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Hamiltonian Form of Integral Spin Wave Equations*

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The Hamiltonian forms of the spin-zero and spin-one wave equations are obtained simultaneously by starting from the Duffin-Kemmer form of the equations using only algebraic properties of the β matrices. From the mode of derivation, the Hamiltonian forms for all integral-spin equations of the Dirac-Fierz-Pauli type follow immediately.

I. INTRODUCTION

RECENTLY[1,2] it has been shown that both the vector and scalar wave equations can be put in the Hamiltonian form

$$i\partial\Psi/\partial t=H\Psi, \tag{1}$$

where

$$H=\boldsymbol{\beta}\cdot\mathbf{P}+\kappa\beta_4, \tag{2}$$

and the wave function is subject to the initial condition

$$(H\beta_4-\kappa)\Psi=0. \tag{3}$$

The β_μ form a 10- or 5-dimensional representation of the Duffin-Kemmer algebra depending on whether it is the vector or scalar equation that is being considered.

The deductions of Eq. (1) were obtained starting from special representations of the β_μ. However, since it is seen to hold for both irreducible representations it would seem that it must be possible to obtain this form using only algebraic properties of the β_μ. This is done below. The present method has the advantages of handling spin 0 and 1 simultaneously, of being shorter, and of making the proofs of the properties of the β_μ in Eq. (2) almost trivial. Lastly this present method can be carried over to obtain the Hamiltonian form for all integral-spin equations of the Dirac-Fierz-Pauli[3] type.

II. SPIN ZERO AND ONE

For simplicity we restrict ourselves to field-free equations. The Duffin-Kemmer[4] equations are

$$(\beta_\mu\partial_\mu+\kappa)\Psi=0. \tag{4}$$

Here the β_μ satisfy the relations

$$\beta_\mu\beta_\nu\beta_\rho+\beta_\rho\beta_\nu\beta_\mu=\delta_{\mu\nu}\beta_\rho+\delta_{\rho\nu}\beta_\mu. \tag{5}$$

If we put

$$\partial_4=-i\partial_t, \tag{6}$$

and separate the term involving β_4, Eq. (4) becomes

$$i\beta_4\partial_t\Psi=(\boldsymbol{\beta}\cdot\boldsymbol{\partial}+\kappa)\Psi. \tag{7}$$

Multiplying Eq. (7) first by β_4 and second by $(1-\beta_4^2)$ and using Eq. (A1), we obtain the two equations

$$i\partial_t\beta_4^2\Psi=\beta_4(\boldsymbol{\beta}\cdot\boldsymbol{\partial}+\kappa)\Psi, \tag{8}$$

and

$$(1-\beta_4^2)=-[(1-\beta_4^2)/\kappa](\boldsymbol{\beta}\cdot\boldsymbol{\partial})\Psi. \tag{9}$$

It is seen that only the time derivative of $\beta_4^2\Psi$ is determined by the equations. The components $(1-\beta_4^2)\Psi$ are given in terms of these by means of the supplementary condition (9). However, if we take the time derivative of Eq. (9) and use this as one of our equations along with Eq. (8), we need merely require that Eq. (9) hold as an initial condition. It will then hold at all later times as a consequence of the equations of motion. Thus Eq. (4) can be replaced by the Eqs. (8) and

$$i\partial_t(1-\beta_4^2)\Psi=-[(1-\beta_4^2)/\kappa](\boldsymbol{\beta}\cdot\boldsymbol{\partial})i\partial_t\Psi. \tag{10}$$

Admissible solutions are subject to the *initial* condition of Eq. (9).

Equation (10) can be simplified by writing Ψ on the right as

$$\Psi=\beta_4^2\Psi+(1-\beta_4^2)\Psi. \tag{11}$$

Using Eq. (A2), we see that the second term in (11) does not contribute to the right-hand side of (10) which is then

$$i\partial_t(1-\beta_4^2)\Psi=-[(1-\beta_4^2)/\kappa](\boldsymbol{\beta}\cdot\boldsymbol{\partial})i\partial_t\beta_4^2\Psi. \tag{12}$$

The time derivative of $\beta_4^2\Psi$ may be eliminated from (12) by using Eq. (8). Hence we obtain

$$i\partial_t(1-\beta_4^2)\Psi=-[(1-\beta_4^2)/\kappa](\boldsymbol{\beta}\cdot\boldsymbol{\partial})\beta_4(\boldsymbol{\beta}\cdot\boldsymbol{\partial}+\kappa)\Psi. \tag{13}$$

Equation (A3) shows the term second order in spatial derivatives in (13) vanishes. This then becomes

$$i\partial_t(1-\beta_4^2)\Psi=-(1-\beta_4^2)\boldsymbol{\beta}\cdot\boldsymbol{\partial}\beta_4\Psi. \tag{14}$$

Adding (8) and (14) we obtain

$$i\partial\Psi/\partial t=H\Psi, \tag{15}$$

where

$$H=\boldsymbol{\beta}'\cdot\mathbf{P}+\beta_4'\kappa. \tag{16}$$

Here

$$\mathbf{P}=-i\boldsymbol{\partial},\quad \beta_4'=\beta_4, \tag{17}$$

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[1] E. Schrödinger, Proc. Roy. Soc. (London) **A229**, 39 (1955).

[2] K. M. Case, Phys. Rev. **99**, 1572 (1955).

[3] P. A. M. Dirac, Proc. Roy. Soc. (London) **A155**, 447 (1936); M. Fierz, Helv. Phys. Acta **12**, 3 (1938); M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939). Hereafter this will be abbreviated as D-F-P.

[4] N. Kemmer, Proc. Roy. Soc. (London) **A173**, 91 (1939).

and

$$\beta_k' = -i(1-\beta_4^2)\beta_k\beta_4 + i\beta_4\beta_k, \quad (k=1,2,3). \tag{18}$$

Using (A1) Eq. (18) can be simplified to

$$\beta_k' = i(\beta_4\beta_k - \beta_k\beta_4), \quad (k=1,2,3). \tag{19}$$

Thus the Duffin-Kemmer equation (4) has been put into the Hamiltonian form of Eq. (1) and (2). It is readily verified that the *initial condition* eq. (9) is just Eq. (3) with the H given by (16).

We must still, however, show that the β_μ' satisfy the algebraic relations (5). This is readily done by exhibiting a similarity transformation expressing the β_μ' in terms of the β_μ. Such a transformation is

$$\beta_\mu' = S\beta_\mu S^{-1}, \quad (\mu=1,2,3,4) \tag{20}$$

where

$$S = e^{i\pi\beta_4/2} = (1-\beta_4^2+i\beta_4). \tag{21}$$

We note that if the matrices β_μ are chosen Hermitian, the matrix S is unitary and hence the β_μ' are also Hermitian.

The situation is remarkably similar to that occurring in the case of the Dirac equation. Here we can pass from the covariant form,

$$(\gamma_\mu\partial_\mu+\kappa)\Psi=0, \tag{22}$$

to the Hamiltonian form

$$i\partial\Psi/\partial t = \{\boldsymbol{\gamma}'\cdot\mathbf{P}+\gamma_4'\kappa\}\Psi, \tag{23}$$

by multiplying by γ_4. (Here we have denoted the conventional $(\boldsymbol{\alpha},\beta)$ matrices by $(\boldsymbol{\gamma}',\gamma_4')$ to stress the analogy.) The γ_μ' are then given by

$$\gamma_4' = \gamma_4, \tag{17a}$$

and

$$\gamma_k' = i\gamma_4\gamma_k, \quad (k=1,2,3). \tag{19a}$$

These γ_μ' are related to the γ_μ by the similarity transformation

$$\gamma_\mu' = S\gamma_\mu S^{-1} \quad (\mu=1,2,3,4), \tag{20a}$$

where

$$S = e^{i\pi\gamma_4/4} = (1+i\gamma_4)/\sqrt{2}, \tag{21a}$$

and hence satisfy the same anticommutation relations as the γ_μ.

III. HIGHER INTEGRAL SPINS

It has been shown by Moldauer[5] that the integral-spin equations describing spin k of the D-F-P type can be written in the form

$$(\beta_\rho\partial_\rho+\kappa)\Psi_{\mu_1,\mu_2\cdots\mu_{k-1}}=0, \tag{24}$$

where the function Ψ is subject to the subsidiary condition

$$\beta_\nu\Psi_{\nu\mu_2\cdots\mu_{k-1}}=0. \tag{25}$$

Here the β matrices satisfy the relations (5) and act on a suppressed index. Ψ is symmetric and traceless in the indicated tensor indices. Following precisely the same procedure as in Sec. II, we obtain the Hamiltonian form of these equations:

$$i\partial\Psi_{\mu_1\cdots\mu_{k-1}}/\partial t = \{H\}\Psi_{\mu_1\cdots\mu_{k-1}}, \tag{26}$$

with

$$H = \boldsymbol{\beta}'\cdot\mathbf{P}+\beta_4'\kappa. \tag{27}$$

The admissible solutions are subject to the initial condition

$$(H\beta_4'-\kappa)\Psi_{\mu_1\cdots\mu_{k-1}}=0, \tag{28}$$

and the supplementary condition (25). This last can be expressed in terms of the β' by multiplying by S. Thus

$$0=S\beta_\nu\Psi_{\nu\mu_2\cdots\mu_{k-1}}=S\beta_\nu S^{-1}S\Psi_{\nu\mu_2\cdots\mu_{k-1}}$$

or

$$0=\beta_\nu'S\Psi_{\nu\mu_2\cdots\mu_{k-1}}, \tag{29}$$

where S is given by Eq. (21).

APPENDIX

Some special consequences of the relations (5) that are used are:

$$(\beta_4)^3=\beta_4, \tag{A1}$$

$$(1-\beta_4^2)\boldsymbol{\beta}\cdot\boldsymbol{\partial}(1-\beta_4^2)=0, \tag{A2}$$

$$\boldsymbol{\beta}\cdot\boldsymbol{\partial}\beta_4\boldsymbol{\beta}\cdot\boldsymbol{\partial}=0. \tag{A3}$$

One obtains (A1) by putting $\mu=\nu=\rho=4$ in Eq. (3). To prove the remaining relations, we will let Latin letters denote indices which run only from one to three. The identity

$$(1-\beta_4^2)\beta_i(1-\beta_4^2)=\beta_i-(\beta_4^2\beta_i+\beta_i\beta_4^2)+\beta_4^2\beta_i\beta_4^2 \tag{A4}$$

becomes, on using (A1) and the relation

$$\beta_4^2\beta_i+\beta_i\beta_4^2=\beta_i \tag{A5}$$

[obtained by putting $\mu=\nu=4$, $\rho=i$ in Eq. (5)],

$$(1-\beta_4^2)\beta_i(1-\beta_4^2)=0. \tag{A6}$$

Multiplying by ∂_i and summing yields (A2). Putting $\mu=i$, $\nu=4$, $\rho=k$ in (5) gives

$$\beta_i\beta_4\beta_k+\beta_k\beta_4\beta_i=0. \tag{A7}$$

Multiplying by $\partial_i\partial_k$ and summing yields (A3).

[5] P. A. Moldauer, Ph.D. thesis, University of Michigan, 1955 (unpublished).