31)]^2\times\left[\frac{2x_{12}^2-x_{23}^2}{2x_{12}x_{31}}\right]\left(1+\frac{1}{x_{12}}\right)\left(1+\frac{1}{x_{31}}\right)\times e^{-(x_{12}+x_{31})}x_{23}dx_{12}dx_{23}dx_{31}$$

for single-term wave functions. Note that the potential V_{3a} contains two terms, in addition to the term explicitly written. This accounts for the factor three in the above integral.

As in our preceding work, we split the integral into four terms. From tabulated values of the exponential integral, I_0 and I_1 [see Eqs. (4) and (5)] may be evaluated without great difficulty. An accurate computation of I_2 and I_3 would be laborious. Because of the effect of the potential, however, I_2 and I_3 are small compared to I_0 and I_1. Accurate values of I_2 and I_3 are not required. We have computed upper and lower limits for these latter quantities. As an example, we discuss the treatment of the integral I_2.

[16] Drell and Huang give derivations for these terms. The potentials V_{3a} and V_{3b} (not V_{3c} nor V_{3d}, however) have also been derived by A. Klein.[6]

TABLE III. Contributions of the various three-body terms, computed with single term wave functions. Each entry represents a percentage of the two-body potential energy. A negative sign indicates that the potential is attractive. Uncertainty in last column is negligible. These results were computed with $G^2/4\pi=11.8$.

ωx_0	$\langle V_3\rangle$	$\langle V_{3a}{}^c\rangle$	$\langle V_{3b}\rangle$	$\langle V_{3c}{}^c\rangle$	$\langle V_{3d}\rangle$
1.0	6.7	$\sim$	≈-3.2	$\sim$	-9.3
1.2	9.7	≈-2.5	$-6.6\rightarrow-7.0$	$\sim$	-14
1.7	18	$-3.2\rightarrow-3.4$	$-14\rightarrow-15$	$-1.1\rightarrow+1.9$	-30

$$I_2=I_2{}^A-I_2{}^B,$$

$$I_2{}^A=\int_{x_0}^\infty dx_{12}\int_{x_0}^\infty dx_{23}\int_{x_{12}+x_{23}}^\infty[\varphi(12)\varphi(23)\varphi(31)]^2\times\frac{2x_{12}^2}{2x_{12}x_{31}}\left(1+\frac{1}{x_{12}}\right)\left(1+\frac{1}{x_{31}}\right)e^{-(x_{12}+x_{31})}dx_{31},$$

$$I_2{}^B=\int_{x_0}^\infty dx_{12}\int_{x_0}^\infty dx_{23}\int_{x_{12}+x_{23}}^\infty[\varphi(12)\varphi(23)\varphi(31)]^2\times\frac{x_{23}^2}{2x_{12}x_{31}}\left(1+\frac{1}{x_{12}}\right)\left(1+\frac{1}{x_{31}}\right)e^{-(x_{12}+x_{31})}dx_{31}.$$

We may write:

$$I_2{}^A=\int_{x_0}^\infty f(x_{12})dx_{12}\int_{x_0}^\infty g(x_{23})dx_{23}\times\int_{x_{12}+x_{23}}^\infty\frac{1}{x_{31}}\left(1+\frac{1}{x_{31}}\right)(x_{31}-x_0)^2e^{-(\omega+1)x_{31}}dx_{31},$$

where f and g are larger than zero. Let $x_{12}+x_{23}=S$. Consider the integration with respect to x_{31}:

$$\int_S^\infty\frac{1}{x}\left(1+\frac{1}{x}\right)(x-x_0)^2e^{-(\omega+1)x}dx=e^{-(\omega+1)S}\left\{\frac{1-2x_0+S}{\omega+1}+\frac{1}{(\omega+1)^2}\right\}-(\omega x_0{}^2+2x_0)\,\mathrm{Ei}[-(\omega+1)S]+\frac{x_0{}^2e^{-(\omega+1)S}}{S}=A-B+C.$$

Suppose $\omega x_0=1.7$. Direct calculation shows that $A/1.72\leq A-B+C<A$ for $2x_0\leq S<\infty$. If we neglect B and C in computing $I_2{}^A$, we shall get a result which is too large, by a factor 1.72 at most. Thus we get an upper and lower limit for $I_2{}^A$, and similarly, for $I_2{}^B$. The integral I_3 has been treated in the same way.

We find that the expectation value of V_{3a} is negative, and lies between 3.16 and 3.55 percent of the two-body potential energy, for $G^2/4\pi=11.8$ and $\omega x_0=1.7$. A rough calculation shows that $\langle V_{3a}{}^c\rangle\cong 2.5$ percent of the two-body contribution when $\omega x_0=1.2$. Since the term $V_{3a}{}^c$ is small, more accurate values will not be computed.

For the second of the above three "minor" terms we have:

$$V_{3b}=\lambda^3\frac{6\mu^2}{M}\left[\frac{\nabla_{12}\cdot\nabla_{23}+\nabla_{23}\cdot\nabla_{31}+\nabla_{31}\cdot\nabla_{12}}{\mu^2}\right]\times\frac{e^{-(x_{12}+x_{23}+x_{31})}}{x_{12}x_{23}x_{31}},$$

$$V_{3b}=\lambda^3\frac{6\mu^2}{M}\left\{\frac{\mathbf{r}_{12}\cdot\mathbf{r}_{13}}{r_{12}r_{13}}\left(1+\frac{1}{x_{12}}\right)\left(1+\frac{1}{x_{23}}\right)\times\frac{e^{-(x_{12}+x_{23}+x_{31})}}{x_{12}x_{23}x_{31}}+\text{symmetric terms}\right\},$$

$$\langle V_{3b}\rangle=-\lambda^3\frac{18}{M\mu^4}\frac{C}{N}\int_V[\varphi(12)\varphi(23)\varphi(31)]^2\left[\frac{ex_{12}{}^2-x_{31}{}^2}{2x_{12}x_{23}}\right]\times\left(1+\frac{1}{x_{12}}\right)\left(1+\frac{1}{x_{23}}\right)e^{-(x_{12}+x_{23}+x_{31})}dx_{12}dx_{23}dx_{31},$$

for single term wave functions.

The integrals which occur in $\langle V_{3a}{}^c\rangle$ and in $\langle V_{3b}\rangle$ are very similar, and essentially the same procedure may be used to evaluate both quantities. The results of these two calculations, however, are considerably different. One sees from Table III that $\langle V_{3b}\rangle$ and $\langle V_3\rangle$ do not differ greatly in absolute magnitude. In fact $\langle V_{3a}{}^c\rangle$ and $\langle V_{3b}\rangle$, taken together, practically cancel the repulsive contribution of the leading term.

Using the refined wave function, we find that the contribution of V_{3b} is 13.6 percent of the expectation value of the two-body potential, when $G^2/4\pi=11.8$. $V_{3a}{}^c$ contributes about 3 percent as much energy as the central two-body forces. Again, $V_{3a}{}^c$ and V_{3b}, taken together, practically cancel the leading term.

The potential V_{3c} is given by the expression:

$$V_{3c}=\lambda^3\frac{6\mu^2}{\pi^2M}\left[\frac{(\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3)(\boldsymbol{\sigma}_2\cdot\nabla_{23})(\boldsymbol{\sigma}_2\cdot\nabla_{23})}{2}F(x_{12},x_{23})+\text{symmetric terms}\right],$$

where

$$F(x_{12},x_{23})=F=\frac{K_0(x_{23})K_1(2x_{12})}{x_{12}{}^2}+\frac{e^{-x_{23}}}{x_{23}}\frac{1}{x_{12}{}^2}\int_{x_{12}}^{\infty}xK_0{}^2(x)dx=F_1+F_2.$$

After averaging out the tensor term, we find:

$$V_{3c}{}^c=\frac{\lambda^3}{3}\frac{6\mu^2}{\pi^2M}(\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3)(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3)(\nabla_{23}{}^2+\nabla_{31}{}^2+\nabla_{12}{}^2)(F_1+F_2).$$

Replacing the spin and charge operators by their expectation values gives

$$V_{3c}{}^c=-\lambda^3\frac{6\mu^2}{\pi^2M}(\nabla_{23}{}^2+\nabla_{31}{}^2+\nabla_{12}{}^2)(F_1+F_2)=V_1+V_2.$$

We treat the term in F_1 first. Using well-known properties of the cylindrical functions, one can show that:

$$\nabla_{23}{}^2K_0(x_{23})=K_0(x_{23})-(1/x_{23})K_1(x_{23}),$$

$$V_1=-\lambda^3\frac{6\mu^2}{\pi^2M}\left\{K_0(x_{23})-\frac{1}{x_{23}}K_1(x_{23})\right\}\frac{K_1(2x_{12})}{x_{12}{}^2}=\left(\frac{\pi^2}{4}\right)\lambda^3\frac{6\mu^2}{M}\frac{G_2(x_{23})G_1(x_{12})}{x_{23}x_{12}}.$$

Through the analysis of a semilog plot, we get:

$$G_1(x)=\frac{2}{\pi}\frac{K_1(2x)}{x}\cong 0.76e^{-2.642x}+6.0e^{-5.149x}+36e^{-11.61x}=g_1(x).$$

Similarly:

$$G_2(x)=-(2/\pi)x\{K_0(x)-(1/x)K_1(x)\}\cong 4.74e^{-3.660x}=g_2(x)\quad\text{for } 0.38<x<1$$

(with an error of about 5 percent), and

$$G_2(x)<g_2(x)\quad\text{for } x>1.$$

$G_2(x)$ remains positive till $x>1.4$. It is clear, then, that $\langle V_1\rangle>0$. To calculate an upper limit for $\langle V_1\rangle$, substitute g_1 for G_1, and g_2 for G_2. Call the resulting potential v_1. As in previous calculations

$$\langle v_1\rangle=I_0-I_1-I_2-I_3,\quad\langle v_1\rangle<I_0.$$

Evaluating I_0, we find that $\langle v_1\rangle$ is less than 3 percent as large as the magnitude of the two-body potential energy. $\langle V_1\rangle$ must be even smaller.

In order to calculate $\langle V_2\rangle$, we make use of another approximating function derived, again, from a semilog plot:

$$\frac{4}{\pi^2}[xK_0{}^2(x)]\cong 0.528e^{-1.964x}-0.230e^{-4.478x}.$$

After substituting this approximating function into the potential V_2, all the integrations may be carried out exactly. Using the upper and lower limits we have computed for $\langle V_1\rangle$, and the value of $\langle V_2\rangle$ determined by the above procedure, we arrive at the following conclusion: taking $\omega x_0=1.7$, $\langle V_{3c}{}^3\rangle$ may be a repulsive

potential 1.9 percent as strong as the two-body potential (at most), or an attractive potential 1.1 percent as strong as the two-body potential (at most). Since this term is so small, we shall not discuss it further.

VII. V_{3d}

The $ps-ps$ Hamiltonian transforms, by the method of Dyson and Foldy[17,18] into the following expression:

$$H=(g/2M)\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}(\boldsymbol{\tau}\cdot\boldsymbol{\varphi})\rho(r)+(g^2/2M)\varphi^2\rho(r), \quad (13)$$

where ρ is the nucleon density, and φ the meson wave function. In a recent article on many-body forces,[15] Wentzel discusses those potentials which arise from the quadratic, or pair theoretical, term alone. For a system consisting of three nucleons, the pair theoretical potential energy becomes:

$$U_3=-3(8\pi i)^{-1}\oint d\xi\xi^{-\frac{1}{2}} \times\ln[1-(\beta_{12}^2+\beta_{12}^2+\beta_{23}^2)+2\beta_{12}\beta_{23}\beta_{13}].$$

Here β_{ij} is a function of the distance r_{ij}, and of the integration variable ξ (see references 9, 15). Expanding the logarithm in increasing powers of the β's (and each β carries a factor λ) one finds:

$$\begin{aligned}&\ln[1-(\beta_{12}^2+\beta_{13}^2+\beta_{23}^2)+2\beta_{12}\beta_{23}\beta_{13}]\\&\quad=\ln[1+x]=x-\tfrac{1}{2}x^2+\cdots\\&\quad=-(\beta_{12}^2+\beta_{13}^2+\beta_{23}^2)+2(\beta_{12}\beta_{23}\beta_{13})-\tfrac{1}{2}(\beta_{12}^4+\beta_{13}^4\\&\qquad+\beta_{23}^4)-(\beta_{12}^2\beta_{23}^2+\beta_{23}^2\beta_{13}^2+\beta_{13}^2\beta_{12}^2)+\cdots.\end{aligned}$$

Terms not explicitly written contain powers of λ higher than the fourth.

The first and third parentheses above contain terms which give rise to two-body potentials. The term in the second parenthesis corresponds to the leading three-body potential. But the last group of terms leads to a three-body potential which we have not yet discussed. In Wentzel's notation, this potential has the form:

$$V_{3d}=-\frac{1}{2\pi}\mu\left[\frac{\lambda_A}{4\mu}\right]^4\frac{K_1[2(x_{12}+x_{13})]}{r_{12}^2r_{13}^2}+\text{symmetric terms.}$$

Here λ_A represents an "effective" or "reduced" coupling constant, smaller than the actual coupling constant λ_0. This reduction in the effective value of the coupling constant is similar to the damping effect discussed in the introduction. If damping is to be neglected, we must also neglect the reduction factor, and replace λ_A by λ_0, which is equal to $8\pi\lambda/\mu$ in our notation. The potential becomes:

$$V_{3d}=-\frac{16}{2\pi}\lambda^4\frac{K_1[2(x_{12}+x_{13})]}{x_{12}^2x_{13}^2}+\text{symmetric terms.}$$

[17] F. J. Dyson, Phys. Rev. **73**, 929 (1948).
[18] L. L. Foldy, Phys. Rev. **84**, 168 (1951).

This expression contains three terms in all. Because of the symmetry of the wave function we have:

$$\langle V_{3d}\rangle=-\frac{C}{N}\frac{48}{2\pi}\lambda^4\int_V[\varphi(12)\varphi(23)\varphi(31)]^2 \times\frac{K_1[2(x_{12}+x_{13})]}{x_{12}x_{31}}dx_{12}dx_{23}dx_{31},$$

$$(2/\pi)K_1[2(x_{12}+x_{31})]\equiv(2/\pi)K_1[2S] \cong0.610e^{-2.194S}+1.27e^{-4.143S}+5.50e^{-11.75S}.$$

The error is less than 1.5 percent for $S\leq3$. Even when $S=3.5$, the error is only 2.2 percent. We substitute the approximating function for the Hankel function in the integrand above. The integral may now be written in the form:

$$\langle V_{3d}\rangle=-\frac{C}{N}\frac{48}{2\pi}\lambda^4[I_0-I_1-I_2-I_3].$$

I_0 may be expressed in terms of tabulated functions, including the exponential integral. I_1, I_2, and I_3 may be approximated by the same procedure which we have used in connection with the potentials V_{3a} and V_{3b}. We show the results of this calculation in Table III. It will be seen that the expectation values $\langle V_{3d}\rangle$ are greater, in absolute value, than the expectation values of the "leading" three-body term.

Using the refined wave functions, we find that the contribution of the term V_{3d} is 28 percent as large as the two-body potential energy, for $G^2/4\pi=11.8$. Again, the sum of V_3 and V_{3d} is attractive, not repulsive.

VIII. CONCLUSIONS

We have seen that $|\langle V_{3d}\rangle|$ is larger than $|\langle V_3\rangle|$, while $|\langle V_{3b}\rangle|$ is only slightly smaller than $|\langle V_3\rangle|$. This is true for a wide range of values of the parameter ω, if we work with simple, single-term wave functions. We reach the same conclusion if we use the refined wave functions, taking $11.3\leq G^2/4\pi\leq11.8$. Therefore this conclusion seems reasonably secure, despite our uncertainty as to the exact form of the S-state wave function. We should stress, here, that damping has so far been neglected.

In all calculations discussed above we have taken $G^2/4\pi\geq11.3$. Such large coupling constants were used to compensate for our neglect of tensor forces. But we have also computed expectation values for $9.7\leq G^2/4\pi\leq11.3$. This was done by progressively diminishing the coupling constant, while adjusting the wave function parameters to maximize the *ratio* of potential to kinetic energy. For parameter values fixed in this way, the percentage change, in kinetic or potential energy, required to produce binding, is a minimum. We find that decreasing the coupling constant to 9.7 leads to no essential change in the relation between $|\langle V_3\rangle|$, $|\langle V_{3b}\rangle|$, and $|\langle V_{3d}\rangle|$.

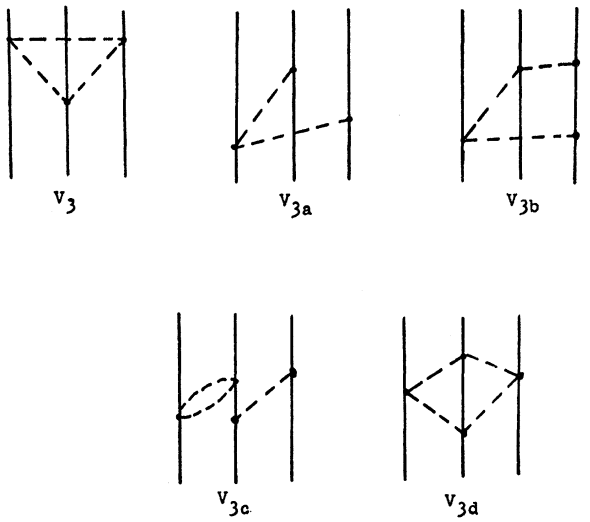

FIG. 2. Feynman diagrams for various three-body terms.

It appears, then, that the character of the "leading" term in the three-body potential gives no clear picture of the properties of the three-body potential as a whole. Since the three-body potential contains an infinite number of terms (some of which may also be large) we cannot even tell whether this potential is attractive or repulsive. The series may, in fact, diverge. There is no reason to believe that the "minor" terms are less important in the other many-body potentials than they are in the three-body potential. We must, then, consider the sign, as well as the form, of all the many-body potentials as uncertain.

At this point we note that the terms V_{3a}, V_{3b}, and V_{3c} have been treated by Drell and Huang in their calculation of the binding energy of heavy nuclei. The expectation values of V_{3a} and V_{3b} are zero, within the framework of approximations adopted by the authors. These expectation values vanish because of the angular dependence of the potentials. In agreement with our results, Drell and Huang find that $\langle V_{3c}\rangle$ is negligible. However, the potential V_{3d} is not considered in their calculation. V_{3d} depends, explicitly, only on the internucleon distances. The corresponding expectation value will not vanish; it may very well be larger, in absolute value, than $|\langle V_3\rangle|$, since this is true for tritium. If the attractive term V_{3d} is included in the three-body potential, we may find that the many-body forces no longer produce saturation.

A rigorous treatment of the damping effect may be expected to change this picture. Indeed, Wentzel's work on pair theory[4,9] indicates that the repulsive many-body terms predominate for large nuclear densities. The pair-theoretical potentials alone exhibit saturation properties, even without a hard core. It appears, then, that the effects produced by higher order terms are an essential feature of the problem of heavy nuclei.

TABLE IV. Effect of damping on three-body potentials. Each entry indicates a percentage of the two-body potential energy.

ωx_0	$\langle V_3\rangle$	$\langle V_{3a}{}^c\rangle$	$\langle V_{3b}\rangle$	$\langle V_{3c}{}^c\rangle$	$\langle V_{3d}\rangle$
1.0	4.2		≈ -4.1		-4.0
1.2	6.4	≈ -5.0	-9.0		-6.3
1.7	13	-6.8	-20	$-1.6 \rightarrow +2.6$	-14

Returning, now, to our discussion of the tritium problem, we find that damping influences the various three-body potentials in different ways. The diagrams in Fig. 2 may serve to classify the three-body potentials. We shall assume that the two-body potential $V^{(4)}$ [see Eq. (12)] carries a damping factor α. Then it follows from Wentzel's discussion[9,15] that every purely pair-theoretical potential (like V_3 or V_{3d}, for example) should be multiplied by $\alpha^{\frac{1}{2}n}$, where n is the number of double vertices in the corresponding diagram. It seems reasonable to suppose, here, that the same procedure will be roughly correct for the other potentials. Comparing the various three-body terms with the leading term, we see that damping increases the relative importance of V_{3a}, V_{3b}, and V_{3c}, while partially suppressing V_{3d}.

To make our argument more concrete, we have again taken $G^2/4\pi=14$, $\alpha=\frac{1}{4}$, and $x_0=0.38$. With these parameters we have computed the expectation values of the various three-body terms, each multiplied by the appropriate power of α. The results of this calculation are shown in Table IV. Increasing the coupling constant partially counteracts the effect of damping. As a result, both $|\langle V_{3b}\rangle|$ and $|\langle V_{3d}\rangle|$ are approximately equal to $|\langle V_3\rangle|$. An increase in core radius will, probably, further decrease the relative importance of V_{3d}, because of the strong r dependence in this term. But, comparing the forms of V_{3b} and V_3, the author sees no reason to believe that an increase in core radius will drastically reduce the importance of the "minor" term, V_{3b}.

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