Scattering of Pions by Nucleons in Intermediate Coupling*

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An intermediate-coupling method of calculation is applied to the meson-nucleon scattering problem for the case of symmetric pseudoscalar mesons, coupled to a fixed extended source through derivative coupling. It is found that the experimentally observed P-wave phase shifts can be explained by taking the coupling constant $f^2=0.712$ and the cutoff $\omega_{\max}=6.21$ meson masses. (This corresponds to a renormalized coupling constant $f_r^2=0.105$.)

I. INTRODUCTION

RECENT experimental evidence[1] on the scattering of mesons from nucleons indicates that in the energy range 50–200 Mev, the cross section can be completely accounted for by considering the P-wave state of isotopic spin $\frac{3}{2}$ and angular momentum $\frac{3}{2}$, plus small contributions from S-wave phase shifts. In order to understand this simple behavior, Chew[2] has investigated the consequences of a symmetrical pseudoscalar meson theory. In this model the meson field interacts with a fixed extended nucleon through pseudovector coupling. In the weak-coupling limit, the predictions of this model are in complete contradiction with experiment. However, using a method of approximation similar to that of Tamm and Dancoff, Chew was able to show that for stronger couplings, the experimental P-wave phase shifts are explainable. One may understand this by recalling, that in the strong-coupling calculations of Pauli and Dancoff[3] the first isobar state is the $J=I=\frac{3}{2}$ state. Chew's results indicate that while the coupling is not strong enough for the existence of a stable isobar, it is sufficient to give a resonance at approximately 190 Mev. In view of this it seems desirable to investigate the above model by using a method that, unlike the Tamm-Dancoff treatment, does not limit the number of mesons in the field. With this in mind, we have applied a previously described[4] intermediate-coupling method to the meson-nucleon scattering problem.

The problem divides into two parts. The first is the solution for the ground state (i.e., physical nucleon) using the Tomonaga variational method.[5] The details of this procedure are discussed in Sec. III. The scattering state is then obtained by again using a variational technique. Here, we construct a trial function by multiplying the ground-state wave function by a scattering function for the meson. The determination of this scattering function and its consequences are discussed in Sec. IV.

Our results seem to substantiate the principal findings of Chew. The δ_{33} phase shift agrees with experiment, if the coupling constant $f^2=0.712$ and the cutoff $\omega_{\max}=6.21\mu$, where μ is the meson mass. (This corresponds to a renormalized constant $f_r^2=0.105$.) The δ_{31} and δ_{11} phase shifts are small, being, e.g., $-1.60°$ and $5.99°$ respectively at 200 Mev.

II. THE HAMILTONIAN

We describe the interaction of pions and nucleons by using the following Hamiltonian:

$$H=\frac{1}{2}\int\sum_{\alpha=1}^{3}[\pi_\alpha^2+(\nabla\phi_\alpha)^2+\mu^2\phi_\alpha^2]d^3r + (4\pi)^{\frac{1}{2}}(f/\mu)\int U(r)\tau_\alpha(\sigma\cdot\nabla\phi_\alpha)d^3r, \quad (1)$$

where the ϕ_α and π_α are the three Hermitean field variables and conjugate momenta describing the mesons. $U(r)$ is a spherically symmetric, normalized source function. Since only P-wave mesons will interact with the nucleon in this model, it is convenient to expand π_α and ϕ_α in spherical waves. Thus:

$$\phi_\alpha(r)=\sum_{k,i}(2\omega_k)^{-\frac{1}{2}}\psi_{i,k}[a_{i\alpha}(k)+a_{i\alpha}^*(k)] + S\text{-waves}+D\text{-waves}+\cdots,$$

and

$$\pi_\alpha(r)=-\sum_{k,i}(\tfrac{1}{2}\omega_k)^{\frac{1}{2}}\psi_{i,k}i[a_{i\alpha}(k)-a_{i\alpha}^*(k)] + S\text{-waves}+D\text{-waves}+\cdots, \quad (2)$$

where

$$\psi_{i,k}=(3/2\pi R)^{\frac{1}{2}}(\gamma_i/kr^3)(\sin kr-kr\cos kr), \quad i=1,2,3 \quad (3)$$

corresponding to the three P-wave functions normalized in a sphere of radius R, with k as the magnitude of the meson momentum and $\omega_k=(k^2+\mu^2)^{\frac{1}{2}}$ the meson energy. $a_{i\alpha}(k)$ and $a_{i\alpha}^*(k)$ represent the annihilation and creation operators respectively.

Substituting Eq. (2) into Eq. (1), we obtain for the

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[1] M. Glicksman, Phys. Rev. **95**, 1045 (1954); de Hoffmann, Metropolis, Alei, and Bethe, Phys. Rev. **95**, 1586 (1954); Bodansky, Sachs, and Steinberger, Phys. Rev. **93**, 1367 (1954).

[2] G. F. Chew, Phys. Rev. **95**, 285 (1954); **95**, 1669 (1954).

[3] W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942).

[4] T. D. Lee and R. Christian, Phys. Rev. **94**, 1760 (1954).

[5] S. Tomonaga, Progr. Theoret. Phys. (Japan) **2**, 6 (1947).

P-wave part of the Hamiltonian:

$$H=\sum_{k,i,\alpha}\{\omega_k a_{i\alpha}^*(k)a_{i\alpha}(k)+(f/\mu)(3\omega_k R)^{-\frac{1}{2}}\times u(k)k^2\sigma_i\tau_\alpha[a_{i\alpha}(k)+a_{i\alpha}^*(k)]\},\quad (4)$$

where

$$u(k)=\int U(r)\exp(i\mathbf{k}\cdot\mathbf{r})d^3r.\quad (5)$$

The four ground states of this Hamiltonian correspond to the physical nucleon with its four possible values of spin and isotopic spin. The scattering of a meson by a nucleon will be described by the first excited states.[6]

III. PHYSICAL NUCLEON

The ground-state wave function will be obtained by using a variational method, similar to one introduced by Tomonaga[5] for the charged scalar field. In this Tomonaga method, one takes advantage of the Bose-Einstein statistics obeyed by mesons, by assuming a trial function in which there is no limitation on the total number of mesons, but in which all mesons are required to have the same radial distribution.

Let $|N_\rho\rangle$ be the state vector of a physical nucleon in a spin, isotopic spin state ρ ($\rho=1,\cdots,4$). In the Fock representation of such a state the probability amplitude for finding a bare nucleon in a spin, isotopic spin state ρ' and $n_{i\alpha}$ mesons of the i,α type with a radial momentum distribution $k_1^{i\alpha},\cdots k_{n_{i\alpha}}^{i\alpha}$ is assumed to be

$$\langle k_1^{i\alpha},\cdots k_{n_{i\alpha}}^{i\alpha};\rho'|N_\rho\rangle=C_\rho(\rho',n_{i\alpha})\prod_{i,\alpha,m}f(k_m^{i\alpha}),\quad (6)$$

where i and α run from 1 to 3 signifying the possible angular momentum and isotopic angular momentum states of the mesons. Invariance under rotations in the ordinary space and isotopic spin space requires that the radial distribution $f(k_m^{i\alpha})$ be the same for all i and α. The function $f(k)$ is chosen to be normalized to unity.

$$\sum_k|f(k)|^2=1$$

where the sum extends over the magnitude of k only.

The functional form $f(k)$ and the constants $C_\rho(\rho',n_{i\alpha})$ will now be determined by the variational method:

$$\delta\langle N_\rho|H-E_0|N_\rho\rangle=0,\quad (7)$$

with E_0 as the self-energy of the nucleon. To carry out this variation we shall take advantage of the special functional form assumed in Eq. (6). It can be shown[7] that it is convenient to use a reduced Hamiltonian $\mathcal{H}$ instead of the original H in the variation. This reduced Hamiltonian is obtained by replacing $a_{i\alpha}(k)$ and $a_{i\alpha}^*(k)$ in the Hamiltonian H, Eq. (4), by

$$a_{i\alpha}(k)\rightarrow f(k)a_{i\alpha},$$

and

$$a_{i\alpha}^*(k)\rightarrow f^*(k)a_{i\alpha}^*,\quad (8)$$

where the $a_{i\alpha}$ and $a_{i\alpha}^*$ are independent of k and obey the usual commutation relations for annihilation and creation operators. We thus obtain for the reduced Hamiltonian[8]:

$$H\rightarrow\mathcal{H}=\Omega a_{i\alpha}^*a_{i\alpha}+(G/\sqrt{2})\sigma_i\tau_\alpha(a_{i\alpha}+a_{i\alpha}^*),\quad (9)$$

where

$$\Omega=\sum_k\omega_k|f(k)|^2,$$

and

$$G=f(2/3R)^{\frac{1}{2}}\sum_k[f(k)u(k)k^2/(\omega_k)^{\frac{1}{2}}\mu].$$

The best values of the constants $C_\rho(\rho',n_{i\alpha})$ of Eq. (6) are then directly given by the ground state $|\mathfrak{N}_\rho\rangle$ of the reduced Hamiltonian $\mathcal{H}$. Furthermore, the value of the self-energy E_0 is precisely the corresponding eigenvalue, given by

$$\mathcal{H}|\mathfrak{N}_\rho\rangle=E_0|\mathfrak{N}_\rho\rangle,\quad (10)$$

where $\rho=1,\cdots,4$ representing four degenerate spin, isotopic spin ground states.

By minimizing E_0 with respect to an arbitrary functional form $f(k)$, we obtain

$$f(k)=-[fu(k)k^2\langle\mathfrak{N}_\rho|\sigma_i\tau_\alpha a_{i\alpha}|\mathfrak{N}_\rho\rangle]\times[\mu(3R\omega)^{\frac{1}{2}}(\omega+\lambda)\langle\mathfrak{N}_\rho|a_{i\alpha}^*a_{i\alpha}|\mathfrak{N}_\rho\rangle]^{-1},\quad (11)$$

where λ is a constant determined by the normalization condition on $f(k)$.

In the limit of $f\rightarrow 0$ and $f\rightarrow\infty$, Eq. (10) can be easily solved. As pointed out by Harlow and Jacobsohn,[9] the values of E_0 in these two limits are identical with that calculated by the rigorous weak- and strong-coupling methods using the original Hamiltonian H. In order to solve Eq. (10) for an intermediate value of f, it is convenient to introduce the canonically conjugate variables $x_{i\alpha}$ and $p_{i\alpha}$, given by

$$x_{i\alpha}=(a_{i\alpha}+a_{i\alpha}^*)/\sqrt{2},$$

and

$$p_{i\alpha}=(a_{i\alpha}-a_{i\alpha}^*)/\sqrt{2}i.\quad (12)$$

Equation (10) then becomes

$$(p_{i\alpha}^2+x_{i\alpha}^2+\bar{f}\sigma_i\tau_\alpha x_{i\alpha})|\mathfrak{N}_\rho\rangle=\epsilon|\mathfrak{N}_\rho\rangle,\quad (13)$$

where $\bar{f}=2G/\Omega$ and $\epsilon=2(E_0/\Omega)+9$. The state vector $|\mathfrak{N}_\rho\rangle$ is seen to be a spinor with 4 components each of which is a function of nine variables. The problem is thus reduced to the solution of one involving nine coupled harmonic oscillators.

Further simplification can be achieved by making use of the symmetry properties of Eq. (13). In order to study this question we introduce the following rotation operators. The components of the orbital angular momentum of the mesons are given by

$$L_i=x_{j\alpha}p_{k\alpha}-x_{k\alpha}p_{j\alpha};\quad i,j,k\text{ in cyclic order}\quad (14)$$

with the total angular momentum

$$L^2=L_i^2.\quad (15)$$

[6] For the scattering problem we restrict ourselves to those values of f for which there are no stable isobar states.

[7] See T. D. Lee and D. Pines, Phys. Rev. **92**, 883 (1953).

[8] In Eq. (8) as well as in the following, we shall use the contraction convention with respect to the indices i and α such that a sum is required whenever the index i or α appears twice.

[9] F. Harlow and B. Jacobsohn, Phys. Rev. **93**, 333 (1954).

The components of the isotopic angular momentum of the mesons are similarly given by

$$T_\alpha = x_{i\beta}p_{i\gamma} - x_{i\gamma}p_{i\beta}; \quad \alpha, \beta, \gamma \text{ in cyclic order} \tag{16}$$

with the total isotopic angular momentum

$$T^2 = T_\alpha^2. \tag{17}$$

The components of the total angular momentum J_i and total isotopic angular momentum I_α are then

$$J_i = L_i + \tfrac{1}{2}\sigma_i \tag{18}$$

and

$$I_\alpha = T_\alpha + \tfrac{1}{2}\tau_\alpha.$$

One may readily verify that J^2, J_3, I^2, I_3 commute with the Hamiltonian $\mathcal{H}$. The ground state $|\mathfrak{N}_p\rangle$ corresponds to a state with $J = I = \frac{1}{2}$. Thus, it is only necessary to consider meson clouds with T and L equal to zero and one.

Since L and T are themselves each the sum of three individual angular momentum operators, it is interesting to find that their eigenfunctions are restricted to a few simple classes of functions. It is useful to define the quantities s_1, s_2 and s_3 as follows:

$$s_1 = x_{i\alpha}x_{i\alpha},$$

$$s_2 = \epsilon_{ijk}\epsilon_{\alpha\beta\gamma}x_{i\alpha}x_{j\beta}x_{k\gamma},$$

and

$$s_3 = x_{i\alpha}x_{j\beta}x_{i\beta}x_{j\alpha}, \tag{19}$$

where ϵ_{ijk} and $\epsilon_{\alpha\beta\gamma}$ are the usual isotropic antisymmetric tensors of third rank [i.e., $\epsilon_{ijk} = (+1, -1)$ for an (even, odd) permutation of the indices, and is zero otherwise]. It is easy to see that s_1, s_2, and s_3 are three functionally independent eigenfunctions with $L = T = 0$. In fact, one can prove that the most general form of any eigenfunction of L and T with L and T equal to 0 or 1 can be expressed in terms of these quantities s_i and their derivatives. This is stated by the following theorems (proved in Appendix I).

Theorem I.—If $S(x_{i\alpha})$ is an eigenfunction of L and T with $L = T = 0$, then

$$S = S(s_1, s_2, s_3). \tag{20}$$

Theorem II.—If $V_j(x_{i\alpha})$ $(j = 1,2,3)$ are eigenfunctions with $L = 1$, $T = 0$, then

$$V_j = 0; \tag{21}$$

similarly for eigenfunctions with $L = 0$ and $T = 1$.

Theorem III.—If $\mathcal{V}_{j\beta}(x_{i\alpha})$ are eigenfunctions with $L = T = 1$, then

$$\mathcal{V}_{j\beta}(x_{i\alpha}) = \sum_{\lambda=1}^{3} \left(\frac{\partial s_\lambda}{\partial x_{j\beta}}\right) S^\lambda(s_1, s_2, s_3), \tag{22}$$

where the S^λ are three scalar functions.

With the aid of the above theorems, and the requirement that the wave function be an eigenfunction of $I = J = \frac{1}{2}$, we may immediately express the physical nucleon in terms of the bare nucleon as

$$|\mathfrak{N}_p\rangle = \left[S^0 + \sum_{\lambda=1}^{3} S^\lambda \left(\frac{\partial s_\lambda}{\partial x_{i\alpha}}\right)\sigma_i\tau_\alpha\right]|n_p\rangle, \tag{23}$$

where $|n_p\rangle$ is the bare nucleon spinor. The S^λ $(\lambda = 0, 1, 2, 3)$ are scalar functions. They are determined by substituting (23) into (10). This yields four coupled partial differentials equations in three independent variables s_1, s_2, s_3. Unfortunately, these equations still appear to be too complicated for analytic solution, and even for numerical solution by present electronic computing machines. Therefore, in the spirit of a variational calculation, we restrict ourselves to the subset of functions given by limiting the S^λ in Eq. (23) to be functions of the nine-dimensional radial variable s_1 only.[10] With this approximation, the equations that determine the S^λ now become four coupled differential equations in one variable.

For practical purposes, it is convenient to choose a set of orthogonal angular functions which are linear combinations of the $\partial s_\lambda / \partial x_{i\alpha}$. They are:

$$Y_{i\alpha}^1 = \left(\frac{9!}{2^9 4! \pi^4}\right)^{\frac{1}{2}} \frac{x_{i\alpha}}{r},$$

$$Y_{i\alpha}^2 = \left(\frac{11\times 9!}{2^{10} 4! \pi^4}\right)^{\frac{1}{2}} \left(\frac{1}{6}\right) \left(\frac{\partial s_2}{\partial x_{i\alpha}}\right), \tag{24}$$

$$Y_{i\alpha}^3 = \left(\frac{13\times 9!}{2^9 4! \pi^4}\right)^{\frac{1}{2}} \left(\frac{11}{10}\right) \left(\frac{1}{4r^3}\frac{\partial s_3}{\partial x_{i\alpha}} - \frac{7}{11}\frac{x_{i\alpha}}{r}\right),$$

where

$$r = (x_{i\alpha}x_{i\alpha})^{\frac{1}{2}}. \tag{25}$$

These satisfy the following orthonormal relations on a unit sphere in the nine-dimensional $x_{i\alpha}$ space:

$$\int Y_{i\alpha}^\lambda Y_{j\beta}^{\lambda'} d^8\Omega = \delta_{\lambda\lambda'}\delta_{ij}\delta_{\alpha\beta}, \tag{26}$$

where $d^8\Omega$ is the solid angle in this unit sphere. The state vector now becomes:

$$|\mathfrak{N}_p\rangle = [F_0(r) + \sum_{\lambda=1}^{3} F_\lambda(r) Y_{i\alpha}^\lambda \sigma_i\tau_\alpha]|n_p\rangle. \tag{27}$$

The differential equations for the F_λ are then obtained by applying the Rayleigh-Ritz variational pro-

[10] In Appendix I, we introduce the three scalar functions Q_1, Q_2, and Q_3. If the four partial differential equations referred to above are written in terms of these variables, then the resulting system of equations exhibits cubic symmetry in the Q_1, Q_2, Q_3 space. This fact was pointed out by Stuart P. Lloyd in private communication.

Since $s_1 = Q_1^2 + Q_2^2 + Q_3^2$, the dependence of S^λ on s_1 only essentially assumes that, for this problem, it is a reasonable approximation to replace cubically symmetric functions by spherically symmetric ones. The best form of the latter is determined by a variational principle.

cedure to Eq. (10). Thus:

$$-\frac{1}{r^8}\frac{d}{dr}\left(r^8\frac{dF_0}{dr}\right)+r^2F_0+\bar{f}rF_1=\epsilon F_0$$

$$-\frac{1}{r^8}\frac{d}{dr}\left(r^8\frac{dF_1}{dr}\right)+\frac{8}{r^2}F_1+r^2F_1+\bar{f}r\left[F_0-\left(\frac{8}{11}\right)^{\frac{1}{2}}F_2\right]=\epsilon F_1$$

$$-\frac{1}{r^8}\frac{d}{dr}\left(r^8\frac{dF_2}{dr}\right)+\frac{18}{r^2}F_2+r^2F_2$$
$$+\bar{f}r\left[-\left(\frac{8}{11}\right)^{\frac{1}{2}}F_1+\left(\frac{50}{143}\right)^{\frac{1}{2}}F_3\right]=\epsilon F_2$$

$$-\frac{1}{r^8}\frac{d}{dr}\left(r^8\frac{dF_3}{dr}\right)+\frac{30}{r^2}F_3+r^2F_3+\bar{f}r\left(\frac{50}{143}\right)^{\frac{1}{2}}F_2=\epsilon F_3, \quad (28)$$

where $\bar{f}$ and ϵ were introduced in Eq. (13).

These differential equations were then solved numerically by a relaxation method,[11] by using the electronic digital computer at the University of Illinois. In this method we approximate the continuous values of r by two hundred discrete lattice points. The problem was then solved for ten different values of $\bar{f}$ ranging from zero to twenty. The $r^4F_\lambda(r)$ are plotted in Fig. 1 for $\bar{f}=3$. This value of $\bar{f}$ will be found to give the best agreement between the scattering calculation and experiment.

It is of interest to examine the accuracy of the approximation introduced by restricting the F_λ to be functions of r only. In both the limits of very small and very large values of f, one may solve Eq. (13) exactly. One then easily verifies that the resulting rigorous wave function is in fact identical to that of (27), in the weak-coupling limit. However, in the strong-coupling limit the approximate form (27) does not go over to the exact solution. In Table I we have tabulated some typical matrix elements, calculated by using both the approximate wave function (27) and the rigorous strong-coupling solutions. One sees that in the strong-coupling limit the agreement is fairly good.

TABLE I. Some typical matrix elements for various values of $\bar{f}=2G/\Omega$.[a]

	$\bar{f}=0$	$\bar{f}=3$	$\bar{f}\to\infty$	$\bar{f}\to\infty$ (exact solution)
$\frac{\langle\mathfrak{N}_{\rho'}\|\tau_\alpha\|\mathfrak{N}_\rho\rangle}{\langle n_{\rho'}\|\tau_\alpha\|n_\rho\rangle}$	1	0.073	−0.014	0
$\frac{\langle\mathfrak{N}_{\rho'}\|\sigma_i\tau_\alpha\|\mathfrak{N}_\rho\rangle}{\langle n_\rho\|\sigma_i\tau_\alpha\|n_\rho\rangle}$	1	0.381	0.323	$\frac{1}{3}$
$\frac{\langle\mathfrak{N}_{\rho'}\|x_{i\alpha}\|\mathfrak{N}_\rho\rangle}{\langle n_{\rho'}\|\sigma_i\tau_\alpha\|n_\rho\rangle}$	$-\frac{1}{2}\bar{f}$	$-0.152\bar{f}$	$-0.14\bar{f}$	$-\bar{f}/6$

[a] Matrix elements in Column 1 are calculated by using the exact weak-coupling solution. These are identical with that obtained by using the solution of Eq. (28). Matrix elements in Column 2 and Column 3 are calculated by using the solution of Eq. (28). Matrix elements in Column 4 are calculated by using the exact strong-coupling solution.

[11] See G. E. Kimball and G. H. Shortley, Phys. Rev. 45, 815 (1934).

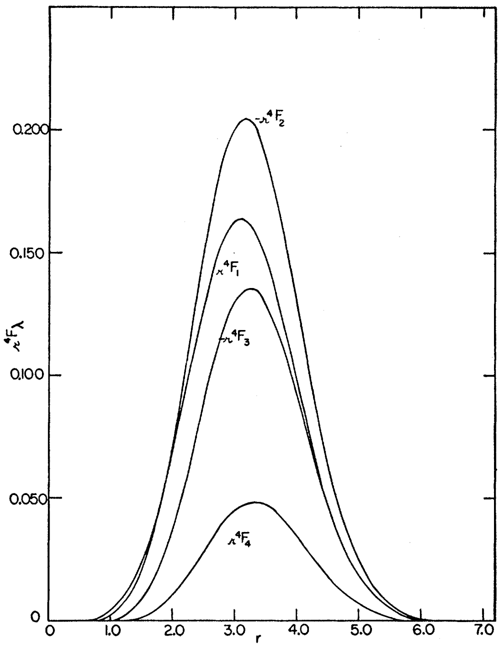

FIG. 1. Plot of $r^4F_\lambda(r)$ for $\bar{f}=3$. Note that F_2 and F_3 are negative. These functions are so normalized that

$$\sum_\lambda \int_0^\infty F_\lambda^2 r^8 dr = 0.6718.$$

IV. SCATTERING STATE

In order to describe the scattering of a meson by a physical nucleon, we shall make use of a variational principle for the calculation of phase shifts. This procedure has been discussed by Lee and Christian,[4] in connection with the charged scalar theory. In a similar manner, we assume that the trial function for the scattering state of total angular momentum J and total isotopic spin I is

$$\Psi_{IJ}=\sum_{k,i,\alpha,\rho}\chi_{IJ}(k)C_{i\alpha}{}^{\rho}(I,J)a_{i\alpha}{}^{*}(k)|N_\rho\rangle+\sum_\rho D_{IJ}{}^{\rho}|N_\rho\rangle, \quad (29)$$

where $|N_\rho\rangle$ is the state vector of a physical nucleon in the spin, isotopic spin state ρ ($\rho=1,\cdots 4$). $C_{i\alpha}{}^{\rho}(I,J)$ are the appropriate numerical factors for constructing a state of total angular momentum J and total isotopic angular momentum I from a nucleon in the state ρ and a meson in the state i, α. The $C_{i\alpha}{}^{\rho}(I,J)$ may be directly obtained from the Clebsch-Gordan coefficients.[12]

The scattering wave function of the π meson is described by the function $\chi_{IJ}(k)$ which together with the constants $D_{IJ}{}^{\rho}$ are determined by the variational procedure,

$$\delta\langle\Psi_{IJ}|H-E_0-\omega_0|\Psi_{IJ}\rangle=0, \quad (30)$$

[12] $C_{i\alpha}{}^{\rho}$ and $D_{i\alpha}{}^{\rho}$ are functions of I_z and J_z also. However χ_{IJ} *is not.*

where E_0 is the self-energy of the physical nucleon and ω_0 is the total energy of the incident meson.

On taking the variation (30) with respect to $D_{IJ}{}^{\rho}$, one obtains

$$D_{IJ}{}^{\rho}=-\sum_{k,i,\alpha,\rho'}\langle N_\rho|\chi_{IJ}(k)C_{i\alpha}{}^{\rho'}(I,J)a_{i\alpha}{}^*(k)|N_{\rho'}\rangle, \quad (31)$$

which of course is the same as the requirement that the ground states be orthogonal to the scattering states.

The variation with respect to an arbitrary functional form of $\chi_{IJ}(k)$ is performed in a manner identical to that in the charged scalar case.[4] We then obtain the following integral equations:

$$(\omega_k-\omega_0)\chi_{IJ}(k)=\sum_{k'}K_{IJ}(k,k')\chi_{IJ}(k'), \quad (32)$$

where

$$K_{IJ}(k,k')=\tfrac{1}{2}A_{IJ}[f(k)\omega'f(k')+f(k')\omega f(k)] +B_{IJ}f(k)f(k'), \quad (33)$$

and A_{IJ} and B_{IJ} are constants which depend only on the structure of the physical nucleon. They are most conveniently given in terms of the matrix elements K, L, M, O, P, Q, R which in turn are defined as follows:

$$\langle\mathfrak{N}_{\rho'}|a_{i\alpha}{}^*a_{j\beta}|\mathfrak{N}_\rho\rangle=K\delta_{ij}\delta_{\alpha\beta}\delta_{\rho\rho'} +iL[\delta_{ij}\epsilon_{\alpha\rho\gamma}\langle n_{\rho'}|\tau_\gamma|n_\rho\rangle+\delta_{\alpha\beta}\epsilon_{ijk}\langle n_{\rho'}|\sigma_k|n_\rho\rangle] +M[\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\langle n_{\rho'}|\sigma_k\tau_\gamma|n_\rho\rangle],$$

$$\langle\mathfrak{N}_{\rho'}|\sigma_i\tau_\alpha a_{j\beta}|\mathfrak{N}_\rho\rangle=O\delta_{ij}\delta_{\alpha\beta}\delta_{\rho\rho'} +iP[\delta_{ij}\epsilon_{\alpha\beta\gamma}\langle n_{\rho'}|\tau_\gamma|n_\rho\rangle+\delta_{\alpha\beta}\epsilon_{ijk}\langle n_{\rho'}|\sigma_k|n_\rho\rangle] +Q[\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\langle n_{\rho'}|\sigma_k\tau_\gamma|n_\rho\rangle],$$

and

$$\langle\mathfrak{N}_{\rho'}|a_{i\alpha}{}^*|\mathfrak{N}_\rho\rangle=R\langle n_{\rho'}|\sigma_i\tau_\alpha|n_\rho\rangle, \quad (34)$$

where $|n_\rho\rangle$ is the bare-nucleon spinor and $|\mathfrak{N}_\rho\rangle$ is the corresponding physical-nucleon wave function in the reduced space.

The constants A_{IJ} and B_{IJ} can then be expressed for various $IJ=\frac{1}{2},\frac{3}{2}$ states as

$$A_{\frac{3}{2},\frac{3}{2}}=-(K+2L-M)+\frac{K}{O}(O+2P-Q),$$

$$B_{\frac{3}{2},\frac{3}{2}}=\lambda\frac{K}{O}(O+2P-Q)+\omega_0(K+2L-M),$$

$$A_{\frac{1}{2},\frac{3}{2}}=A_{\frac{3}{2},\frac{1}{2}}=-(K-L+2M)+\frac{K}{O}(O-P+2Q),$$

$$B_{\frac{1}{2},\frac{3}{2}}=B_{\frac{3}{2},\frac{1}{2}}=\lambda\frac{K}{O}(O-P+2Q)+\omega_0(K-L+2M),$$

$$A_{\frac{1}{2},\frac{1}{2}}=-(K-4L-4M)+\frac{K}{O}(O-4P-4Q),$$

$$B_{\frac{1}{2},\frac{1}{2}}=\lambda\frac{K}{O}(O-4P-4Q)+\omega_0(K-4L-4M)-9R^2\omega_0, \quad (35)$$

where λ is defined in (11).

We note that the kernel (33) is separable and hence Eq. (32) can be solved by elementary means. The resulting phase shifts δ_{IJ} for various IJ are given by

$$\tan\delta_{IJ}=\frac{\omega_0 f^2(k_0)}{k_0}\left[\frac{A_{IJ}\omega_0+B_{IJ}+(A_{IJ}{}^2/4)[\omega_0{}^2\langle f^2(k)\rangle+\langle\omega^2f^2(k)\rangle-2\omega_0\langle\omega f^2(k)\rangle]}{1-A_{IJ}\langle\omega f^2(k)\rangle-B_{IJ}\langle f^2(k)\rangle+(A_{IJ}{}^2/4)[\langle\omega f^2(k)\rangle^2-\langle\omega^2f^2(k)\rangle\langle f^2(k)\rangle]}\right], \quad (36)$$

where

$$\langle g(k)\rangle\equiv\frac{1}{\pi}\mathcal{P}\int_0^\infty\frac{g(k)}{\omega-\omega_0}dk$$

for any function $g(k)$ and $\mathcal{P}$ signifies that the principal value of the integral is to be taken.

In the weak-coupling limit these phase shifts are exactly correct as is expected. It should be note that in using a trial function of the form (29) we have implicitly assumed that no stable isobar exists. This of course does correspond to what one actually finds in nature. Unfortunately, it makes direct comparison with the exact strong coupling limit impossible.

In order to compare the phase shifts (36) with recent experimental results,[1] it is necessary to choose a form for the source function $u(k)$. Following Chew, we take $u(k)$ to be a step function:

$$u(k)=1 \quad \text{for} \quad k<k_{\max}$$
$$=0 \quad \text{for} \quad k>k_{\max}. \quad (37)$$

This form of $u(k)$ enables us to analytically perform all the integrations that appear in (36). The phase shifts then depend on only two parameters, the cutoff $k_{\max}$ and the coupling constant f. These were determined by requiring that $\delta_{\frac{3}{2},\frac{3}{2}}$ passes through the experimentally deduced values for incident kinetic energies of 65 Mev and 189 Mev (in the laboratory system). They are

$$\omega_{\max}=(k_{\max}{}^2+\mu^2)^{\frac{1}{2}}=6.21\mu, \qquad f^2=0.712, \quad (38)$$

where μ is the meson mass. Using these values, all four phase shifts were then computed over the energy range 0 to 217 Mev. They are plotted in Fig. 2. One sees that only $\delta_{\frac{3}{2},\frac{3}{2}}$ is appreciable throughout most of the energy range. It is of some interest to notice that, although it is very small, $\delta_{\frac{1}{2},\frac{1}{2}}$ is positive, while it would be negative if the coupling constant were much smaller.

One may now ask if in using the parameters of Eq. (38) we are near either the weak- or strong-coupling limit. As far as the distribution of the number of mesons in the Fock space is concerned, the relevant quantity is $\bar{f}$, which in this case is

$$\bar{f}=3. \tag{39}$$

Referring to Table I, one sees that the various matrix elements in the reduced space are very close to the strong-coupling limit values.[13] On the other hand, for this case the λ that appears in Eq. (11) has the value

$$\lambda=3.39. \tag{40}$$

However, in the strong- and weak-coupling limits λ is identically zero. Thus we see that while the distribution of the various numbers of mesons resembles that of the strong-coupling limit, the meson orbital momentum distribution is quite different from either the strong- or weak-coupling limits.

In order to compare with other methods of calculation for the same problem, it is convenient to compute the mesonic charge renormalization Z_2/Z_1. This can be done by utilizing the identity[14]

$$f_r/f=Z_2/Z_1=\langle \mathfrak{N}_{p'}|\sigma_i\tau_\alpha|\mathfrak{N}_p\rangle/\langle n_{p'}|\sigma_i\tau_\alpha|n_p\rangle, \tag{41}$$

where f_r is the renormalized coupling constant. Using the value of this matrix element given in Table I, we find that

$$f_r^2=0.105 \tag{42}$$

in the present case.

It is of interest to notice that the same f_r^2 may be obtained in a semiempirical way. By using a general result obtained by Chew and Low,[15] it is possible to show[16] that the rigorous scattering solution of the original Hamiltonian H, Eq. (1), has the following

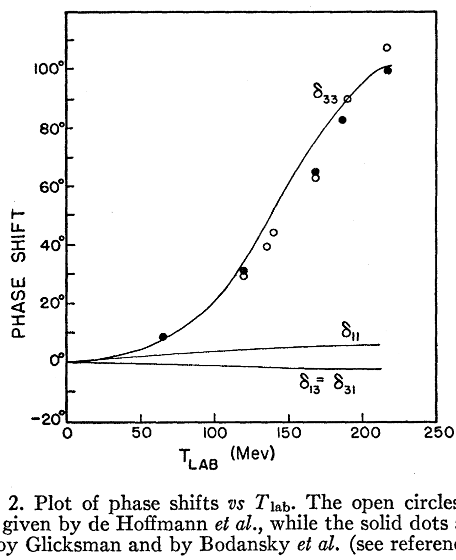

FIG. 2. Plot of phase shifts *vs* T_{lab}. The open circles are the values given by de Hoffmann *et al.*, while the solid dots are those given by Glicksman and by Bodansky *et al.* (see reference 1).

[13] See Appendix II for numerical values of other matrix elements.
[14] T. D. Lee, Phys. Rev. **95**, 1329 (1954); G. F. Chew, reference 2.
[15] F. E. Low, Phys. Rev. **97**, 1392 (1955); G. F. Chew and F. E. Low (to be published).
[16] R. Serber (private communication).

property. If the function $F_{IJ}(\omega)$ is defined as

$$F_{IJ}(\omega)\equiv u^2(k)\left(\frac{k^3}{\omega}\cot\delta_{IJ}-\frac{\mu^3}{\omega}\right)\frac{1}{\mu^2}, \tag{43}$$

then at $\omega=0$

$$F_{IJ}(0)=\lambda_{IJ}^{-1} \tag{44}$$

exactly, where

$$\lambda_{\frac{3}{2},\frac{3}{2}}=(4/3)f_r^2,$$
$$\lambda_{\frac{3}{2},\frac{1}{2}}=\lambda_{\frac{1}{2},\frac{3}{2}}=-\tfrac{2}{3}f_r^2, \tag{45}$$

and

$$\lambda_{\frac{1}{2},\frac{1}{2}}=-(8/3)f_r^2.$$

By using the experimentally deduced values of the δ_{33} phase shift up to about 175-Mev incident meson energy, the function $F_{\frac{3}{2},\frac{3}{2}}(\omega)$, plotted against ω, is found to be a remarkably straight line. One may graphically extrapolate the function $F_{\frac{3}{2},\frac{3}{2}}(\omega)$ back to the point $\omega=0$. The renormalized coupling constant f_r^2 can then be determined by using Eq. (44). Thus one obtains

$$f_r^2=0.10,$$

which agrees with Eq. (42). Inclusion of phase shift data beyond 175 Mev would add a curvature to $F_{\frac{3}{2},\frac{3}{2}}(\omega)$ in a direction as to make f_r^2 slightly larger.

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APPENDIX I

In this Appendix we wish to study the general functional form of the variables $x_{i\alpha}$, which are simultaneous eigenfunctions of the operators L_3, L^2, T_3, and T^2 with eigenvalues $L=0,1$ and $T=0,1$. These operators are given in Eqs. (14) through (17). Since the polynomials $x_{i\alpha}$, $x_{i\alpha}x_{j\beta}$, $\cdots$, form a complete set of functions (Weierstrass' theorem) and since the operators L_i, T_α commute with the operator $x_{i\alpha}\partial/\partial x_{i\alpha}$ (thereby preserving the degree of any homogeneous polynomial), it is only necessary to consider homogeneous polynomials.

It is convenient to define the following tensors (all repeated indices to be summed):

$$C_{\alpha\beta}=x_{i\alpha}x_{i\beta}, \tag{1A}$$

$$T_{\alpha\beta\gamma}=\epsilon_{ijk}x_{i\alpha}x_{j\beta}x_{k\gamma} \tag{2A}$$

$$=T\epsilon_{\alpha\beta\gamma}, \tag{3A}$$

where T is evidently a scalar quantity given by,

$$T=\tfrac{1}{6}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}x_{i\alpha}x_{j\beta}x_{k\gamma}. \tag{4A}$$

Since $C_{\alpha\beta}$ is a real, symmetric matrix it may be diagonalized by an orthogonal transformation. Let us denote its eigenvalues by Q_1^2, Q_2^2, Q_3^2. Then, since T is the determinant of the matrix $(x_{i\alpha})$, one can write

$$T=Q_1Q_2Q_3. \tag{5A}$$

The three scalar quantities s_1, s_2, s_3 introduced in Eq. (19) are then related to the Q_l by

$$s_1=C_{\alpha\alpha}=Q_1^2+Q_2^2+Q_3^2,$$
$$s_2=6Q_1Q_2Q_3,$$
$$s_3=Q_1^4+Q_2^4+Q_3^4. \quad (6A)$$

We now wish to prove the following theorems.

Theorem I: *If $S(x_{i\alpha})$ is an eigenfunction of L and T with $L=T=0$, then*

$$S=S(s_1,s_2,s_3). \quad (7A)$$

Proof: Let us consider a homogeneous polynomial of nth degree,

$$S_n=A_{i_1\cdots i_n,\alpha_1\cdots\alpha_n}x_{i_1\alpha_1}\cdots x_{i_n\alpha_n}. \quad (8A)$$

We thus wish to contract a tensor of nth rank (in both the i and α indices) into a tensor of rank zero. Thus the coefficients $A_{i_1\cdots i_n,\alpha_1\cdots\alpha_n}$ must be made from products of the isotopic tensors δ_{ij} and ϵ_{ijk} (similarly for the α indices). Furthermore, we note the identity:

$$\epsilon_{ijk}\epsilon_{lmn}=\delta_{il}\delta_{jm}\delta_{kn}+\delta_{im}\delta_{jn}\delta_{kl}+\delta_{in}\delta_{jl}\delta_{km} -\delta_{il}\delta_{jn}\delta_{km}-\delta_{in}\delta_{jm}\delta_{kl}-\delta_{im}\delta_{jl}\delta_{kn}. \quad (9A)$$

Thus we will never have to use more than one ϵ_{ijk} and $\epsilon_{\alpha\beta\gamma}$ in contracting the indices.

(a) We first consider homogeneous polynomials of even degree. Using the above statement we take $A_{i_1\cdots i_n,\alpha_1\cdots\alpha_n}$ to be a product of δ_{ij} and $\delta_{\alpha\beta}$ only. After contracting on the i indices, we find that the polynomial consists of sums of terms, each of the form

$$C_{\alpha_1\alpha_2}C_{\alpha_3\alpha_4}\cdots C_{\alpha_{n-1}\alpha_n}.$$

Contracting with respect to the α indices, we find this polynomial can be written as (or sums of)

$$S_n=\prod_\lambda \text{Trace}(C_{\alpha\beta})^{m_\lambda}=\prod_\lambda(Q_1^{2m_\lambda}+Q_2^{2m_\lambda}+Q_3^{2m_\lambda}) \quad (10A)$$

(or sums of such terms), where

$$\textstyle\sum_\lambda m_\lambda=n/2.$$

Thus (7A) is true.

(b) We now consider polynomials of odd degree. After contracting on the i indices, we find the polynomial to be a sum of terms of the type

$$T\epsilon_{\alpha_1\alpha_2\alpha_3}C_{\alpha_4\alpha_5}\cdots C_{\alpha_{n-1}\alpha_n}.$$

Upon contracting with respect to the α indices, these terms become

$$T\epsilon_{\alpha_1\alpha_2\alpha_3}C_{\alpha_1\beta_1}{}^kC_{\alpha_2\beta_2}{}^lC_{\alpha_3\beta_3}{}^m\epsilon_{\beta_1\beta_2\beta_3}S_{n'},$$

where $S_{n'}$ is an even polynomial of the form (10A) and

$$2(k+l+m)+n'=n-3.$$

Thus, upon using (5A), (6A), and (9A), we complete the proof.

Theorem II: *If $V_\beta(x_{i\alpha})$ $(\beta=1,2,3)$ are eigenfunctions of $L=0$, $T=1$, then $V_\beta=0$. Similarly, if $V_j(x_{i\alpha})$ $(j=1,2,3)$ are eigenfunctions of $L=1$, $T=0$, then $V_j=0$.*

It is evident that we need explicitly consider the $L=0$, $T=1$ case only.

Proof: (a) Consider a polynomial of even degree. We must then simultaneously contract an nth-rank tensor in the i indices to a zero-rank tensor and an nth rank tensor in the α indices to a first-rank tensor.

After contracting with respect to the i indices, the polynomials consist of sum of terms of the type

$$C_{\alpha_1\alpha_2}C_{\alpha_3\alpha_4}\cdots C_{\alpha_{n-1}\alpha_n}.$$

After contracting with respect to the α indices we find, V_β can be written as (or sums of)

$$V_\beta=\epsilon_{\alpha_1\alpha_2\alpha_3}C_{\alpha_1\beta}{}^kC_{\alpha_2\alpha_3}{}^lS_m,$$

with

$$2(k+l)+m=n, \quad (11A)$$

where, if k is zero, then $C_{\alpha_1\beta}{}^0=\delta_{\alpha_1\beta}$. Since $C_{\alpha\beta}$ is a symmetric matrix, we have $V_\beta=0$.

(b) Let V_β be a polynomial of odd degree. After contracting with respect to the i indices, V_β consists of sums of terms of the type

$$T\epsilon_{\alpha_1\alpha_2\alpha_3}C_{\alpha_4\alpha_5}\cdots C_{\alpha_{n-1}\alpha_n},$$

which, as in the above case, becomes zero after contracting with respect to the α indices.

Theorem III: *If $\mathcal{V}_{i\alpha}(x_{j\beta})$ are eigenfunctions of $L=T=1$, then $\mathcal{V}_{i\alpha}$ can be written as*

$$\mathcal{V}_{i\alpha}=\sum_{\lambda=1}^{3}\left(\frac{\partial s_\lambda}{\partial x_{i\alpha}}\right)S^\lambda(s_1,s_2,s_3), \quad (12A)$$

where the S^λ are three scalar functions.

Proof: (a) Let $\mathcal{V}_{i\alpha}$ be a polynomial of odd degree. Then we need only use δ_{ij} and $\delta_{\alpha\beta}$ in order to contract the indices. After contracting with respect to the i indices, $\mathcal{V}_{i\alpha}$ consists of sums of terms of the type

$$x_{i\alpha_1}C_{\alpha_2\alpha_3}C_{\alpha_4\alpha_5}\cdots C_{\alpha_{n-1}\alpha_n},$$

which, after contracting with respect to the α indices, becomes

$$x_{i\alpha_1}C_{\alpha_1\alpha}{}^mS_{n-2m-1}.$$

Using the identity,

$$\left(\frac{\partial}{\partial x_{i\alpha}}\right)C_{\beta\beta}{}^l=2lx_{i\alpha_1}C_{\alpha_1\alpha}{}^{l-1}, \quad (13A)$$

one completes the proof for this case.

(b) Let $\mathcal{V}_{i\alpha}$ be a polynomial of even degree. We must use one ϵ_{jkl} and one $\epsilon_{\beta\gamma\delta}$. By writing the polynomial explicitly in terms of the isotopic tensors, it is clear that the index i will be carried by either $x_{i\beta}$ or ϵ_{ijk}. Similarly for α. Thus we only need to consider the following three cases:

$$(1)\quad \mathcal{V}_{i\alpha}{}^{(1)}=x_{i\beta}C_{\beta\alpha}{}^lS_m$$
$$(2)\quad \mathcal{V}_{i\alpha}{}^{(2)}=x_{i\lambda}\epsilon_{\alpha\beta\gamma}C_{\lambda\mu}{}^lC_{\beta\nu}{}^mC_{\gamma\omega}{}^n\epsilon_{\mu\nu\omega}S_p,$$
$$(3)\quad \mathcal{V}_{i\alpha}{}^{(3)}=\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}x_{j\lambda}C_{\lambda\beta}{}^mx_{k\mu}C_{\mu\gamma}{}^nS_q. \quad (14A)$$

Using (13A) immediately completes the proof for $\mathcal{V}_{i\alpha}^{(1)}$.

Using (9A), we notice that $\mathcal{V}_{i\alpha}^{(2)}$ can be written as a linear combination of the type $\mathcal{V}_{i\alpha}^{(1)}$.

For the third case, we introduce the scalar function

$$S=x_{j\beta}\mathcal{V}_{j\beta}^{(3)}. \tag{15A}$$

Without loss of generality, we need only, in the definition of $\mathcal{V}_{i\alpha}^{(3)}$, to consider the case $S_q=1$.

Differentiating (15A), we obtain

$$\begin{aligned}\mathcal{V}_{i\alpha}^{(3)}=(\partial S/\partial x_{i\alpha})-x_{j\beta}\epsilon_{jil}\epsilon_{\beta\gamma\delta}C_{\alpha\gamma}{}^{m}x_{l\mu}C_{\mu\delta}{}^{n}\\ -x_{j\beta}\epsilon_{jki}\epsilon_{\beta\gamma\delta}x_{k\lambda}C_{\lambda\gamma}{}^{m}C_{\alpha\delta}{}^{n}\\ -x_{j\beta}x_{k\lambda}x_{l\mu}\epsilon_{jkl}\epsilon_{\beta\gamma\delta}(\partial/\partial x_{i\alpha})(C_{\lambda\gamma}{}^{m}C_{\mu\delta}{}^{n}).\end{aligned} \tag{16A}$$

We notice that the second and third terms on the right-hand side of (16A) can be reduced to a linear combination of $\mathcal{V}_{i\alpha}^{(1)}$. By using Eq. (3A), the last term can be explicitly written as

$$-T\epsilon_{\beta\lambda\mu}\epsilon_{\beta\gamma\delta}(\partial/\partial x_{i\alpha})(C_{\lambda\gamma}{}^{m}C_{\mu\delta}{}^{n}),$$

which, on using (9A), can be written as the product of scalar functions by the derivatives of scalar functions. Thus we complete the proof.

APPENDIX II

In this section, we list the numerical values of matrix elements and other quantities used in the scattering calculations for $f^2=0.712$ and $\omega_{\max}=6.21\mu$.

$$f_r^2=0.1048,\ K=0.19906,\ L=0.07704,\ M=-0.05538,$$
$$O=-0.32145,\ P=-0.08626,\ Q=0.17631,$$
$$R=-0.32145,\ \bar{f}=3.0,\ \lambda=3.39,\ \Omega=4.7557,$$

where $K, L, M, \cdots, R$ are defined in Eq. (34) and $\bar{f}$, λ, Ω are defined in Eqs. (9), (11), and (13).

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Anomalous Magnetic Moment of the Nucleon

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The ground-state solution of the physical nucleon problem in the Tomonaga approximation is used to compute the anomalous magnetic moment of nucleons. When computed on the basis of parameters that make the phase shift calculations in the Tomonaga approximation consistent with meson-nucleon scattering data, the values obtained are +1.48 for the proton and −1.48 for the neutron.

I. INTRODUCTION

THE meson-nucleon scattering calculation in the Tomonaga approximation[1] seems to give a correct description of the experimentally observed data in the low-energy range. Furthermore, a weak-coupling treatment of the same Hamiltonian[2] (with a gauge-invariant electromagnetic interaction added) yields ±1.44 nucleon magnetons for the anomalous magnetic moment of the nucleon (plus for the proton and minus for the neutron). This result is also fairly close to the experimental one, and hence it becomes of interest to do this last calculation again, but now using the Tomonaga approximation.

II. METHOD OF CALCULATION

The Hamiltonian used will be the same as that of reference 2. As was done there, we will not include the effect of the nucleon current or the Dirac moment in intermediate states. We will proceed by evaluating the matrix element

$$\langle H^{em}\rangle=\langle\Psi_0|\mathbf{j}\cdot\mathbf{A}|\Psi_0\rangle, \tag{1}$$

where Ψ_0 is the state vector representing the physical nucleon. (Natural units to be used throughout.) We write

$$\mathbf{j}\cdot\mathbf{A}=e\int A_l\left(\phi_1\frac{\partial\phi_2}{\partial x_l}-\phi_2\frac{\partial\phi_1}{\partial x_l}\right)d\mathbf{x}, \tag{2}$$

where ϕ_1, ϕ_2 are the first two components of the meson field and are real. (Repeated indices are to be summed over all values from one to three, throughout.) $\mathbf{A}$ is the electromagnetic vector potential and is chosen to be transverse for the purposes of this problem. For

$$\mathbf{A}=V^{-\frac{1}{2}}\sum_{\mathbf{q}}\mathbf{A}(\mathbf{q})\exp[-i\mathbf{q}\cdot\mathbf{x}], \tag{3}$$

and

$$\phi_\alpha=V^{-\frac{1}{2}}\sum_{\mathbf{k}}[2\omega(|\mathbf{k}|)]^{-\frac{1}{2}}\{a_\alpha(\mathbf{k})\exp[i\mathbf{k}\cdot\mathbf{x}] +a_\alpha{}^*(\mathbf{k})\exp[-i\mathbf{k}\cdot\mathbf{x}]\} \tag{4}$$

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