Dispersion Relations for Pion-Nucleon Scattering.* I. The Spin-Flip Amplitude

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Dispersion relations are derived for the derivative with respect to $\sin\vartheta$, taken at zero angle, of the spin-flip amplitude for pion-nucleon scattering. The derivation of these relations is based on general field-theoretical concepts. It is shown that the condition of microscopic causality is sufficient and essentially also necessary for the existence of these equations. The special form of the dispersion relations depends on the assumptions about the high-frequency behavior of the spin-flip amplitude.

The exact dispersion formulas, which are derived under the more stringent boundedness condition for the amplitude, can be reduced to the spin-flip part of Low's equations for P-wave scattering. This reduction involves approximations which correspond to those underlying the direct derivation of Low's equations.

Under certain conditions the dispersion relations may hold approximately at low energies even if the causality condition is not valid in small but finite regions. This possibility is discussed briefly.

INTRODUCTION

EXACT dispersion relations for pion-nucleon scattering have been derived so far only for the forward scattering amplitude.[1-3] In this paper we will discuss corresponding relations for a function $S(\omega)$ which is the derivative of the spin-flip amplitude with respect to $\sin\vartheta$ at zero angle.[3a] One can also obtain dispersion relations for higher derivatives of both spin-flip and nonspin-flip amplitudes at $\vartheta=0$; these will be discussed in a following paper.

We derive the dispersion relations on the basis of covariant field theory, but we do not need to make specific assumptions about the form of the interaction between pions and nucleons. The asymptotic condition for field operators is sufficient to find an expression for the scattering amplitude in terms of field operators and to exhibit the energy dependence of the function $S(\omega)$. The essential tool for the derivation of the dispersion relations is the condition of microscopic causality which is assumed here in the following form: two measurements shall be independent of each other if they are performed at points which have a finite space-like separation. It is of course not certain that this principle holds in very small domains, but in order to find out whether this principle is valid we must study its consequences. The special form of the dispersion relations depends on the extent to which the causality condition is violated in infinitesimal space-like regions, because these regions are responsible for the high-frequency behavior of the amplitudes. As a connecting link between dispersion relations and the causality principle, we use in this paper the Titchmarch theorems about Hilbert transforms.[4] These theorem state, roughly speaking, that under certain boundedness conditions the vanishing of the Fourier transform $\mathfrak{M}(\xi_0)$ of a complex function $M(\omega)$ for $\xi_0<0$ is necessary and sufficient for the relation

$$M(\omega+i\nu)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{M(\omega')}{\omega'-(\omega+i\nu)}d\omega'$$

to hold for $\nu>0$ and for the existence of the limit $M(\omega+i\nu)\rightarrow M(\omega)$ for $\nu\rightarrow 0$. In order to make use of these theorems, we use a function $N(\omega)$ which is equal to $S(\omega)/\mathbf{q}^2$ for $\omega\geq\mu$ (μ=pion mass) and prove that the causality condition is sufficient and essentially also necessary for the vanishing of its Fourier transform $\mathfrak{N}(\xi_0)$ for negative values of ξ_0. Thus we obtain for $N(\omega)$ a relation of the type

$$N(\omega)=\frac{1}{\pi i}\mathrm{P}\int_{-\infty}^{+\infty}\frac{N(\omega')}{\omega'-\omega}d\omega',$$

where P denotes the Cauchy principal value. This relation has to be converted into equations involving dispersive and absorptive parts of the physical amplitude. For the region of integration from $-\infty$ to $-\mu$ this is achieved by use of the invariance of the theory under charge conjugation; the region from $-\mu$ to $+\mu$ leads to a contribution resulting from the neutron as a possible intermediate state of the pion-proton system.

In this paper we have restricted ourselves to pion-nucleon scattering, but the results can of course be generalized to other boson-fermion scattering processes.

I. EXPRESSIONS FOR THE SPIN-FLIP AMPLITUDE

The elastic scattering amplitude for pions on nucleons can be written as matrix in spin space in the form

$$T(\omega,\vartheta,\varphi)=F(\omega,\cos\vartheta)+iG(\omega,\cos\vartheta)\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{k}),\quad(1)$$

where $\mathbf{q}=\mathbf{m}(\omega^2-\mu^2)^{\frac{1}{2}}$ is the momentum of the incoming pion and $\mathbf{k}=\boldsymbol{\kappa}(k_0{}^2-\mu^2)^{\frac{1}{2}}$ the momentum of the scattered

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[1] M. L. Goldberger, Phys. Rev. 99, 979 (1955). See also Gell-Mann, Goldberger, and Thirring, Phys. Rev. 95, 1612 (1954); in this paper the existence of dispersion relations for the derivatives of the scattering amplitude with respect to $\cos\vartheta$ has been suggested.

[2] Goldberger, Miyazaea, and Oehme, Phys. Rev. 99, 986 (1955).

[3] R. Karplus and M. A. Ruderman, Phys. Rev. 98, 771 (1955).

[3a] Calculations about this problem have been made independently by W. Thirring; private communication from M. L. Goldberger.

[4] E. C. Titchmarch, *Fourier Integrals* (Oxford University Press, Oxford, 1937), p. 119.

meson. We restrict ourselves in this paper to the laboratory system. By $\mathbf{m}$ and $\boldsymbol{\kappa}$ we denote unit vectors in the corresponding directions. We have $\mathbf{m}\cdot\boldsymbol{\kappa}=\cos\vartheta$ and define in addition the unit vectors $\mathbf{n}$ and $\mathbf{l}$ by $\mathbf{n}\sin\vartheta=\mathbf{m}\times\boldsymbol{\kappa}$ and $\mathbf{l}=\mathbf{n}\times\mathbf{m}$. The functions F and G are matrices in isotopic spin space. We indicate by $S_{\pi^\pm}$, S_{π^0}, etc., the amplitudes corresponding to the reactions $\pi^\pm+p\rightarrow p+\pi^\pm$ and $\pi^0+p\rightarrow p+\pi^0$ respectively. Throughout this paper we consider only the interaction due to nonelectromagnetic forces. The quantity which will be mainly discussed here is defined by

$$S(\omega)=\mathbf{q}^2G(\omega)=\lim_{\vartheta\rightarrow 0}|\mathbf{q}|\,|\mathbf{k}|G(\omega,\cos\vartheta); \qquad (2)$$

for simplicity we call $S(\omega)$ just the spin-flip amplitude. In order to express this function by the full scattering amplitude T, we note that

$$|\mathbf{q}|\,|\mathbf{k}|G(\omega,\cos\vartheta)=(1/2i)\ \mathrm{Tr}\{\boldsymbol{\sigma}\cdot\mathbf{n}T(\omega,\vartheta,\varphi)\}.$$

Because $G(\omega,\cos\vartheta)$ and $|\mathbf{k}|=k(\omega,\cos\vartheta)$ are even functions of ϑ or $\xi\equiv\sin\vartheta$, we find

$$S(\omega)=\frac{1}{2i}\mathrm{Tr}\left\{\boldsymbol{\sigma}\cdot\mathbf{n}\left(\frac{\partial}{\partial\xi}T(\omega,\vartheta,\varphi\right)_{\xi=0}\right\}. \qquad (3)$$

This expression will be useful in the derivation of the dispersion relations.

We can describe the pion field by three linear Hermitian operators $\phi_\alpha(x)$ in Hilbert space. For our purposes it seems to be of some advantage to use the equivalent set

$$\phi_\pm(x)=\{\phi_1(x)\pm i\phi_2(x)\}/\sqrt{2},\quad \phi_0(x)=\phi_3(x).$$

It has been shown by several authors[5] that one can express the scattering amplitude T for charged pions on nucleons in the form

$$T_{\pi^\pm}{}^{fi}(p_f,k;p_1,q)=Ci\int\int d^4xd^4ye^{-ikx}e^{iqy}(\Box_x-\mu^2)(\Box_y-\mu^2)\times\langle p_f|T(\phi_\pm{}^*(x),\phi_\pm(y))|p_i\rangle, \qquad (4)$$

where $|p_i\rangle$ and $|p_f\rangle$ denote initial and final state of the proton with four-momentum p_i and p_f respectively; the spin states are not indicated explicitly. The symbol $T(\phi^*(x),\phi(y))$ stands for the time-ordered product of the field operators and the coefficient C is unity in the laboratory system if we use Gaussian units for the meson field; all operators are in the Heisenberg representation. One obtains a corresponding equation for neutral pions. Equation (4) can be derived on the basis of the well known condition about the asymptotic behavior of field operators. It states, physically speaking, that the particles of the interacting system behave essentially as free particles if one only waits a sufficiently long time. This means that the particles tend to become far separated from each other and do not form stable bound states. For the elastic scattering processes we are dealing with, this seems to be a reasonable assumption. Mathematically the asymptotic condition is usually expressed in the form

$$\lim_{x_0\rightarrow\mp\infty}\phi(x)=\phi_{\mathrm{in,out}}(x),$$

where ϕ_{in} and ϕ_{out} are creation and destruction operators for free physical particles. In the case of pions we have then $(\Box-\mu^2)\phi_{\mathrm{in,out}}(x)=0$.

In addition to the asymptotic condition, we assume invariance of the theory under translations in time and space (this not necessarily), rotations in isotopic spin space, etc., and Lorentz invariance. The essential condition for the derivation of the dispersion relations is the principle of microscopic causality: operators corresponding to physically measurable quantities shall commute at points which are separated by a finite space-like distance. Because all these operators are built up by field operators, this condition is satisfied if all boson field operators commute and all fermion field operators anticommute at space-like points. The causality condition is closely connected with Lorentz invariance, because it guarantees that the time ordered product in Eq. (4) does not depend on the choice of the time axis.

We describe the interaction between pions and nucleons by a current operator $j(x)$ so that

$$\begin{aligned}(\Box-\mu^2)\phi_\pm(x)&=-j_\pm(x),\\ (\Box-\mu^2)\phi_0(x)&=-j_0(x).\end{aligned} \qquad (5)$$

These current operators are functionals of the boson field operators $\phi(x)$ and the nucleon field operators $\psi(x)$, $\bar{\psi}(x)$, and they will in general also contain derivatives of these quantities. For simplicity let us assume in the following that the currents do not depend on time derivatives of the pion field. The generalization of the discussions is straightforward.

We perform now the differentiations in Eq. (4) and obtain, using translation invariance and Eqs. (5),

$$T^{fi}(\omega,\vartheta,\varphi)=i\int_{-\infty}^{+\infty}dy_0e^{-i\omega y_0}\int d^3ye^{i\mathbf{q}\mathbf{y}}\times\left\langle p_f\right|T(j^*(0),j(\mathbf{y},y_0))-\delta(y_0)\left[j^*(0),\frac{\partial\phi}{\partial y_0}(\mathbf{y},y_0)\right]\left|p_i\right\rangle, \qquad (6)$$

where

$$(2\pi)^4\delta(p_i-p_f+q-k)T^{fi}(\omega,\vartheta,\varphi)=T^{fi}(p_f,k;p_i,q).$$

[5] Lehmann, Symanzik, and Zimmerman, Nuovo cimento **1**, 1 (1955); M. L. Goldberger, Phys. Rev. **97**, 508 (1955); F. E. Low, Phys. Rev. **97**, 1392 (1955); Y. Nambu, Phys. Rev. **98**, 803 (1955).

Here and in the following we omit the subscripts $\pi^{\pm}$ and π^0 wherever it is clear what has to be inserted for j and ϕ in order to obtain the special amplitudes. Furtheremore we have in the laboratory system $p_i=(0,M)$ and $p_f=(\mathbf{p},p_0)=(\mathbf{q}-\mathbf{k},\ M+\omega-k_0)$, where

$$|\mathbf{k}|=|\mathbf{q}|\frac{(\mu^2+M\omega)\cos\vartheta+(\omega+M)(M^2-\mu^2\sin^2\vartheta)^{\frac{1}{2}}}{(\omega+M)^2-(\omega^2-\mu^2)\cos^2\vartheta}$$

and $k_0=(\mathbf{k}^2+\mu^2)^{\frac{1}{2}}$.

The spin-flip amplitude $S(\omega)$ can be obtained from Eq. (6) by the prescription given in Eq. (3). Let us first discuss the term proportional to $\delta(y_0)$ in Eq. (6). The causality condition demands that the commutator

$$\left[j^*(0),\frac{\partial\phi}{\partial y_0}(\mathbf{y},0)\right]$$

vanish for $\mathbf{y}^2>0$. Therefore the matrix elements of this commutator can only lead to expressions of the form

$$\Delta^n\delta(\mathbf{y})\langle\mathbf{p},p_0|O(\mathbf{y})|0,M\rangle$$

or

$$\mathbf{grad}\Delta^n\delta(\mathbf{y})\langle\mathbf{p},p_0|\mathbf{O}(\mathbf{y})|0,M\rangle,$$

where Δ is the three-dimensional Laplace operator and $n=0,1,\cdots$. The term proportional to $\Delta^n\delta(\mathbf{y})$ does not contribute to the spin flip amplitude $S(\omega)$. This is evident for $n=0$, because in that case the matrix element cannot contain terms proportional to $\boldsymbol{\sigma}$; it depends only on the one polar vector $\mathbf{p}$ and must be invariant under rotations and space inversions. If $n>0$ we find by use of Eq. (3) for the contributions to $S(\omega)$:

$$\int d^3ye^{i\mathbf{q}\mathbf{y}}\,\mathrm{Tr}\left\{\boldsymbol{\sigma}\cdot\mathbf{n}\left[\frac{\partial}{\partial\xi}\langle\mathbf{p},p_0|O(\mathbf{y})|0,M\rangle\right]_{\xi=0}\right\}\Delta^n\delta(\mathbf{y})$$
$$=\int d^3ye^{i\mathbf{q}\mathbf{y}}f(\mathbf{y}^2)(\mathbf{q}\cdot\mathbf{y})\Delta^n\delta(\mathbf{y})$$
$$=[\Delta^n\{e^{i\mathbf{q}\mathbf{y}}(\mathbf{q}\cdot\mathbf{y})f(\mathbf{y}^2)\}]_{\mathbf{y}=0}=0.$$

Those terms which contain $\mathbf{grad}\Delta^n\delta(\mathbf{y})$ lead to contributions to $S(\omega)$ of the form $q^2q^{2n}\cdot C$, where C is a constant. We will see later that the special form of the dispersion relations depends on the high-frequency behavior of the spin-flip amplitude. It turns out that the terms $q^2q^{2n}\cdot C$ either cannot occur at all because of the boundedness condition, or they drop out in the final dispersion relations. To avoid unnecessary complications we will therefore neglect these terms in the following considerations.

Let us assume now that the positive energy states of the interacting system form a complete set.[6] Then we can decompose the matrix element in Eq. (6) with respect to these states and find, using translation invariance,

$$T^{fi}(\omega,\vartheta,\varphi)=\sum_{n,\mathbf{p}_n=\mathbf{q}}\frac{\langle\mathbf{p},p_0|j^*(0)|n\rangle\langle n|j(0)|0,M\rangle}{E_n-M-\omega-i\epsilon}+\sum_{n,\mathbf{p}_n=-\mathbf{k}}\frac{\langle\mathbf{p},p_0|j(0)|n\rangle\langle n|j^*(0)|0,M\rangle}{E_n-p_0+\omega-i\epsilon}.\quad(7)$$

The numerators are matrices in spin space and must be of the form

$$\sum_{m_n}\langle\mathbf{p},p_0|j^*(0)|m_n,\mathbf{p}_n,E_n\rangle\langle m_n,\mathbf{p}_n,E_n|j(0)|0,M\rangle=a_++ib_+\boldsymbol{\sigma}\cdot(\mathbf{p}\times\mathbf{p}_n),$$
$$\sum_{m_n}\langle\mathbf{p},p_0|j(0)|m_n,\mathbf{p}_n,E_n\rangle\langle m_n,\mathbf{p}_n,E_n|j^*(0)|0,M\rangle=a_-+ib_-\boldsymbol{\sigma}\cdot(\mathbf{p}\times\mathbf{p}_n),\quad(8)$$

where a_+, b_+, a_-, and b_- are functions of $(\mathbf{p}\cdot\mathbf{p}_n)$, p_0, $\mathbf{p}_n^2$, and E_n. $\sum m_n$ denotes the sum over all remaining quantum numbers of an intermediate state with fixed energy E_n and momentum $\mathbf{p}_n$. Inserting Eq. (8) into Eq. (7), we obtain

$$T(\omega,\vartheta,\varphi)=\sum_{E_n}\left\{\frac{a_+(\mathbf{q}^2,\mathbf{q}\cdot\mathbf{k},p_0;E_n)+ib_+(\mathbf{q}^2,\mathbf{q}\cdot\mathbf{k},p_0;E_n)\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{k})}{E_n-M-\omega-i\epsilon}+\frac{a_-(\mathbf{q}^2,\mathbf{q}\cdot\mathbf{k},p_0;E_n)-ib_-(\mathbf{q}^2,\mathbf{q}\cdot\mathbf{k},p_0;E_n)\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{k})}{E_n-p_0+\omega-i\epsilon}\right\}.\quad(9)$$

Here the quantities $a_{\pm}$ and $b_{\pm}$, as well as the energy p_0 of the recoiling nucleon, are even functions of $\xi=\sin\vartheta$. Thus, using Eq. (3) and the relation $(\partial\mathbf{k}|\partial\xi)_{\xi=0}=|\mathbf{q}|\mathbf{b}$, we find for the spin-flip amplitude

$$S(\omega)=\mathbf{q}^2\sum_{E_n}\left\{\frac{b_+(\mathbf{q}^2,E_n)}{E_n-M-\omega-i\epsilon}-\frac{b_-(\mathbf{q}^2,E_n)}{E_n-M+\omega-i\epsilon}\right\}.\quad(10)$$

II. ANALYTIC CONTINUATION

The spin-flip amplitude $S(\omega)$ is *a priori* only defined on the real axis and for $\omega\geq\mu$. In order to derive dispersion relations we want to continue this function analytically into the upper half-plane. More specifically, it is our aim to find a function $N(\lambda)=N(\omega+i\nu)$ such that

(*a*) $N(\omega+i\nu)$ is analytic for $\nu>0$,

(*b*) the Lesbesgue integral $\int_{-\infty}^{+\infty}d\omega|N(\omega+i\nu)|^2$ exists and is bounded for $\nu>0$, (*A*)

(*c*) $\lim_{\nu\to0}N(\omega+i\nu)\to G(\omega)=S(\omega)/\mathbf{q}^2$ for $\omega\geq\mu$.

[6] Naturally we assume also that the one nucleon state is stable.

It has been shown by Titchmarch[4] that the conditions (a) and (b) are necessary and sufficient for the relation

$$N(\omega+i\nu)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{N(\omega')}{\omega'-(\omega+i\nu)}d\omega' \qquad (11)$$

to hold for $\nu>0$ and for the existence of the limit

$$N(\omega+i\nu)\to N(\omega) \quad \text{for} \quad \nu\to 0 \qquad (11a)$$

almost everywhere, i.e., except possibly for a set of measure zero. Performing this limit in Eq. (11), we find that the conditions (A) are necessary and sufficient for the relation

$$N(\omega)=\frac{1}{\pi i}\mathrm{P}\int_{-\infty}^{+\infty}\frac{N(\omega')}{\omega'-\omega}d\omega' \qquad (12)$$

to hold almost everywhere.[7]

Direct continuation of $S(\omega)$ by use of Eq. (10) does certainly not lead to an analytic function in the upper half-plane, because the second term introduces poles slightly above the real axis. But these poles can occur only for $\omega<\mu$ because there are no states of the pion nucleon system with $E_n\leq M-\mu$. The lowest intermediate state is a neutron at rest so that we have always $E_n\geq M_N$. The quantity $-i\epsilon$ in the denominator of the second term in Eq. (10) is therefore irrelevant for the physical region $\omega\geq\mu$. We use this freedom by changing $-i\epsilon$ into $+i\epsilon$ and continue instead of $G(\omega)$ the function

$$N(\omega)=\sum_{E_n}\left\{\frac{b_+(\mathbf{q},{}^2E_n)}{E_n-M-\omega-i\epsilon}-\frac{b_-(\mathbf{q}^2,E_n)}{E_n-M+\omega+i\epsilon}\right\} \qquad (13)$$

into the upper half-plane. This function coincides with $G(\omega)$ in the physical region $\omega\geq\mu$. Writing $N(\omega)$ as a Fourier integral corresponding to Eqs. (3) and (6), we find

$$N(\omega)=\frac{1}{\mathbf{q}^2}\int_{-\infty}^{+\infty}dy_0\eta(-y_0)e^{-i\omega y_0}\int d^3ye^{i\mathbf{q}\mathbf{y}}$$
$$\times\tfrac{1}{2}\mathrm{Tr}\left\{\boldsymbol{\sigma}\cdot\mathbf{n}\left(\frac{\partial}{\partial\xi}\langle\mathbf{p},p_0|[j^*(0),j(\mathbf{y},y_0)]|0,M\rangle\right)_{\xi=0}\right\}. \qquad (14)$$

Here $\eta(-y_0)$ is the step function and defined by

$$\eta(-y_0)=\begin{cases}0 & \text{for } y_0>0\\ 1 & \text{for } y_0<0.\end{cases}$$

Because of the appearance of the commutator and the step function, the space-time integration in Eq. (14) extends only over the region inside and on the past light cone, provided the causality condition holds. We assume in the following that these space and time integrations exist and that they are bounded. For finite ω their existence is guaranteed if the matrix element of the commutator is sufficiently well behaved and bounded at time-like points $\mathbf{y}^2-y_0^2<0$, which is certainly the case for physically reasonable interactions. The high-frequency behavior of $N(\omega)$ depends on the matrix elements at the light cone, which may have there singularities of the form $\delta(\mathbf{y}^2-y_0^2)$ or even derivatives thereof. The causality condition demands that the commutator $[j^*(0),\ j(\mathbf{y},y_0)]$ vanishes for every finite space like distance $\mathbf{y}^2-y_0^2>0$, and therefore only derivatives of finite order can occur. These singularities at the light cone correspond to violations of causality in infinitesimal regions. The order of the pole of $N(\omega)$ at infinity depends on the order of derivatives of $\delta(\mathbf{y}^2-y_0^2)$ occurring at the light cone, which in turn is determined by the properties of the interaction. There may of course occur cancellations due to terms like that involving $\delta(y_0)$ in Eq. (6).

Let us first assume that the interaction is such that $N(\omega)$ is $L^2(-\infty,+\infty)$, i.e., that N is Lesbesgue square-integrable from $-\infty$ to $+\infty$. In this case we can use another theorem of Tichmarch[4] which states that the vanishing of the Fourier transform

$$\mathfrak{N}(\xi_0)=\int_{-\infty}^{+\infty}d\omega e^{-i\omega\xi_0}N(\omega) \qquad (15)$$

for $\xi_0<0$ is necessary and sufficient for the conditions (A) to hold. In order to prove that $\mathfrak{N}(\xi_0)=0$ for $\xi_0<0$ we start from the representation of $N(\omega)$ given in Eq. (14). The matrix element of the commutator can be written as matrix in spin space in the form

$$\langle\mathbf{p},p_0|[j^*(0),j(\mathbf{y},y_0)]|0,M\rangle$$
$$=A(\mathbf{y}^2-y_0^2,\ y_0;\ \mathbf{p}\cdot\mathbf{y},\ p_0)$$
$$+iB(\mathbf{y}^2-y_0^2,\ y_0;\ \mathbf{p}\cdot\mathbf{y},\ p_0)\boldsymbol{\sigma}\cdot(\mathbf{p}\times\mathbf{y}). \qquad (16)$$

Because of $\mathbf{p}(\xi=0)=0$ and $p_0(\xi=0)=M$, we find for the derivative with respect to ξ at $\xi=0$:

$$\left(\frac{\partial}{\partial\xi}\langle\mathbf{p},p_0|[j](0),j(\mathbf{y},y_0)^*|0,M\rangle\right)_{\xi=0}$$
$$=|\mathbf{q}|iB(\mathbf{y}^2-y_0^2,\ y_0;\ 0,M)\times\boldsymbol{\sigma}\cdot(\mathbf{y}\times\mathbf{l}), \qquad (17)$$

where $B(\mathbf{y}^2-y_0^2,\ y_0)$ retains the property of the commutator to vanish for space-like points $\mathbf{y}^2-y_0^2>0$. Insertion of Eq. (17) into Eq. (14) yields

$$N(\omega)=\frac{i}{\mathbf{q}^2}\int_{-\infty}^{+\infty}dy_0\eta(-y_0)e^{-i\omega y_0}\int d^3ye^{i\mathbf{q}\mathbf{y}}$$
$$\times B(\mathbf{y}^2-y_0^2,\ y_0)(\mathbf{y}\cdot\mathbf{q}). \qquad (18)$$

Here we can perform the angle integrations and find,

[7] This function $N(\omega)$ corresponds to the function $M(\omega)$ which Goldberger uses in the case of the forward scattering amplitude (see reference 1).

making use of the causality condition,

$$N(\omega)=(2\pi)^{\frac{3}{2}}\int_0^\infty dy_0 e^{i\omega y_0}\int_0^{y_0} y^4 dy B(y^2-y_0^2,-y_0)\times J_{\frac{3}{2}}(y(\omega^2-\mu^2)^{\frac{1}{2}})/[y(\omega^2-\mu^2)^{\frac{1}{2}}]^{\frac{3}{2}}, \quad (19)$$

where $y\equiv|\mathbf{y}|$. To calculate the Fourier transform $\mathfrak{N}(\xi_0)$, we may use either Eq. (18) or Eq. (19) and interchange the ω integration with the space-time integrations. This interchange is certainly permitted because $N(\omega)$ has been assumed to be $L^2(-\infty,+\infty)$. By use of Eq. (19), we obtain

$$\mathfrak{N}(\xi_0)=(2\pi)^{\frac{3}{2}}\int_0^\infty dy_0\int_0^{y_0} y^4 dy B(y^2-y_0^2,-y_0)\times\int_{-\infty}^{+\infty} d\omega e^{i\omega(y_0-\xi_0)}\frac{J_{\frac{3}{2}}(y(\omega^2-\mu^2)^{\frac{1}{2}})}{[y(\omega^2-\mu^2)^{\frac{1}{2}}]^{\frac{3}{2}}}. \quad (20)$$

The ω integration can be performed by using, for instance, the Laplace transformation,

$$J_\nu(z)=\frac{z^\nu}{2^\nu 2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-\nu-1}\exp(t-z^2/4t)dt, \quad c>0,$$

and yields

$$\int_{-\infty}^{+\infty} d\omega e^{i\omega(y_0-\xi_0)}\frac{J_{\frac{3}{2}}(y(\omega^2-\mu^2)^{\frac{1}{2}})}{[y(\omega^2-\mu^2)^{\frac{1}{2}}]^{\frac{3}{2}}} =\begin{cases} 0 & \text{for } |y_0-\xi_0|>y, \\ \dfrac{(2\pi)^{\frac{1}{2}}}{\mu y^3}[(y_0-\xi_0)^2-y^2]^{\frac{1}{2}}J_1(\mu[(y_0-\xi_0)^2-y^2]^{\frac{1}{2}}) & \text{for } |y_0-\xi_0|<y. \end{cases} \quad (21)$$

If $y_0\geq 0$ and $\xi_0<0$ the integral (21) is zero for $y_0\geq y$ and therefore it vanishes just for the whole region of the space and time integrations in Eq. (20). Thus we have $\mathfrak{N}(\xi_0)=0$ for $\xi_0<0$; the causality condition is sufficient for the statements (A) to hold and consequently for the validity of the dispersion relation (12). We cannot prove directly that it is necessary because this depends on the specific behavior of the function $B(y^2-y_0^2,-y_0)$ for $y^2-y_0^2>0$. But suppose B does not vanish in a space-like region of nonzero measure. In this case we have for $\xi_0<0$

$$\mathfrak{N}(\xi_0)=(2\pi)^2\int_0^\infty dy_0\int_{y_0-\xi_0}^\infty y dy B(y^2-y_0^2,-y_0)\times i/\mu[y^2-(y_0-\xi_0)^2]^{\frac{1}{2}}J_1(i\mu[y^2-(y_0-\xi_0)^2]^{\frac{1}{2}}), \quad (22)$$

and it is difficult to believe that one find a physically reasonable function B such that the integral (22) vanishes for all $\xi_0<0$. In this sense we say that the causality condition is also necessary for the vanishing of the Fourier transform of $N(\omega)$ for negative ξ_0. If the behavior of the function B at spacelike points should be such that $\mathfrak{N}(\xi_0)=0$ for $\xi_0<-a$, where a is a positive constant, then $N(\omega)$ is still the limit as $\nu\rightarrow 0+$ of an analytic function $N(\omega+i\nu)$, but we have instead of $(Ab)^4$:

$$\int_{-\infty}^{+\infty}|N(\omega+i\nu)|^2 d\omega=O(e^{2a\nu}) \quad \text{for } \nu\rightarrow\infty. \quad (23)$$

In fact the condition $\mathfrak{N}(\xi_0)=0$ for $\xi_0<-a$ is necessary and sufficient for $N(\omega+i\nu)$ to have an essential singularity at infinity. Under these circumstances the dispersion relation (12) holds for the functions $e^{i\alpha\omega}N(\omega)$ where $\alpha>a$:

$$e^{i\alpha\omega}N(\omega)=\frac{1}{\pi i}\mathrm{P}\int_{-\infty}^{+\infty} d\omega'\frac{e^{i\alpha\omega'}N(\omega')}{\omega'-\omega}. \quad (23\text{a})$$

So far we have restricted our discussion to functions $N(\omega)$ which are $L^2(-\infty,+\infty)$. Actually the connection between the validity of the conditions (A) [and consequently of the dispersion relation (12)] and the vanishing of the Fourier transform is more general. Of course the function $N(\omega)$ must be sufficiently bounded to guarantee the convergence of the integral in Eq. (12).

Let us suppose now that the interaction is such that $N(\omega)$ is not sufficiently bounded for $\omega\rightarrow\infty$. Then we can go through the same considerations as given above if we use a function

$$K(\omega)=N(\omega)\Big/\prod_{n=1}^{K}(\omega-\omega_n), \quad (24)$$

with $\mathrm{Im}\,\omega_n<0$, provided we have supplied sufficient powers of ω in the denominator. In the following we will consider only the case that

$$K_2(\omega)=N(\omega)/(\omega-\omega_1)(\omega-\omega_2)$$

is $L^2(-\infty,+\infty)$. Instead of Eq. (12) we obtain the dispersion relation

$$\frac{N(\omega)}{(\omega-\omega_1)(\omega-\omega_2)}=\frac{1}{\pi i}\mathrm{P}\int_{-\infty}^{+\infty}\frac{N(\omega')}{(\omega'-\omega)(\omega'-\omega_1)(\omega'-\omega_2)}d\omega'. \quad (24\text{a})$$

Again here the square integrability could be too strong a condition and it may be sufficient to assume that the integral in Eq. (24a) exists.

III. THE PHYSICAL DISPERSION RELATIONS

The dispersion relations (12) and (24a) involve the function $N(\omega)$ on the whole real axis, whereas *a priori* it has a physical meaning only for $\omega\geq\mu$. Invariance of the theory under charge conjugation will enable us to interpret $N(\omega)$ for $\omega\leq-\mu$ by physically meaningful quantities. The region $|\omega|<\mu$ will yield a "bound state contribution" containing an undetermined constant which can be related to the coupling constant of the Chew-Low theory.[8] This constant is therefore essentially a measure for the strength of the P-wave pion-nucleon interaction.

By use of Eq. (13) we divide $N(\omega)$ into a dispersive and an absorptive part:

$$N(\omega)=D(\omega)+iA(\omega). \tag{25}$$

For charged mesons D and A are given by

$$\begin{aligned}D_{\pi^\pm}(\omega)&=\sum_{E_n}\left\{\mathrm{P}\frac{b_\pm(\mathbf{q}^2,E_n)}{E_n-M-\omega}-\mathrm{P}\frac{b_\mp(\mathbf{q}^2,E_n)}{E_n-M+\omega}\right\},\\ A_{\pi^\pm}(\omega)&=\pi\sum_{E_n}\{b_\pm(\mathbf{q}^2,E_n)\delta(E_n-M-\omega)\\&\qquad+b_\mp(\mathbf{q}^2,E_n)\delta(E_n-M+\omega)\},\end{aligned} \tag{26}$$

where the numerators can be defined according to Eqs. (3) and (8) by

$$\begin{aligned}b_+(\mathbf{q}^2,E_n)&=\frac{-i}{2\mathbf{q}^2}\mathrm{Tr}\left\{\boldsymbol{\sigma}\cdot\mathbf{n}\left[\frac{\partial}{\partial\xi}\sum_{mn}\langle\mathbf{p},p_0|j^*(0)|m_n,\,\mathbf{p}_n=\mathbf{q},\,E_n\rangle\langle m_n,\,\mathbf{p}_n=\mathbf{q},\,E_n|j(0)|0,M\rangle\right]_{\xi=0}\right\},\\ b_-(\mathbf{q}^2,E_n)&=\frac{i}{2\mathbf{q}^2}\mathrm{Tr}\left\{\boldsymbol{\sigma}\cdot\mathbf{n}\left[\frac{\partial}{\partial\xi}\sum_{mn}\langle\mathbf{p},p_0|j(0)|m_n,\,\mathbf{p}_n=-\mathbf{k},\,E_n\rangle\langle m_n,\,\mathbf{p}_n=-\mathbf{k},\,E_n|j^*(0)|0,M\rangle\right]_{\xi=0}\right\}.\end{aligned} \tag{27}$$

From Eq. (26) we find immediately the relations

$$D_{\pi^+}(-\omega)=-D_{\pi^-}(\omega),\quad A_{\pi^+}(-\omega)=+A_{\pi^-}(\omega). \tag{28}$$

For neutral mesons we have to replace the current operators $j(0)$ and $j^*(0)$ in Eq. (27) by the Hermitian operator $j_0(0)=j_3(0)$ and it can be easily shown that we obtain in this case $b_+=b_-=b_0$. Thus we find for neutral pions the symmetry relations

$$D_{\pi^0}(-\omega)=-D_{\pi^0}(\omega),\quad A_{\pi^0}(-\omega)=+A_{\pi^0}(\omega). \tag{28a}$$

It is clear that the relations (28) and (28a) are intrinsically a consequence of the invariance of the theory under charge conjugation.[9] The functions b_+, b_-, and b_0 and therefore also the dispersive and the absorptive part are real functions. This will be shown in the appendix.

From Eq. (28) we see that the dispersive and absorptive parts of the combinations $\frac{1}{2}(N_{\pi^+}(\omega)\pm N_{\pi^-}(\omega))$ are even or odd functions of ω. Therefore we write for charged pions instead of Eq. (12)

$$\tfrac{1}{2}\{N_{\pi^+}(\omega)\pm N_{\pi^-}(\omega)\}=\frac{1}{\pi i}\mathrm{P}\int_{-\infty}^{+\infty}\frac{\frac{1}{2}\{N_{\pi^+}(\omega')\pm N_{\pi^-}(\omega')\}}{\omega'-\omega}d\omega', \tag{29}$$

and by use of the Eqs. (25), (28), and (28a), this leads to the relations

$$\begin{aligned}&\tfrac{1}{2}\{D_{\pi^+}(\omega)+D_{\pi^-}(\omega)\}+\tfrac{1}{2}i\{A_{\pi^+}(\omega)+A_{\pi^-}(\omega)\}\\&\quad=\frac{2\omega}{\pi}\mathrm{P}\int_0^\infty\frac{\frac{1}{2}\{A_{\pi^+}(\omega')+A_{\pi^-}(\omega')\}}{\omega'^2-\omega^2}d\omega'-\frac{2i}{\pi}\mathrm{P}\int_0^\infty\frac{\frac{1}{2}\{D_{\pi^+}(\omega')+D_{\pi^-}(\omega')\}\omega'}{\omega'^2-\omega^2}d\omega',\\&\tfrac{1}{2}\{D_{\pi^+}(\omega)-D_{\pi^-}(\omega)\}+\tfrac{1}{2}i\{A_{\pi^+}(\omega)-A_{\pi^-}(\omega)\}\\&\quad=\frac{2}{\pi}\mathrm{P}\int_0^\infty\frac{\frac{1}{2}\{A_{\pi^+}(\omega')-A_{\pi^-}(\omega')\}\omega'}{\omega'^2-\omega^2}d\omega'-\frac{2i\omega}{\pi}\mathrm{P}\int_0^\infty\frac{\frac{1}{2}\{D_{\pi^+}(\omega')-D_{\pi^-}(\omega')\}}{\omega'^2-\omega^2}d\omega',\\&D_{\pi^0}(\omega)+iA_{\pi^0}(\omega)\\&\quad=\frac{2\omega}{\pi}\mathrm{P}\int_0^\infty\frac{A_{\pi^0}(\omega')}{\omega'^2-\omega^2}d\omega'-\frac{2i}{\pi}\mathrm{P}\int_0^\infty\frac{D_{\pi^0}(\omega')\omega'}{\omega'^2-\omega^2}d\omega'.\end{aligned} \tag{30}$$

These relations involve only integrations over positive values of ω. We can split every one of the Eqs. (30) into two separate relations, one for the even and one for the odd part. Thus we obtain relations which express dispersive parts by integrals involving absorptive parts and vice versa. In the remainder of this paper we will be interested only in the first-mentioned formulas, which resemble the well-known Kramers-Kronig dispersion relations for the forward scattering amplitude.

It remains to discuss the region $0<\omega<\mu$, where the pion nucleon system has an intermediate state corresponding to a single neutron. Because of charge conservation only the matrix elements $\langle\mathbf{p},p_0|j_+(0)|n\rangle$ and $\langle n|j_-(0)|0,M\rangle$ in Eqs. (27) are different from zero in this case. Therefore b_+(neutron) vanishes for charged pions. If we equate neutron and proton mass, we obtain

[8] G. F. Chew and F. E. Low, Fifth Rochester Conference on High Energy Physics, 1955 (Interscience Publishers, Inc., New York, 1955).

[9] We might mention that invariance under charge conjugation also yields $S_{\pi^\pm,p}(\omega)=S_{\pi^\mp,[p]}(\omega)$, where $S_{\pi,[p]}$ is the spin flip amplitude for the process $\pi^\pm$+antiproton→$\pi^\pm$+antiproton. (We use the notation [p] for antiproton.) Therefore we can write instead of Eqs. (28)

$$D_{\pi^+,p}(-\omega)=-D_{\pi^+,[p]}(\omega),\quad A_{\pi^+,p}(-\omega)=+A_{\pi^+,[p]}(\omega).$$

from Eq. (27)

$$b_-(\text{neutron})=b_-(\mathbf{q}^2,\, E_n=(\mathbf{q}^2+M^2)^{\frac{1}{2}}),$$

because in this case conservation of momentum demands that $E_n=(\mathbf{p}_n{}^2+M^2)^{\frac{1}{2}}=(\mathbf{k}^2+M^2)^{\frac{1}{2}}$ and for $\xi=0$ we have $\mathbf{k}(\xi=0)=\mathbf{q}$. Furthermore conservation of energy yields $\mathbf{q}^2=-\mu^2+(\mu^2/2M)^2$ and consequently

$$E_n-M=-\mu^2/2M.$$

Thus we obtain for the absorptive parts in the energy region $0<\omega<\mu$:

$$A_{\pi^+}(\omega)=\pi b_-(\mu,M)\delta(\omega-\mu^2/2M),\quad A_{\pi^-}(\omega)=0, \tag{31}$$

and the corresponding considerations for neutral mesons lead to

$$A_{\pi^0}(\omega)=2\pi b_0(\mu,M)\delta(\omega-\mu^2/2M), \tag{31a}$$

where we have $2b_0=b_-$ because of charge independence. If we now introduce Eqs. (31) and (31a) into Eqs. (30), we find the physical dispersion relations for the forward spin-flip amplitude $S(\omega)$ divided by $\mathbf{q}^2$:

$$\begin{aligned}
\tfrac{1}{2}\{D_{\pi^+}(\omega)+D_{\pi^-}(\omega)\}
&=\frac{2\omega}{\pi}\mathrm{P}\int_\mu^\infty \frac{\frac{1}{2}\{A_{\pi^+}(\omega')+A_{\pi^-}(\omega')\}}{\omega'^2-\omega^2}d\omega' +\frac{\omega}{\omega^2-(\mu^2/2M)^2}\frac{1}{\mu^2}\Gamma_s,\\
\tfrac{1}{2}\{D_{\pi^+}(\omega)-D_{\pi^-}(\omega)\}
&=\frac{2}{\pi}\mathrm{P}\int_\mu^\infty \frac{\frac{1}{2}\{A_{\pi^+}(\omega')-A_{\pi^-}(\omega')\}\omega'}{\omega'^2-\omega^2}d\omega' +\frac{1}{\omega^2-(\mu^2/2M)^2}\frac{1}{2M}\Gamma_s,\\
D_{\pi^0}(\omega)&=\frac{2\omega}{\pi}\int_\mu^\infty \frac{A_{\pi^0}(\omega')}{\omega'^2-\omega^2}d\omega'+\frac{\omega}{\omega^2-(\mu^2/2M)^2}\frac{1}{\mu^2}\Gamma_s;
\end{aligned} \tag{32}$$

here we have introduced the constant

$$\Gamma_s\equiv-\mu^2 b_-(\mu,M).$$

We will show now that for pseudoscalar mesons this constant must be positive. Let $|\mathbf{p}_n,E_n\rangle$ denote a free neutron state and $|\mathbf{p},p_0\rangle$ a free proton state (positive energy). Because of the invariance of the theory under Lorentz transformations the matrix element of the pseudoscalar operator $j_-(0)$ must be of the form

$$\begin{aligned}
\langle\mathbf{p}_n,E_n|j_-(0)|\mathbf{p},p_0\rangle&=\bar{u}(\mathbf{p}_n)H_-(p_\mu,p_{n\nu})u(\mathbf{p})\\
&=\bar{u}(\mathbf{p}_n)\{\gamma_5h_1+i\gamma_5\gamma_\mu(p_{n\mu}-p_\mu)h_2\\
&+i\gamma_5\gamma_\mu(p_{n\mu}+p_\mu)h_3+i\gamma_5(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)p_{n\mu}p_\nu h_4\}u(\mathbf{p}),
\end{aligned} \tag{33}$$

where the quantities h_k are invariant functions of $p_{n\mu}p_\mu$, $p_{n\mu}{}^2$, $p_\mu{}^2$, and where $u(\mathbf{p})$, $\bar{u}(\mathbf{p}_n)$ are solutions of the free Dirac equation for positive energy. Calculating the matrix elements of the γ operators in Eq. (33) we find that $\langle\mathbf{p}_n,E_n|j_-(0)|\mathbf{p},p_0\rangle$ can be written:

$$\begin{aligned}
\langle\mathbf{p}_n,E_n|j_-(0)|\mathbf{p},p_0\rangle&=i\sqrt{2}g(\mathbf{p}_n{}^2,\mathbf{p}^2,\mathbf{p}\cdot\mathbf{p}_n,E_n,p_0)\\
&\times\frac{1}{(2M)^2}\boldsymbol{\sigma}\cdot[\mathbf{p}_n(p_0+M)-\mathbf{p}(E_n+M)].
\end{aligned} \tag{33a}$$

If we now insert Eq. (33a) into Eq. (27) for $b_-(\mathbf{q}^2,E_n)$ and perform the differentiation and the trace we obtain with $\mathbf{q}^2=-\mu^2+(\mu^2/2M)^2$:

$$\begin{aligned}
\Gamma_s=-\mu^2b_-(\mu,M)&=\frac{2\mu^2}{(2M)^2}\left\{|g[\mathbf{q}^2,0,0,(\mathbf{q}^2+M^2)^{\frac{1}{2}},M]|^2\times\frac{M+(\mathbf{q}^2+M^2)^{\frac{1}{2}}}{2M}\right\}_{\mathbf{q}^2=-\mu^2+(\mu^2/2M)^2}\\
&=\frac{2\mu^2}{(2M)^2}|g|^2[1-(\mu/2M)^2]=2f^2[1-(\mu/2M)^2].
\end{aligned} \tag{33b}$$

The constant f^2 can be interpreted directly as the coupling constant of the Chew-Low theory and therefore is of the order 0.08.[8]

The coefficient appearing in the bound state contributions to the dispersion relations for the forward scattering amplitude[2] can be defined by

$$\Gamma_f=\left\{\frac{\mu^2}{2\mathbf{q}^2}\mathrm{Tr}\langle 0,M|j_-^*(0)|\mathbf{p}_n=\mathbf{q},E_n\rangle\langle\mathbf{p}_n=\mathbf{q},E_n|j_-(0)|0,M\rangle\right\}_{\mathbf{q}^2=-\mu^2+(\mu^2/2M)^2}, \tag{34}$$

and this gives by use of Eq. (33a)

$$\begin{aligned}
\Gamma_f&=\frac{2\mu^2}{(2M)^2}|g(\mathbf{q}^2,0,0,(\mathbf{q}^2+M^2)^{\frac{1}{2}},M)|^2{}_{\mathbf{q}^2=-\mu^2+(\mu^2/2M)^2}\\
&=2f^2.
\end{aligned} \tag{34a}$$

Thus we have for pseudoscalar mesons $\Gamma_s=\Gamma_f$ if we neglect in Eq. (33b) the term $(\mu/2M)^2$ against one; we conclude that there appear essentially the same coefficients in the bound state terms of the dispersion relations for the spin flip amplitude and in the corresponding terms of the dispersion relations for the forward scattering amplitude.

In case $N(\omega)$ is not sufficiently bounded for Eq. (12) to be valid we have to rely on dispersion relations of the type given in Eq. (24a). In order to reduce this formula to physical dispersion relations we choose $\omega_1=\omega_0-i\epsilon$ and $\omega_2=-\omega_0-i\epsilon$, where $\omega_0\geq\mu$ and $\epsilon>0$ is a small real constant which finally shall go to zero. By use of the relation

$$\frac{1}{(\omega'-\omega_0+i\epsilon)(\omega'+\omega_0+i\epsilon)}=\mathrm{P}\frac{1}{\omega'^2-\omega_0^2}-\frac{\pi i}{2\omega_0}\{\delta(\omega'-\omega_0)-\delta(\omega'+\omega_0)\}$$

we can write Eq. (24a) in the form

$$N(\omega)-\tfrac{1}{2}\{N(\omega_0)+N(-\omega_0)\}-\frac{\omega}{\omega_0}\tfrac{1}{2}\{N(\omega_0)-N(-\omega_0)\}=\frac{\omega^2-\omega_0^2}{\pi i}\mathrm{P}\int_{-\infty}^{+\infty}\frac{N(\omega')(\omega'+\omega)}{(\omega'^2-\omega^2)(\omega'^2-\omega_0^2)}d\omega'. \quad (35)$$

From Eq. (35) we can obtain physical dispersion relations using the same considerations which led to Eq. (32). We find

$$\tfrac{1}{2}\{D_{\pi^+}(\omega)+D_{\pi^-}(\omega)\}-\frac{\omega}{\omega_0}\tfrac{1}{2}\{D_{\pi^+}(\omega_0)+D_{\pi^-}(\omega_0)\}=2\omega\frac{\omega^2-\omega_0^2}{\pi}\mathrm{P}\int_\mu^\infty\frac{\frac{1}{2}\{A_{\pi^+}(\omega')+A_{\pi^-}(\omega')\}}{(\omega'^2-\omega^2)(\omega'^2-\omega_0^2)}d\omega'-\frac{\omega(\omega^2-\omega_0^2)}{[\omega^2-(\mu^2/2M)^2][\omega_0^2-(\mu^2/2M)^2]}\frac{1}{\mu^2}\Gamma_s,$$

$$\tfrac{1}{2}\{D_{\pi^+}(\omega)-D_{\pi^-}(\omega)\}-\tfrac{1}{2}\{D_{\pi^+}(\omega_0)-D_{\pi^-}(\omega_0)\}=2\frac{\omega^2-\omega_0^2}{\pi}\mathrm{P}\int_\mu^\infty\frac{\frac{1}{2}\{A_{\pi^+}(\omega')-A_{\pi^-}(\omega')\}\omega'}{(\omega'^2-\omega^2)(\omega'^2-\omega_0^2)}d\omega'-\frac{\omega^2-\omega_0^2}{[\omega^2-(\mu^2/2M)^2][\omega_0^2-(\mu^2/2M)^2]}\frac{1}{2M}\Gamma_s, \quad (36)$$

$$D_{\pi^0}(\omega)-\frac{\omega}{\omega_0}D_{\pi^0}(\omega_0)=2\omega\frac{\omega^2-\omega_0^2}{\pi}\mathrm{P}\int_\mu^\infty\frac{A_{\pi^0}(\omega')}{(\omega'^2-\omega^2)(\omega'^2-\omega_0^2)}d\omega'-\frac{\omega(\omega^2-\omega_0^2)}{[\omega^2-(\mu^2/2M)^2][\omega_0^2-(\mu^2/2M)^2]}\frac{1}{\mu^2}\Gamma_s.$$

If Eqs. (32) are valid, then Eqs. (35) are certainly correct, but the reverse is not true.

IV. DISCUSSION AND CONNECTION WITH LOW'S EQUATIONS

By adding and subtracting the first two relations of Eq. (32) we can obtain dispersion relations for the individual amplitudes D_{π^+} and D_{π^-}. If we use instead of $G(\omega)$ the forward spin-flip amplitude $S(\omega)$ and write

$$S_{\pi^\pm}(\omega)=\mathbf{q}^2G_{\pi^\pm}(\omega)=D_{\pi^\pm}{}^{(s)}(\omega)+iA_{\pi^\pm}{}^{(s)}(\omega), \quad (37)$$

we find the dispersion relations

$$D_{\pi^+}{}^{(s)}(\omega)=\frac{\omega^2-\mu^2}{\pi}\mathrm{P}\int_\mu^\infty\frac{d\omega'}{\omega'^2-\mu^2}\left\{\frac{A_{\pi^+}{}^{(s)}(\omega')}{\omega'-\omega}-\frac{A_{\pi^-}{}^{(s)}(\omega')}{\omega'+\omega}\right\}+2f^2\frac{1}{\mu^2}\frac{\omega^2-\mu^2}{\omega-\mu^2/2M},$$
$$D_{\pi^-}{}^{(s)}(\omega)=\frac{\omega^2-\mu^2}{\pi}\mathrm{P}\int_\mu^\infty\frac{d\omega'}{\omega'^2-\mu^2}\left\{\frac{A_{\pi^-}{}^{(s)}(\omega')}{\omega'-\omega}-\frac{A_{\pi^+}{}^{(s)}(\omega')}{\omega'+\omega}\right\}+2f^2\frac{1}{\mu^2}\frac{\omega^2-\mu^2}{\omega+\mu^2/2M}. \quad (38)$$

In these formulas we have written the bound state term with the coefficient f^2, which can be identified with the coupling constant of Chew and Low. Note that Eqs. (38) are written in the laboratory system.

In the center-of-mass system the spin-flip amplitudes $S_{\pi^\pm}(\omega)$ can be easily expressed in terms of phase shifts. We have

$$S_{\pi^+}(\omega)=S_{\frac{3}{2}}(\omega)=\sum_{l=1}^\infty\frac{l(l+1)}{2}\{a^l{}_{3,\,2l+1}(\omega)-a^l{}_{3,\,2l-1}(\omega)\},$$
$$S_{\pi^-}(\omega)=\tfrac{1}{3}\{S_{\frac{3}{2}}(\omega)+2S_{\frac{1}{2}}(\omega)\}=\sum_{l=1}^\infty\frac{l(l+1)}{2}\tfrac{1}{3}\{a^l{}_{3,\,2l+1}(\omega)-a^l{}_{3,\,2l-1}(\omega)+2a^l{}_{1,\,2l+1}(\omega)-2a^l{}_{1,\,2l-1}(\omega)\}, \quad (39)$$

where $S_{\frac{3}{2}}$ and $S_{\frac{1}{2}}$ denote the amplitudes for isotopic spin $\frac{3}{2}$ and $\frac{1}{2}$ respectively. The quantities $a^l{}_{2\tau,\,2j}(\omega)$ are defined in terms of phase shifts by

$$a^l{}_{2\tau,\,2j}(\omega)=\frac{\sin\delta_{\tau l}{}^j(\omega)\exp[i\delta_{\tau l}{}^j(\omega)]}{(\omega^2-\mu^2)^{\frac{1}{2}}}\cdot\frac{W^2(\omega)}{M^2}, \quad (40)$$

where $W(\omega)=(M^2+\mu^2+2M\omega)^{\frac{1}{2}}$; as before ω is the total energy of the incoming pion in the laboratory system. At higher energies the phase shifts are of course complex because of the additional channels describing pion production, nucleon pair production, etc. In order to obtain from the dispersion relations (38) approximate equations for very low energies which involve only P waves, we make the following assumptions:

(1) We can neglect all inelastic process; (2) We can neglect all contributions from phase shifts with $l>1$. (3) We can neglect the recoil of the proton ($M\rightarrow\infty$). Under these restrictions the phase shifts become real,

and we obtain the approximate relations:

$$\tfrac{1}{2}\{\sin 2\alpha_{33}(\omega)-\sin 2\alpha_{31}(\omega)\}\cdot W^2(\omega)/M^2$$
$$=\frac{q^3}{\pi}\mathrm{P}\int_\mu^\infty \frac{d\omega'}{q'^3}\left\{\frac{\sin^2\alpha_{33}(\omega')-\sin^2\alpha_{31}(\omega')}{\omega'-\omega}-\frac{\sin^2\alpha_{33}(\omega')-\sin^2\alpha_{31}(\omega')+2\sin^2\alpha_{13}(\omega')-2\sin^2\alpha_{11}(\omega')}{3(\omega'+\omega)}\right\}\cdot\frac{W^2(\omega')}{M^2}+2f^2\frac{q^3}{\mu^2\omega}, \quad (41)$$
$$\tfrac{1}{6}\{\sin 2\alpha_{33}(\omega)-\sin 2\alpha_{31}(\omega)+2\sin 2\alpha_{13}(\omega)-2\sin 2\alpha_{11}(\omega)\}\cdot W^2(\omega)/M^2$$
$$=\frac{q^3}{\pi}\mathrm{P}\int_\mu^\infty \frac{d\omega'}{q'^3}\left\{\frac{\sin^2\alpha_{33}(\omega')-\sin^2\alpha_{32}(\omega')+2\sin^2\alpha_{13}(\omega')-2\sin^2\alpha_{11}(\omega')}{3(\omega'-\omega)}-\frac{\sin^2\alpha_{33}(\omega')-\sin^2\alpha_{31}(\omega')}{\omega'+\omega}\right\}\cdot\frac{W^2(\omega')}{M^2}+2f^2\frac{\mu^2\omega}{q^3},$$

where $q=(\omega^2-\mu^2)^{\frac{1}{2}}$. These are essentially the spin-flip parts of Low's equations.[10] The nonspin-flip parts can be obtained directly from the dispersion relations for the derivative of the nonspin flip amplitude $F(\omega,\cos\vartheta)$ [see Eq. (1)] with respect to $\cos\vartheta$ at zero angle,[11] provided we make the corresponding approximations. Thus we obtain four approximare relations for the four P-wave amplitudes, and by addition and subtraction we can find four relations for the individual amplitudes which are the same as Eqs. (3.11) of reference 10 if we set $\alpha_{31}=\alpha_{13}$ and $W/M=1$. The imaginary parts of Low's equations are of course identities. We conclude that, apart from the approximations stated above, the Low equations are a consequence of the general assumptions leading to the exact dispersion relations. They are certainly approximate equations, because there do not exist exact dispersion relations for amplitudes corresponding to individual angular momenta.[11] If we make the approximations (1), (2), and (3) in the dispersion relations for the forward scattering amplitude we obtain equations involving S and P waves.[2] The P-wave parts are essentially the nonspin-flip parts of Low's equations which have been obtained separately from the dispersion relations for $[\partial F(\omega,\cos\vartheta)/\partial\cos\vartheta]_{\vartheta=0}$. Thus we find by subtraction approximate equations for S waves only, but the inhomogeneous terms of these relations contain the two zero energy scattering lengths.[11]

Let us finally discuss the possibility that the condition of microscopic causality is not valid in very small but finite spacelike regions. Suppose the commutator $[j^*(0),j(\mathbf{y},y_0)]$ vanishes only if $\mathbf{y}^2-y_0^2>l_0^2$, where l_0 is a finite length. In this case we have according to Eq. (22)

$$\mathfrak{M}(\xi_0)=0 \quad \text{for} \quad \xi_0<-l_0,$$

and, if $N(\omega)$ is sufficiently bounded on the real axis, and relation (23a) holds with $\alpha\geq l_0$. Splitting $N(\omega)$ as before into a dispersive and an absorptive part we obtain for neutral mesons, instead of the dispersion relations (30):

$$D_{\pi^0}(\omega)\cos l_0\omega-A_{\pi^0}(\omega)\sin l_0\omega=\frac{2\omega}{\pi}\mathrm{P}\int_0^\infty d\omega'\frac{\{A_{\pi^0}(\omega')\cos l_0\omega'+D_{\pi^0}(\omega')\sin l_0\omega'\}}{\omega'^2-\omega^2},$$
$$A_{\pi^0}(\omega)\cos l_0\omega+D_{\pi^0}(\omega)\sin l_0\omega=-\frac{2}{\pi}\mathrm{P}\int_0^\infty d\omega'\frac{\omega'\{D_{\pi^0}(\omega')\cos l_0\omega'-A_{\pi^0}(\omega')\sin l_0\omega'\}}{\omega'^2-\omega^2}, \quad (42)$$

and corresponding equations for charged pions. We see that the first of the Eqs. (42) reduces approximately to the usual dispersion relation if l_0 and $\omega(\geq\mu)$ are such that $\omega l_0\ll 1$, and if the functions $A(\omega)$ and $D(\omega)$ decrease fast enough with increasing energy to allow the approximate replacement of $\cos l_0\omega$ by one and $\sin l_0\omega$ by zero inside the integral. Even for very low energies these approximations are certainly not possible if l_0 is of the order of the pion Compton wavelength. Therefore we cannot expect the dispersion relations to be approximately correct if the dimensions of the acausal region are of the order of 10^{-13} cm.

In the above discussion we have restricted ourselves to the spin flip amplitude, but the corresponding considerations can be made for the forward-scattering amplitude. In this case we know that the dispersion relations are in fairly good agreement with experiments up to about 200 Mev in the laboratory system.[12] We may conclude therefore that, if acausal domains exist at all, one should expect that their dimensions are small compared to the Compton wavelength of the pion.

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APPENDIX

In this appendix we will show that dispersive and absorptive parts of the amplitude $N(\omega)$, which were defined in Eqs. (25) and (26), are real functions of ω. More generally we prove the corresponding statement for the function

$$N(\omega,\vartheta)=\frac{1}{qk}\int_{-\infty}^{+\infty}dy_0\eta(-y_0)e^{-i\omega y_0}\int dy^3e^{i\mathbf{q}\cdot\mathbf{y}}\times\tfrac{1}{2}\sum_{m_s,m_s'}\{(\boldsymbol{\sigma}\cdot\mathbf{n})_{m_s,m_s'}\langle\Psi(\mathbf{p},p_0;m_s')|[j^\dagger(0),j(\mathbf{y},y_0)]|\Psi(0,M;m_s)\rangle\}, \quad (A1)$$

[10] F. E. Low, Phys. Rev. **97**, 1392 (1955).
[11] This problem will be discussed in a forthcoming paper. The dispersion relations for the derivatives of $F(\omega,\cos\vartheta)$ can be only derived under neglect of recoil.
[12] Anderson, Davidon, and Kruse, Phys. Rev. **100**, 339 (1955).

which is equal to $G(\omega, \cos\vartheta)\sin\vartheta$ for $\omega\geq\mu$. If we write $\eta(-y_0)=\frac{1}{2}[1-\epsilon(y_0)]$ and

$$N(\omega,\vartheta)=D(\omega,\vartheta)+iA(\omega,\vartheta), \tag{A2}$$

then the dispersive part D and the absorptive part A are given by Eq. (A1) with $\eta(-y_0)$ replaced by $-\frac{1}{2}\epsilon(y_0)$ and $-\frac{1}{2}i$ respectively. For the complex conjugate of $N(\omega,\vartheta)$ we find, using invariance under space inversions,

$$N^*(\omega,\vartheta)=\frac{1}{qk}\int_{-\infty}^{+\infty} dy_0\tfrac{1}{2}[1+\epsilon(y_0)]e^{-i\omega y_0}\int d^3y e^{i\mathbf{q}\cdot\mathbf{y}} \times\tfrac{1}{2}\sum_{m_s,m_s'}\{(\boldsymbol{\sigma}\cdot\mathbf{n})_{m_s',m_s}\langle\Psi^*(-\mathbf{p},p_0;m_s')|[j^\dagger(0),j(\mathbf{y},-y_0)]^*|\Psi^*(0,M;m_s)\rangle; \tag{A3}$$

note that we distinguish here explicitly between Hermitian conjugation (†) and complex conjugation (*). Now we invoke the invariance of the theory under time reversal or inversion of motion. In order to define this transformation, we follow S. Watanabe[13] and introduce a unitary operator $\mathcal{R}$ such that

$$Q(\mathbf{x},-x_0)=\rho_R(\mathcal{R}^{-1}Q(\mathbf{x},x_0)\mathcal{R})^T,$$

where Q represents operators corresponding to physically measurable quantities and ρ_R is a sign function. State vectors transform according to

$$\Psi^I=\mathcal{R}^T\Psi^*; \tag{A4}$$

here Ψ^I represents the state of inversed motion corresponding to Ψ.

For the boson current operators j appearing in Eq. (A3), we find,[12] with an arbitrary phase factor $e^{i\beta}$,

$$(\mathcal{R}^{-1}j(0)\mathcal{R})^T=e^{i\beta}j^\dagger(0),\quad (\mathcal{R}^{-1}j^\dagger(\mathbf{y},y_0)\mathcal{R})^T=e^{-i\beta}j(\mathbf{y},-y_0). \tag{A5}$$

Thus we can write by use of Eq. (A5) and the relation $\mathcal{R}^T\mathcal{R}^{-1}=1$:

$$[j^\dagger(0),j(\mathbf{y},-y_0)]^*=\mathcal{R}^{-1T}[j^\dagger(0),j(\mathbf{y},y_0)]\mathcal{R}^T. \tag{A6}$$

Introducing Eq. (A6) into Eq. (A3) yields by use of Eq. (A4)

$$N^*(\omega,\vartheta)=\frac{1}{qk}\int_{-\infty}^{+\infty} dy_0\tfrac{1}{2}[1+\epsilon(y_0)]e^{-i\omega y_0}\int dy^3 e^{i\mathbf{q}\cdot\mathbf{y}} \times\tfrac{1}{2}\sum_{m_s,m_s'}\{(\boldsymbol{\sigma}\cdot\mathbf{n})_{m_s',m_s}\langle\Psi^I(-\mathbf{p},p_0;m_s')|[j^\dagger(0),j(\mathbf{y},y_0)]|\Psi^I(0,M;m_s)\rangle. \tag{A7}$$

In order to perform the time inversion of the free single nucleon states Ψ, we write

$$\Psi(\mathbf{p},p_0;m_s)=g^\dagger(\mathbf{p},m_s)\Omega_{\text{vac}}, \tag{A8}$$

where $g^\dagger(\mathbf{p},m_s)$ is the corresponding creation operator which transforms like[12]

$$(\mathcal{R}^{-1}g(\mathbf{p},+\tfrac{1}{2})\mathcal{R})^T=e^{i\alpha}g^\dagger(-\mathbf{p},-\tfrac{1}{2}),\quad (\mathcal{R}^{-1}g(\mathbf{p},-\tfrac{1}{2})\mathcal{R})^T=-e^{i\alpha}g^\dagger(-\mathbf{p},+\tfrac{1}{2}). \tag{A9}$$

From Eq. (A9) we find for the motion inversed state

$$\Psi^I(\mathbf{p},p_0;m_s)=\mathcal{R}^Tg^T(\mathbf{p},m_s)\Omega_{\text{vac}}^*=\frac{m_s}{|m_s|}e^{i\alpha}\Psi(-\mathbf{p},p_0;-m_s). \tag{A10}$$

Application of Eq. (A10) in Eq. (A7) yields finally

$$N^*(\omega,\vartheta)=\frac{1}{qk}\int_{-\infty}^{+\infty} dy_0\tfrac{1}{2}(-1-\epsilon(y_0))e^{-i\omega y_0}\int d^3y e^{i\mathbf{q}\cdot\mathbf{y}} \times\tfrac{1}{2}\sum_{m_s,m_s'}\{(\boldsymbol{\sigma}\cdot\mathbf{n})_{m_s',m_s}\langle\Psi(\mathbf{p},p_0;m_s)|[j^\dagger(0),j(\mathbf{y},y_0)]|\Psi(0,M;m_s')\rangle, \tag{A11}$$

because the trace involves only off-diagonal matrix elements of the commutator in spin space. By comparison with Eq. (A1), we find that $D(\omega,\vartheta)$ and $A(\omega,\vartheta)$ are real functions.

[13] S. Watanabe, Revs. Modern Phys. **27**, 40 (1955); see also G. Lüders, Z. Physik **133**, 325 (1952), and Kgl. Danske Videnskab. Selskab, Mat-fys. Medd. **28**, No. 5 (1954), F. Coester, Phys. Rev. **89**, 619 (1955); these papers contain further references.