That is, the absorption coefficient in nuclear matter equals the particle density times the cross section for scattering of the π meson by a particle in the nucleus. This calculation should, in principle be equivalent to the one above, except that the experimental total cross sections are used rather than the results of the phase shift analysis with reasonably good agreement with the experiment neglecting such other contributions to the absorption. It gives a mean free path at 65, 80, and 120 Mev of 5.5×10^{-13} cm, 4.2×10^{-13} cm, and 1.9×10^{-13} cm respectively. Hence at 80- and 100-Mev incident energy we would obtain mean free paths of 4.2×10^{-13} cm and about 3×10^{-13} cm. However, the cross section for inelastic processes for bound nucleons in a nucleus will be smaller than the total cross section for free nucleons due to the Pauli exclusion principle and binding effects requirement that the final nucleon state be previously empty. Thus these last values should be somewhat increased.

A best speculation as to the true mean free path might be taken as 4×10^{-13} cm for all the above calculations with the particular choice of nuclear model and nuclear size.

ACKNOWLEDGMENTS

The authors wish to thank Professor Gilberto Bernardini, who initiated some of the early phases of this work while at Nevis. Many others of the Columbia Faculty and Nevis staff provided valuable cooperation and assistance; in particular we wish to thank Dr. Val Fitch for many helpful discussions and suggestions on the experimental techniques used in the measurements.

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Phase-Shift Optical Model Calculations for the Elastic Scattering of Pions on Aluminum*†

AIHUD PEVSNER‡ AND JAMES RAINWATER
Department of Physics, Columbia University, New York, New York
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Theoretical optical-model exact phase-shift calculations are given for the predicted elastic scattering of pions from aluminum at the energy used (79 Mev) in the experimental studies described in the preceding paper. The predicted curves show strong diffraction minima, not present in the experimental curves. We have emphasized a comparison of the results of these calculations with those of more approximate methods. Sixteen choices of constant complex potentials were used for $r<R_0$ for both π^+ and π^- mesons, with a Coulomb potential for $r>R_0=3.600\lambdabar$. Here $\hbar k_0=\hbar/\lambdabar$ is the momentum at infinity and $\hbar(k_1+ik_2)$ is the momentum for $r<R_0$. It is found that the predicted incoherent cross section σ_a and positions of the diffraction minima depend strongly on k_1/k_0 for a fixed k_2. Values of $\sigma_a>\pi R_0^2$ are obtained for relatively long nuclear mean free paths $(2k_2)^{-1}$ for incoherent processes for the attractive potentials which give best fit to experiment. The results of various modified Born approximation calculations are compared with the results of the exact calculations.

INTRODUCTION

An experimental program[1] has been undertaken at the Nevis Cyclotron Laboratories to investigate the elastic scattering of π^+ and π^- mesons by nuclei with better angular resolution and statistical accuracy than earlier measurements of this type.[2] Since measurements of this type can, in principle, yield considerable valuable information on the interaction of pions with nuclear matter, we believed that it was important to compare the experimental results with as exact theoretical predictions as would be feasible. This led us to an examination of the methods then favored (1953) for the calculation of such scattering. We have also carried through a program of calculations of the scattering expected on the basis of an "optical model" using an exact phase shift analysis for various complex indices of refraction inside the nucleus, and using Coulomb wave functions outside the nucleus. These calculations provide numerous test cases to compare with the results of approximate methods often employed for this purpose. Several significant features of disagreement were found between the results of our exact calculations and the usual approximate methods, which we believe should be emphasized as giving important limitations on the applicability of the approximate analysis. These matters are discussed in some detail in the following sections. Most of these calculations were done in 1953. Since then many exact phase-shift calculations have

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† This paper contains portions of a thesis submitted by Aihud Pevsner in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Faculty of Pure Science, Columbia University.
‡ Now at the Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts.
[1] Pevsner, Lindenbaum, Williams, and Rainwater, preceding paper [Phys. Rev. **100**, 1419 (1955)].
[2] Byfield, Kessler, and Lederman, Phys. Rev. **86**, 17 (1952); J. O. Kessler and L. M. Lederman, Phys. Rev. **94**, 689 (1954); A. M. Shapiro, Phys. Rev. **84**, 1063 (1951).

been carried out by others for electron and nucleon scattering from nuclei showing similar differences from the results of less exact calculations.

A second general method of approach to the elastic scattering of fast particles by nuclei is based on the Born approximation, either directly or through the use of various modifications which are intended to include effects which are present in the scattering, but are not readily incorporated in the usual first-order Born approximation treatment. As is discussed in more detail below, the formulas resulting from this second approach include many features not present in the usual optical model approach, but which we believe should be present to some extent in a more exact theory of elastic scattering. Similarly, the concept of a change in phase relations due to the real part of the index of refraction inside the nucleus differing from the outside value, and the concept of extinction effects due to a noninfinite mean free path for absorption (noncoherent) processes in nuclear matter are special features of the optical model which are absent from the simple Born approximation analysis, but would be expected to be present to some extent in a more exact theory. We have tested various recipes for altering the usual Born approximation formulas to consider these latter effects and, by treating cases which were also treated exactly by the phase-shift analysis, are able to make detailed comparison with the results from the optical-model phase-shift analysis. These calculations and comparisons are presented in detail in the following sections. We shall not, in this paper, be mainly concerned with the fundamental theory[3] used to establish the approximate validity of the differential equation which forms the basis of the usual optical model. Rather we shall emphasize the comparison of the predictions of the various methods of analysis discussed above using what may be considered a semiempirical approach.

The Optical Model

The introduction of the use of an optical model to the investigation of the elastic nuclear scattering of fast particles was due mainly to Fernbach, Serber, and Taylor[4] (denoted by F.S.T.). They showed that fast-neutron elastic, interaction, and total cross sections when analyzed on this basis seemed to provide an excellent basis for interpretation of the experimental results. In carrying through their derivation of actual final formulas for comparison with experiment, however, they made two important simplifications which are themselves not implied by the concept "optical model," but which facilitated the analysis. These were: (1) A "uniform" nuclear model (U.N.M.) was assumed as has been customary in most treatments of nuclear processes. This considers a spherical nucleus of radius R_0 having constant nuclear density inside and zero density outside. (2) An approximate method was used to solve for diffraction effects similar to the usual approximate methods of dealing with interference and diffraction effects in physical optics, as opposed to exact solutions in terms of differential equations and boundary conditions. By adopting this approximate method, they were able to explore the general features of the theoretical predictions over an extended range of the free parameters to an extent that would require a prohibitively large program of calculations by the exact phase shift method. Thus they were able to present an excellent preliminary survey of the subject which could gradually be improved by testing it against exact calculations for selected sets of parameters. It should be noted that they had in mind situations where $R_0/\lambda\!\!\!^{-}$ was much larger than the case we have studied.

Recent experimental studies of μ-mesonic x-ray transitions,[5] and the recent studies of the elastic nuclear scattering of fast electrons,[6] at Stanford in particular, indicate that the nucleus has a considerably higher density of nuclear matter near the center than was previously believed to be the case, with a gradual dropping off of the density in the outer regions (fuzzy edge). Since the basic "optical model" differential equations are capable of exact solution by a phase shift analysis of the various angular momentum components, the method need not, in principle or fact, limit itself to the approximate method of Fernbach, Serber, and Taylor.

For the purposes of this paper we define the "optical model" as follows: The particle being scattered is represented at a large distance in terms of a modified plane wave plus an outgoing spherical wave from the scatterer, which we take as centered at the origin of coordinates. (When a Coulomb field is present, the plane wave is modified in the usual way.) At all points the wave equation for the particle can be represented as

$$\nabla^2\Psi+(k_1+ik_2)^2\Psi=0, \tag{1}$$

where k_1 and k_2 are real functions. In a region of zero potential energy $k_1=k_0$ and $k_2=0$, so

$$k_1+ik_2=nk_0 \tag{2}$$

introduces a complex index of refraction n. If absorption (incoherent) processes do not occur at r, and the momentum is real (positive kinetic energy), then $k_2=0$, and k_1 differs from k_0 due to a change in momentum (scalar value) from its value at large distances. When absorption (incoherent) processes occur at r, k_2 is positive and is the inverse of the mean free path for absorption (of amplitude rather than intensity). When r represents a classically disallowed point (negative kinetic energy or imaginary momentum) the situation

[3] See, e.g., K. M. Watson, Phys. Rev. **89**, 575 (1953).
[4] Fernbach, Serber, and Taylor, Phys. Rev. **75**, 1352 (1949).
[5] Val L. Fitch and James Rainwater, Phys. Rev. **92**, 789 (1953); L. N. Cooper and E. M. Henley, Phys. Rev. **92**, 801 (1953).
[6] Hofstadter, Hahn, Knudsen, and McIntyre, Phys. Rev. **95**, 512 (1954).

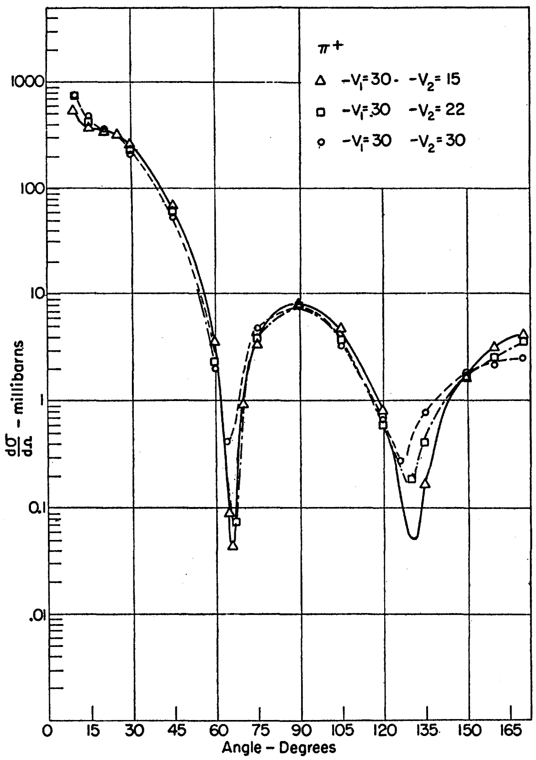

FIG. 1. Differential cross section for 79-Mev π^+ mesons on aluminum calculated using phase-shift analysis on optical model for the nucleus, for $V_1=-30$ Mev, $V_2=-15$, -22, -30 Mev. Plotted points correspond to calculated points.

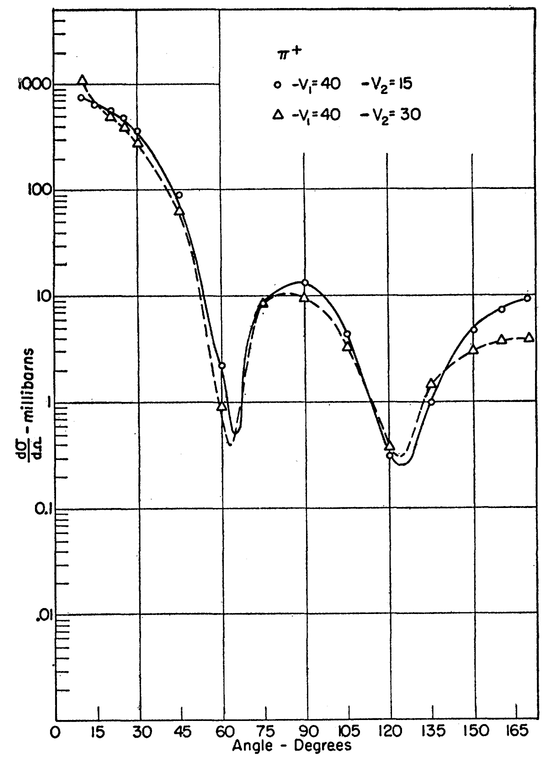

FIG. 2. Differential cross section for 79-Mev π^+ mesons on aluminum calculated using phase-shift analysis on optical model for the nucleus, for $V_1=-40$ Mev, $V_2=-15$, -30 Mev. Plotted points correspond to calculated points.

becomes more complex. When the nonrelativistic Schrödinger equation is used, k_1 and k_2 are related to the real and imaginary parts (V_1+iV_2) of the potential at r and to the energy E.

$$\hbar^2(k_1+ik_2)^2/2m=(E-V_1-iV_2). \qquad (3)$$

If the Klein-Gordon[7] equation is used, then

$$c^2\hbar^2(k_1+ik_2)^2=(E-V_1-iV_2)^2-(mc^2)^2. \qquad (4)$$

where m is the rest mass and E the total energy. We have used (4) for relating k_1 and k_2 to a (V_1+iV_2) for nuclear matter in treating the π-meson scattering. In this paper we treat the potential, in analogy to the Coulomb term, as the fourth component of a four vector. Meson theory suggests that V might better be treated as a masslike term, but it is easy to "translate" our V to a corresponding V introduced in a different manner without altering the results. We normally consider situations where the scattering system is very massive relative to the incident particle and can thus consider laboratory and center-of-mass coordinates as equivalent. Elastic coherent scattering requires that there be no change in the internal wave function of the scatterer. Table IV of the preceding paper lists the parameters studied for which a phase shift calculation was carried out.

When a Coulomb term is present outside the nucleus, there is Coulomb scattering modified by the nuclear interaction. The analysis is similar in the relativistic and nonrelativistic cases if (a) terms quadratic in V are neglected in (4) for the region outside the nucleus, (b) if E/c^2 is used in place of the rest mass m for the outside Coulomb wave functions. The theory in this case can be found in Schiff.[8] His equations (20.24) and (20.10) give for the scattering amplitude:

$$f(\theta)=f_c(\theta)+k_0^{-1}\sum_{l=0}^{\infty}(2l+1)e^{i(2\eta_l+\delta_l)}\sin\delta_l P_l(\cos\theta), \qquad (5)$$

$$f_c(\theta)=\frac{\alpha}{2k_0\sin^2(\theta/2)}\exp\{-i\alpha\ln[\sin^2(\theta/2)]+2i\eta_l\}, \qquad (6)$$

where

$$e^{2i\eta_l}=\Gamma(1+l+i\alpha)/\Gamma(1+l-i\alpha) \quad \text{and} \quad \alpha=Z_1Z_2e^2/\hbar v.$$

[7] Other methods of including the nuclear interaction could also give a complex k in the wave equation. Thus the interaction might better be taken as a mass term instead of the fourth component of a four vector. For a square well this would change the values of V_1 and V_2 for a given k_1 and k_2. It is also possible to construct a wave equation containing a term proportional to the gradient of the nuclear density. This gives a surface effect for a uniform nucleus. (We wish to thank Dr. Kislinger and Dr. Francis for discussions of their work on this matter method prior to publication of their results.) See also K. M. Watson, Phys. Rev. **89**, 575 (1953).

[8] L. I. Schiff, *Quantum Mechanics* (McGraw Hill Book Company, Inc., New York, 1949), Sec. 20.

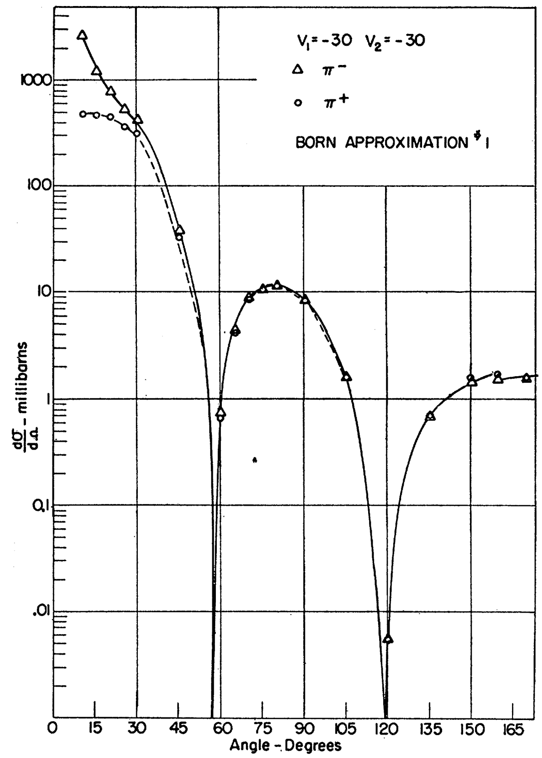

FIG. 3. Differential cross sections for Born approximation No. 1 (described in text). π^+ and π^- curves for $V_1=-30$ Mev, $V_2=-30$ Mev. Plotted points correspond to calculated points.

$Z_1Z_2e^2/r$ is the Coulomb potential and v is the velocity of the particle. The extended actual nucleus introduces an extra phase shift δ_l in addition to the Coulomb phase shift η_l.

To actually carry out the calculation of the δ_l values, a U.N.M. was used with a constant complex potential V_1+iV_2 inside. Regular and irregular Coulomb wave functions were taken from the tables of Bloch, Hull, Broyles, Bouricius, Freeman, and Breit,[9] and the tables of Coulomb wave functions prepared by the National Bureau of Standards.[9] The method of using these tables and the details of the phase-shift calculations are given by A. Pevsner.[10]

Two methods were used by Fernbach, Serber, and Taylor.[4] In the one usually employed for comparison with experiment, the wave front is considered to pass through the nucleus without disturbing initial ray directions, and the relative phase and amplitudes of points on a wave front are considered after traversing the nucleus. All parts not in the geometric shadow of the nucleus have the same amplitude and phase. A point on a part of the surface for which the ray traversed nuclear matter has an additional phase $\int(k_1-k_0)ds$, and is attenuated by a factor $\exp[-\int k_2 ds]$ in amplitude. (Note: Our k_2 is half of their K, which refers to *intensity* attenuation.) The general diffraction formula is now used and the problem treated as the linear combination of a case where the amplitude and phase on the wave front are the same in the geometric shadow as elsewhere, giving no scattering, plus the case where the amplitude is zero outside the shadow, and equal to the (phase) vector change in amplitude due to the nucleus inside the shadow. The scattered radiation pattern is then that due to a flat circular area and depends only on $\sin\theta$ for given k, R_0, etc., so it is symmetric about 90 degrees. The absorption cross section asymptotically approaches πR_0^2 as $K\to\infty$.

It is evident that the above angular pattern for the scattering can only be valid at small angles since symmetry about 90 degrees requires that all odd angular momentum terms be zero. The Born approximation calculation, which should become exact in the limit of weak interactions, depends on the vector change in momentum in scattering, $q=2k_0\sin(\theta/2)$. Thus an improvement might be expected in their scattering formula, by replacing $\sin\theta$ with $2\sin(\theta/2)$. They mention another formula, (9), which is based on a WKB phase shift analysis and involves a series in $P_l(\cos\theta)$. This would be expected to extrapolate better to large angles. The errors in this method were discussed by Pasternack and Snyder.[11]

The F. S. T. approximate expression for σ_a in terms

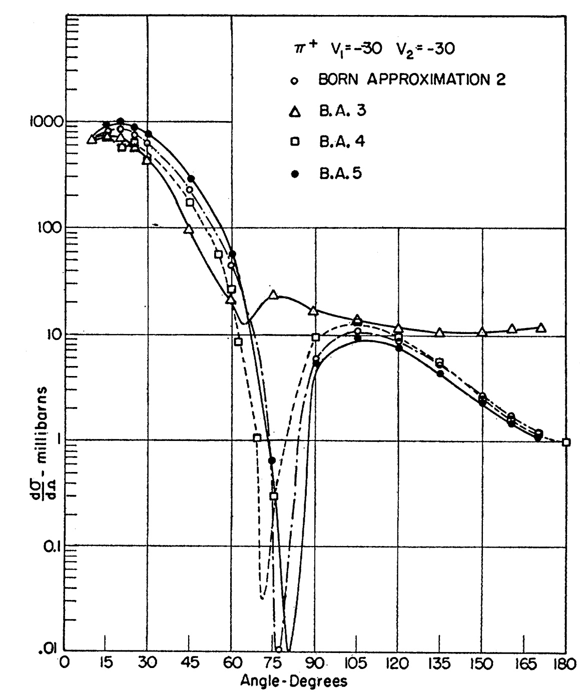

FIG. 4. Differential cross sections for Born approximations Nos. 2, 3, 4, and 5 (described in text). π^+ curves for $V_1=-30$ Mev, $V_2=-30$ Mev. Plotted points correspond to calculated points.

[9] Bloch, Hull, Broyles, Bouricius, Freeman, and Breit, Revs. Modern Phys. 23, 147 (1951); National Bureau of Standards, Applied Mathematics Series, Vol. 17.

[10] A. Pevsner, Nevis Report No. 3, 1954 (unpublished).

[11] S. Pasternack and H. S. Snyder, Phys. Rev. **80**, 921 (1950).

TABLE I. Calculated (total minus coherent-scattering) cross sections in mb, for 79-Mev π^- mesons on aluminum.

Case number	σ_a $l=0$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$\sum_{l=0}^{4} \sigma_{a,l}$	Fernbach, Serber, Taylor σ	σ_{modified} [a]
1	35.4	93.0	184.6	152.1	33.5	3.8	498.6	348.7	474.2
2	27.3	82.5	144.0	88.6	18.7	2.1	361.1	276.2	342.5
3	17.0	51.3	90.0	48.6	9.6	1.1	216.5	164.6	204.1
4	4.1	12.4	22.0	10.5	1.9	0.2	50.9	38.4	47.6
5	26	89.8	125.6	68.5	5.8	2.5	325.0	280.7	314.4
6	0	0	0	0	0	0	0	...	...
7	36.8	121.9	156.5	95.4	24.5	1.5	435.1	414.9	427.3
8	27.5	92.5	104.8	54.4	13.8	3.0	293.0	285.9	285.9
9	17.3	59.1	61.2	28.7	6.7	1.0	173.0	171.6	171.3
10	34.4	78.9	69.2	35.4	7.4		225.3	300.5	231.1
11	39.0	119.4	205.9	162.9	43.8		571.0	443.1	549.4
12	41.0	125.1	213.6	186.1	57.2		623.0	489.3	611.6
13	39.7	108.4	207.4	186.7	47.4		589.6	418.8	569.6
14	41.6	117.2	217.0	208.7	60.6		645.1	476.2	635.4
15	37.6	86.6	180.5	193.0	42.0		539.8	345.5	497.5
16	43.0	112.4	215.0	242.9	73.2		685.5	464.9	688.1
$(2l+1)\pi ƛ^2$	43.7	131.1	218.5	305.9	393.3	480.7	1092.5		

[a] σ_{modified} is obtained by multiplying the Fernbach, Serber, and Taylor cross section by $(k_1/k_0)^{1.5}$.

of λ_a has been used[12] in the analysis of meson scattering to obtain an estimate of the mean free path for absorption of π mesons in nuclear matter from the experimental σ_a. Comparison of the approximate formula for σ_a with the results of our exact calculations shows good agreement for small $\sigma_a/\pi R_0^2$, but serious disagreement for larger K, since the exact calculation allows σ_a to *exceed* πR_0^2 for relatively small values of KR_0. The results are compared in Tables I and II. The discrepancy may be considered as due to the small number of l values required. In F. S. T. a summation over l values is replaced by an integral, and this approximation fails when only the first few l values are involved in the series. The last column of Table I labeled σ_{modified} is obtained by multiplying the F. S. T. cross section by $(k_1/k_0)^{1.5}$. It is seen that the resulting values are in much better agreement with the exact values. Thus an attractive potential increases σ_a and a repulsive potential decreases σ_a. It is not known how this result would be modified for a large k_0R_0. Note that this effect is different from the classical orbit distortion by the outside Coulomb potential, which also affects the probability that a particle will strike the nucleus.

In our phase-shift calculations $k_0R_0=3.600$, so it would be customary to neglect terms for $l\geq 4$. To be certain that no errors were involved, however, we calculated terms through $l=5$, and it was evident from the results that contributions from $l\geq 6$ would really be negligible. The theoretical angular distribution of the cross section has been calculated for the sixteen choices of complex potential each for π^+ and π^- listed in Table IV of the preceding paper. Tables of values for the calculated cross sections, the scattering amplitude f, and of p and q where $f=(p+iq)/ik_0$, are given in Pevsner,[10] from which Figs. 10 and 9 of the preceding paper and Figs. 1 to 7 of this paper can be calculated. It was found to be quite useful to make vector plots of $f=(p+iq)/2ik_0$ *vs* θ, of the type shown in Figs. 8 to 12, to interpolate values of $d\sigma/d\Omega$ between the values calculated. This was particularly true in the region of the diffraction minima, where the exact angle of the minimum, and the value of the minimum cross section, could readily be found. Also, it served as a check on the over-all calculations since the resulting curves behaved in a regular fashion when no errors were made. If an occasional point seemed to be out of line with the general curve, an extra check was initiated and the

[12] H. A. Bethe and R. R. Wilson, Phys. Rev. **83**, 690 (1951); Fowler, Fowler, Shutt, Thorndyke, and Whittemore, Phys. Rev. **91**, 135 (1953).

TABLE II. Calculated (total minus coherent-scattering) cross sections in mb, for 79-Mev π^+ mesons on aluminum.

Case number	$l=0$	$l=1$	$l=2$	$l=3$	$l=4$	$\sum_{l=0}^{4} \sigma_{r,l}$
1	41.3	88.8	189.7	142.7	24.6	487.1
2	28.0	79.2	152.1	81.0	13.7	354.0
3	17.1	48.8	96.7	44.2	7.0	213.8
4	4.2	11.7	24.1	9.5	1.4	50.9
5	26.1	88.4	132.9	61.5	11.6	320.5
6	0	0	0	0	0	0
7	36.2	122.6	160.9	85.3	18.1	423.1
8	26.8	93.7	109.8	48.3	14.3	292.9
9	16.7	60.2	64.8	25.4	4.9	172.0
10	33.5	82.8	70.6	31.0	5.4	223.3
11	39.0	116.5	210.0	149.9	32.4	547.8
12	40.8	122.7	214.9	171.2	42.6	596.1
13	40.0	104.5	210.6	175.0	35.0	565.1
14	41.7	138.5	218.0	195.1	45.0	638.4
15	41.6	82.3	181.7	186.3	30.8	522.7
16	43.1	108.7	212.6	231.0	54.5	649.9
$(2l+1)\pi ƛ^2$	43.7	131.1	218.5	305.9	393.3	1092.5

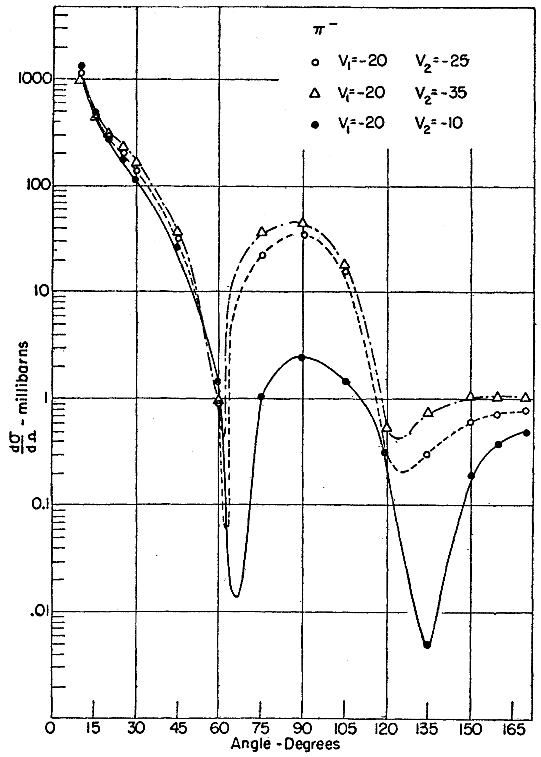

FIG. 5. Differential cross section for 79-Mev π^- mesons on aluminum calculated using phase-shift analysis on optical model for the nucleus, for $V_1=-20$ Mev, $V_2=-10$, -25, -35 Mev. Plotted points correspond to calculated points.

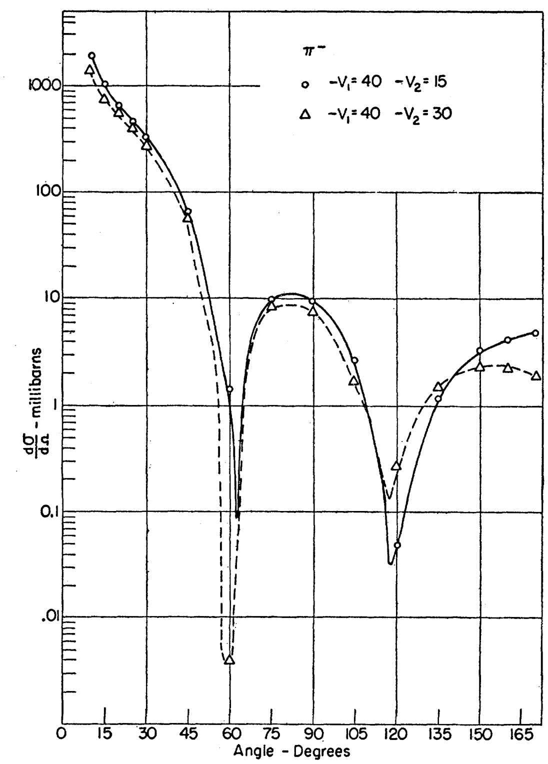

FIG. 6. Differential cross section for 79-Mev π^- mesons on aluminum calculated using phase-shift analysis on optical model for the nucleus, for $V_1=-40$ Mev, $V_2=-15$, -30 Mev. Plotted points correspond to calculated points.

error found and corrected. The results for a number of cases are shown in Figs. 1 to 7.

For comparison we also carried through a number of Born approximation type calculations, using various modifications of the usual Born approximation procedure to try to obtain a better agreement with various features of the phase shift calculation results. The various procedures are listed as Born approximations 1 to 5, and are discussed below. If $f_a(\theta)$ is the basic scattering amplitude for a point nucleus, then $f_a(\theta)f_b(\theta)$ is the Born approximation scattering amplitude from an extended nucleus, where for a spherically symmetric distribution $\rho(r)$ of nuclear density,

$$f_b(\theta)=\int_0^\infty r^2\rho(r)\left(\frac{\sin qr}{qr}\right)dr,$$

where $q=2k\sin(\theta/2)$ is proportional to the momentum change in scattering, and we have the integral of $\rho(r)$ normalized to unity over the nuclear volume.

Born approximation 1 (modified).—We use $f_a(\theta)$ for a point nucleus:

$$f_a(\theta)=-\frac{2m}{\hbar^2}\left\{\frac{Z_1Z_2e^2}{q_0^2}+\tfrac{1}{3}R_0^3(V_1+iV_2)\right\}. \quad (7)$$

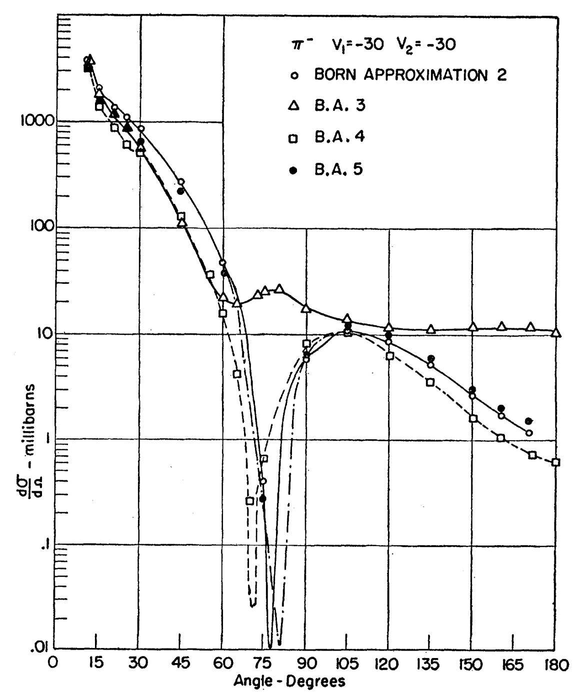

FIG. 7. Differential cross sections for Born approximation No. 1 (described in text). π^- curves for $V_1=-30$ Mev, $V_2=-30$ Mev. Plotted points correspond to calculated points.

For $f_b(\theta)$, the nuclear distribution form factor, use $q_1 = 2k_1 \sin(\theta/2)$ rather than $q_0 = 2k_0 \sin(\theta/2)$, where k_0 and $k = k_1 + ik_2$ are the values of k at $r = \infty$, and inside the nucleus respectively. The use of k_1 rather than k_0 in $f_b(\theta)$ brings the diffraction minima to about the same angle as for the phase-shift calculation.

Also in f_b, we weigh interior regions of the nucleus less than the surface by a factor $e^{-k_2(R_0-r)}$ to try to take account of the attenuation effects at least approximately. Thus

$$f_b = \frac{3}{R_0^3} \int_0^{R_0} r^2 \left(\frac{\sin q_1 r}{q_1 r} \right) e^{-k_2(r_0 - r)} dr. \tag{8}$$

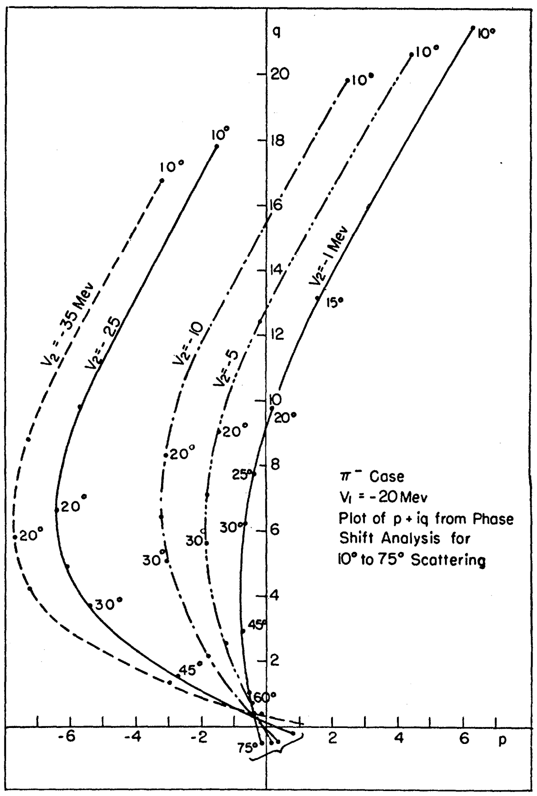

FIG. 8. Plot of $p+iq$ for π^- mesons from phase-shift analysis for 10° to 75° scattering, where $d\sigma/d\Omega = f^2$, $p+iq = 2ik_0 f$. Curves are plotted for $V_1 = -20$ Mev, $V_2 = -1, -5, -10, -25, -35$ Mev.

Born approximation 2 (standard).—Here we use Eq. (7) for $f_a(\theta)$ and use the regular Born approximation also for $f_b(\theta)$. Thus, this is the usual first Born approximation.

$$f_b = \frac{3}{R_0^3} \int_0^{R_0} r^2 \left(\frac{\sin q_0 r}{q_0 r} \right) dr. \tag{9}$$

Born approximation 3 (modified).—Here we use Eq. (10) for $f_a(\theta)$ and use $q = 2(k_1 + ik_2)\sin(\theta/2)$ for $f_b(\theta)$; thus q is complex.

$$f_b = \frac{3}{R_0^3} \int_0^{R_0} r^2 \left(\frac{\sin qr}{qr} \right) dr. \tag{10}$$

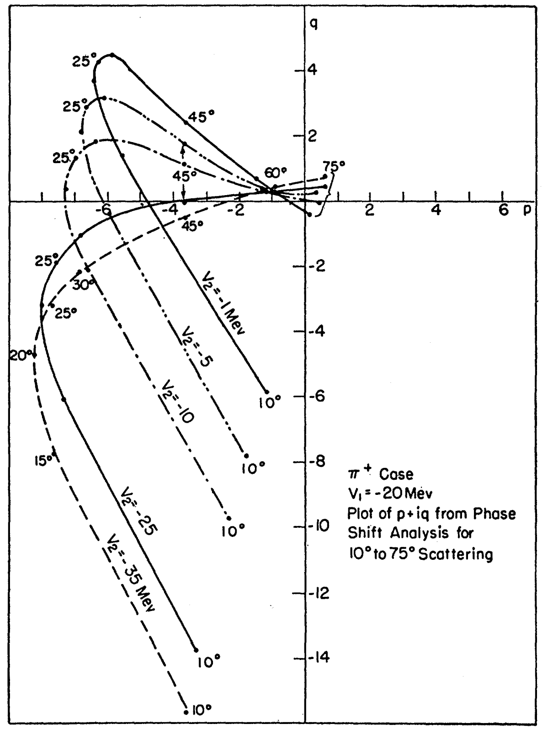

FIG. 9. Plot of $p+iq$ for π^+ mesons from phase-shift analysis for 10° to 75° scattering.

Born approximation 4 (modified).—Here we do not separate $f(\theta)$ into a product of f_a and f_b, since the external Coulomb contribution from the central protons

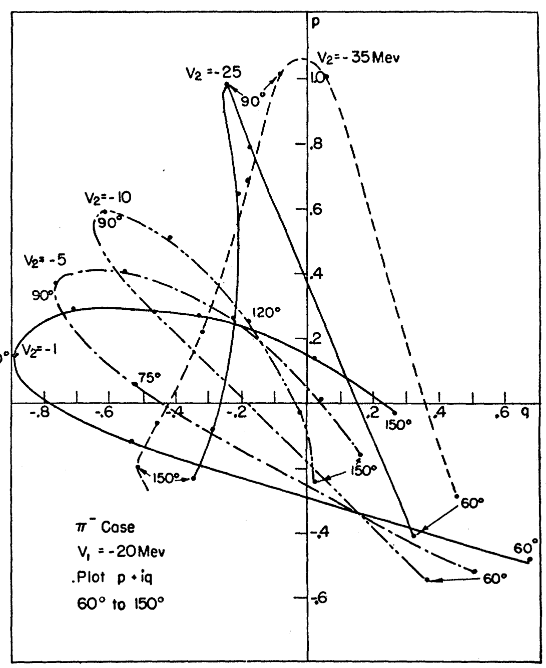

FIG. 10. Plot of $p+iq$ for π^- mesons from phase-shift analysis for 60° to 150° scattering.

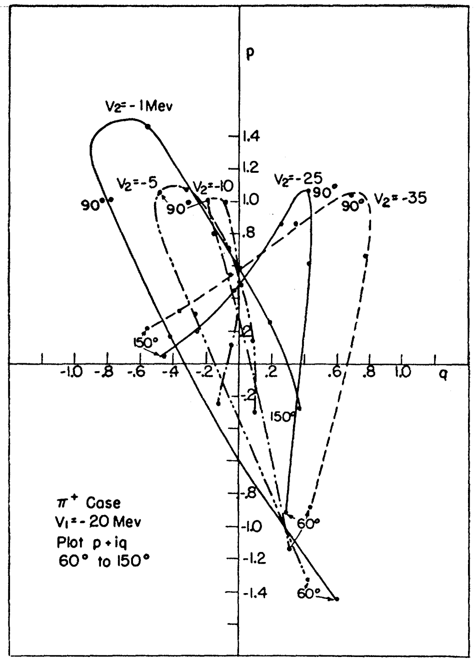

FIG. 11. Plot of $p+iq$ for π^+ mesons from phase-shift analysis for 60° to 150° scattering.

is not attenuated, even if their short-range force effect is attenuated.

$$\begin{aligned}
&\text{(a)}\quad f=f_i+f_0\\
&\text{(b)}\quad f_i=-\frac{2m}{\hbar^2}\int_0^{R_0} r^2\left(\frac{\sin q_0 r}{q_0 r}\right)\times(V_1+iV_2)e^{-k_2(R_0-r)}dr, \qquad (11)\\
&\text{(c)}\quad f_0=-\frac{2m}{\hbar^2}\int_{R_0}^{\infty}\left(\frac{\sin q_0 r}{q_0 r}\right)\frac{Z_1 Z_2 e^2}{r}dr.
\end{aligned}$$

Born approximation 5 (modified).—Here we used the same technique as in approximation 4, but without the attenuation. This differs from the true Born approximation (No. 2) in that the potential is taken as constant for $r<R_0$, while in No. 2, the true inside Coulomb potential also appears. Thus this approximation agrees with the procedure for the phase shift calculations in holding V fixed for $r<R_0$.

The results of the calculations are partly available from inspection of Figs. 3, 4, and 7. In general, the separation into $f_a f_b$ always gives a true zero at the interference minima except in approximation 3, when a complex q is used. When the outside q_0 was used, the diffraction minima occurred at angles independent of the choice of V_1 and V_2, while the phase-shift calculations gave minima at angles that decreased as V_1 became more negative. The angles of the minima were well matched by the Born approximation calculation using q_1 rather than q_0 for f_b.

Approximation 3 represents one attempt to obtain the proper damping of the diffraction minima. The calculations show considerable overdamping in this case. The results of approximations 4 and 5 also should not have true zeros, but they are much less damped than the phase-shift calculations results.

The absence of strong diffraction dips in the experimental curves may represent the true situation for the elastic scattering, but it cannot be stated with certainty that such strong dips are not present in the true elastic scattering. This could occur through a combination of the effects of limited experimental angular resolution and statistical accuracy, plus the contribution from inelastic scattering at the elastic scattering minima. This inelastic scattering could contribute if the final nucleus were left in a state of only a few Mev excitation. In this connection we note that the minima for the elastic scattering in Born approximation occurs when

$$\langle i|\textstyle\sum_j e^{i\mathbf{q}\cdot\mathbf{r}_j}|i\rangle=0,$$

where $|i\rangle$ is the nuclear ground-state function and $\mathbf{r}_j$ is the position coordinate of one of the nucleons responsible for the interaction. This implies that the state $\sum_j e^{i\mathbf{q}\cdot\mathbf{r}_j}|i\rangle$ is orthogonal to the ground state and could correspond to a relative maximum in the inelastic scattering.

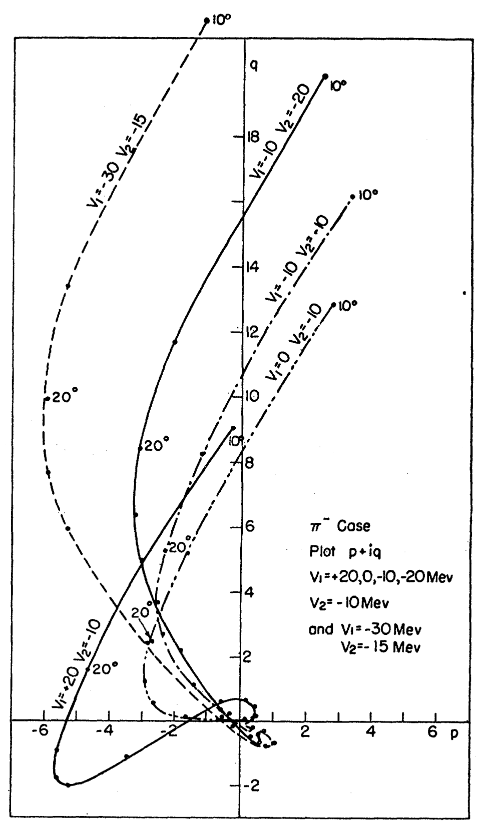

FIG. 12. Plot of $p+iq$ for π^- mesons, for $V_1=+20$, 0, -10, -20 Mev, $V_2=-10$ Mev; and also for $V_1=-30$ Mev, $V_2=-15$ Mev.

Schiff[13] has investigated the Born approximation results for the elastic scattering of high-energy electrons using various nuclear distributions. The Born approximation $f_b(\theta)$ is just the expectation value of $(\sin qr/qr)$ for the responsible nucleons. This is a real oscillatory function of qr. The oscillatory $\sin qr$ factor is multiplied by $r\rho(r)$ so $\langle\sin qr/qr\rangle$ tends to oscillate about zero as q increased if $\rho(r)$ approximates a uniform distribution. The absence of strong diffraction effects in the original electron scattering experiments for Pb and Au were thus originally interpreted as implying that $\rho(r)$ must be very peaked at $r=0$ like an exponentially decreasing function of r. Exact phase-shift calculations of Yennie, Ravenhall, and Wilson[14] showed that the Born approximation results were misleading for high-Z elements. Thus, for a uniform nuclear distribution, $f(\theta)$ does oscillate in sign, but it is no longer a real function and passes some distance from the origin in the complex plane, to an even greater degree than we find in our calculations. This greatly decreased the diffraction effects and has led to their presently favored choice of an approximately uniform density in a main central region followed by a gradual dropping off in an edge region. The higher-energy electron elastic scattering does show damped diffraction minima[15] which are relatively smaller in the regions where large quadruple distortions effectively increase the blurring of the edge region.

For low-Z elements Yennie *et al.* find that the phase shift analysis gives much closer agreement with the Born calculation, as would be expected for weaker interaction. In the case of meson scattering, however, the interaction is much stronger than for the electron scattering. For a U. N. M. (uniform nuclear model) the Born analysis gives minima for $qR=4.49$, 7.73, etc., and this feature has been used in some cases of nuclear scattering to determine the effective value of R. In this connection it is interesting to note that our modified Born calculation using k_1 instead of k_0 to calculate q seems to give the position of the minimum in much improved agreement with the phase-shift value. This implies that the angles of the minima are decreased roughly in proportion to the ratio of the particle momentum inside the nucleus to that outside. For the experiments on the elastic scattering of ~20-Mev protons by nuclei the outside proton momentum is quite close to that of our ~80-Mev mesons and the effective nuclear potential depths are comparable for mesons and nucleons. Thus we might expect a great similarity between the angular distribution of the scattering in the two cases. This is misleading, however, since the 20-Mev protons roughly double or triple their kinetic energy on entering the nuclear potential, while the fractional increase is much smaller for the 80-Mev mesons. Thus the protons would be expected to have their minima at considerably smaller angles than mesons of the same (outside) momentum.

An additional possible qualitative feature of difference in the position of diffraction minima for electrons, mesons, and nucleons come from the absorption term. The nucleus is transparent to electrons but has a mean free path for incoherent processes of the general order of magnitude of nuclear dimensions for ~20-Mev nucleons and fast mesons. The ratio of this mean free path to the nuclear radius is energy-dependent and differs for nucleons and mesons. It is of interest to consider an extreme situation where the mean free path is very small compared to nuclear dimensions. If this situation is treated using a modified Born analysis, where contributions from interior regions are decreased by the attenuation factor of an incoming spherical wave, then $\langle\sin qr/qr\rangle$ will only emphasize the outer regions of the nucleus and the minima will move to smaller values of qR, approaching π rather than 4.49 for the first minimum. For a nuclear distribution having a gradual dropping off of $\rho(r)$ in an extended edge region, this could lead to a continual increase in the effective R as the mean free path decreases. This would lead, according to this reasoning, to a decreasing angle for the minima, and also more pronounced minima when the relative contribution of the central region is decreased.

[13] L. I. Schiff, Phys. Rev. **92**, 988 (1953).
[14] Yennie, Ravenhall, and Wilson, Phys. Rev. **95**, 500 (1954).
[15] R. Hofstadter, Proceedings of Fifth Annual Rochester Conference on High-Energy Physics (Interscience Publishers, Inc., New York, 1955).