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Statistical Theory of Delayed-Coincidence Experiments*

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The consistent use of a general statistical theory makes possible the elimination of ambiguities and of idealizing assumptions from the interpretation of delayed-coincidence experiments. Introduction of the concept of the "total coincidence counting rate" (which can be determined experimentally) makes possible the definition of resolving time, thereby eliminating discrepancies between earlier definitions; it also provides means for relating the coincidence efficiencies directly to the number of source events.

The effect of random time lags on coincidence curves is calculated and experimental methods for the determination of time lags are derived. The statistical errors in the determination of moments of a coincidence curve are calculated and outlined in detail for first moment investigations. It is shown that: (1) the best choice of the resolving time is a (experimentally measurable) weighted rms of all time delays present in the measurement; (2) using the best choice of the resolving time the standard error of the centroid, obtained by successive measurements of the points of a coincidence curve, is approximately twice the least theoretical standard error (that could be obtained for the total time of observation); (3) the moment method can be applied generally for the determination of mean time delays; other methods, while applicable with some restrictions, can lead to similar or greater statistical errors.

I. GENERAL CONSIDERATIONS

A CONSEQUENCE of recent improvements[1–10] in coincidence counting techniques is that a reformulation of some basic definitions has become necessary. For example, two hitherto equivalent definitions of the resolving time of a coincidence circuit are no longer equivalent or even, strictly speaking, meaningful. We shall demonstrate that a consistent and unambiguous set of definitions for all parameters needed in the interpretation of delayed-coincidence measurements can be formulated in terms of experimentally observable quantities without any idealizing assumptions or approximations.

The basic equation for delayed coincidence experiments is given by the convolution integral[11–15]:

$$N(T)=\int_{-\infty}^{+\infty} P(T-t)w(t)dt, \tag{1}$$

where T is the inserted time delay, $N(T)$ is the coincidence curve for the source whose decay time is to be measured, $P(T)$ (usually termed a "prompt" curve) is the coincidence curve for a source of simultaneous events, and $w(t)$ is the normalized probability density for the time interval t between entry of particles of the source to be measured into the respective detectors. $P(T)$ and $N(T)$ are meant to be taken with the same source strength. Equation (1) is valid under the following conditions: (a) the quantities t and T are interchangeable, i.e., the inserted delay mechanism does not materially affect the pulse shapes; (b) the pulse shape distributions in the respective channels are the same for both sources. Condition (a) usually offers no experimental difficulty, as one generally uses short delay cables with negligible attenuation. Condition (b) can be met either by assuring that the same type and energy of radiation enters the detectors from both sources,[9] or by eliminating the effect of any discrepancy between the radiations by proper pulse shaping or pulse height selection.

As has been pointed out earlier,[14] a consequence of Eq. (1) is that the moments of N can be expressed in terms of those of P and w by the relations

$$M_r(N)=\sum_{k=0}^{r} \frac{r!}{k!(r-k)!} M_{r-k}(P)M_k(w), \tag{2}$$

where

$$M_r(N)=\int_{-\infty}^{+\infty} T^r N(T)dT.$$

This set of equations can be solved for the moments of w, thus determining the latter function. If w is known except for the values of a finite set of parameters, as many of Eqs. (2) will be needed as there are parameters to be evaluated. It is convenient to rewrite Eqs. (2) in terms of the normalized moments $\mu_r=M_r/M_0$. Since

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$M_0(N)=M_0(P)$ for equal source strengths, one obtains

$$\mu_r(N)=\sum_{k=0}^{r}\frac{r!}{k!(r-k)!}\mu_{r-k}(P)\mu_k(w). \tag{3}$$

Since the normalized moments are independent of the source strengths, Eq. (3) is valid for any source strength.

Another parameter, the "total coincidence counting rate," to be denoted by C, will be needed for a clear understanding of the phenomena involved.[16] In particular, it is necessary for the definition and determination of resolving time and efficiency and for the investigation of time lags and statistical errors. We define a "coincidence-countable" pulse pair to be a pair of pulses, one from each detector, having the property that they will produce a coincidence count for some range or set of ranges of T of nonzero measure. The property that a pair of pulses be coincidence-countable is then dependent on the shapes of both pulses and the operation of the coincidence circuit but independent of the relative time orientation of the two pulses. The quantity C is then defined to be the number of coincidence-countable pulse pairs which originate in unit time from related events at the source.

The description of the following thought experiment may serve to clarify the definition of C. Let a multichannel delayed-coincidence apparatus observe the pulses from two detectors, using a source of related events. By a multichannel apparatus is meant a system in which the detector outputs are branched and simultaneously observed by a number of delayed-coincidence circuits, each having a different fixed value of T but being otherwise identical. Thus, a finite set of points spanning the delay curve is obtained in one measurement. Now, let the discriminator outputs of all the coincidence circuits go to one scaling circuit of sufficiently great dead time that only one count will be registered for one source event, regardless of how many of the coincidence circuits respond to it. Then the counting rate registered by the scaler will (after substraction of the chance coincidence counting rate in the usual way) asymptotically approach C as the number of channels is increased indefinitely with a proportional decrease in ΔT, the interval between adjacent T values.

Note.—The operational conditions of the coincidence circuit change from time to time because of internal noise. Therefore it may happen (in the case of small pulses) that the same pulse pair which is coincidence countable in one state of noise is not coincidence countable in another state. The quantity C, given by the above definition and appearing as the result of the above thought experiment, is an expectation over the pulse-shape distribution and over the distribution of all noise parameters inherent in the coincidence circuit.

Practical methods for the direct experimental measurement of C have been devised[17,18]; the general principle will be described briefly in Sec. IV and the methods presented in detail in another paper.[19]

The theory can further be developed most simply in terms of $p(T)=P(T)/C$ and $n(T)=N(T)/C$, which will be called "reduced coincidence curves." These functions define the probability of a single coincidence-countable pulse pair producing a delayed coincidence count for a delay of value T. In terms of p and n, Eq. (1) can be rewritten as

$$n(T)=\int_{-\infty}^{+\infty}p(T-t)w(t)dt. \tag{4}$$

By applying the moment theorem of Eq. (2) to p and n, one obtains the important result:

$$M_0(n)=M_0(p), \tag{5}$$

i.e., the area under a reduced coincidence curve is independent of the source strength and independent of time delays at the source. Furthermore, since the coincidence circuit cannot distinguish time delays at the source from those introduced by the detectors or the coincidence device, the above statement can be extended to read; the area under a reduced coincidence curve is independent of all time delays, regardless of origin. For a given coincidence circuit, this area depends only on the pulse-shape distributions.

Application of Eq. (2) to a simple parent-daughter decay is straightforward.[14] If one detector responds only to one of the particles, which case will be called "asymmetric" (as, for example, in a $\beta-\gamma$ experiment), then

$$\mu_1(N)=\mu_1(P)+\theta, \tag{6}$$

i.e., the mean life θ is given by the displacement of the centroids of the N and P curves.

In the event that both detectors have the same response to either particles, which case is termed "symmetric" (as is true in a $\gamma-\gamma$ experiment with γ's of similar energy), and in a coordinate system in which $\mu_1(P)=0$, Eq. (3) yields

$$\mu_2(N)=\mu_2(P)+2\theta^2. \tag{7}$$

The generalization of the theory to the analysis of the mean lives of a radioactive family will be given in Appendix A.

II. RESOLVING TIME

A careful analysis shows that two different time magnitudes are needed to describe the resolution of a coincidence device and the statistical accuracy of time measurements[16] and by their use the inconsistencies between several earlier definitions of the resolving time can be eliminated. The problems encountered in attempting to define the resolving time of a coincidence circuit can best be appreciated by considering some simple cases.

First, let us assume that all pulses in a given channel are of the same shape and that no random time lags between events and pulses are introduced by the equipment. Then there will be a well-defined interval in the time coordinate describing the separation of the members of a pulse pair for which a coincidence will be

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recorded. Without loss in generality the possibility that this interval consists of several separate subintervals can be disregarded here. The resolving time, τ, is customarily defined as half of the magnitude of this interval, and could be measured experimentally in two ways. One could either measure the coincidence curve for a prompt source, obtaining a rectangular curve of width 2τ, or one could observe the chance coincidence counting rate, N_c, for unrelated sequences of pulses in the two channels, obtaining

$$N_c = N_A N_B \cdot 2\tau, \tag{8}$$

where N_A and N_B are the respective "singles" counting rates.

Now, let us introduce random time lags between the actual events and the pulses which come from them. The value of τ will be unchanged, and can be determined as before from Eq. (8), since the chance coincidence counting rate will not be influenced by the random lags. However, the prompt coincidence curve will be broadened and will no longer be of rectangular shape. The maximum of the prompt curve will even be lowered if *any* of the random time lags exceed τ. Despite the distortion of the coincidence curve, there is a convenient functional of it that will yield τ. Noting that in the absence of random time lags, the height of the rectangular coincidence curve will be C, the area of the curve divided by C will be 2τ. Now, invoking the principle expressed by Eq. (5), that the area of a reduced coincidence curve is independent of time delays, we can write

$$\tau = \frac{1}{2C}\int_{-\infty}^{+\infty} P(T)dT = \frac{1}{2}\int_{-\infty}^{+\infty} p(T)dT, \tag{9}$$

which is also valid if $P(T)$ is replaced by $N(T)$, the coincidence curve for any source of related events, prompt or otherwise.

On the other hand, we see that the width of the coincidence curve is no longer given by the resolving time, τ. Nevertheless, the width of the prompt coincidence curve determines the statistical accuracy of the measurement of time delays. Therefore, it is desirable to introduce a second characteristic time, τ', that is a measure of this width. Theoretically, the best choice would be the square root of the normalized second moment about the mean of the prompt curve, $\sigma = \{\mu_2 - \mu_1^2\}^{\frac{1}{2}}$, which appears in the calculations of statistical accuracy. However, in practice, one prefers to choose a quantity which is easier and quicker to calculate from the data. Therefore we choose the definition

$$\tau' = \frac{1}{2p_{\max}}\int_{-\infty}^{+\infty} p(T)dT = \frac{1}{2P_{\max}}\int_{-\infty}^{+\infty} P(T)dT, \tag{10}$$

which has the virtues that τ' reduces to τ in the absence of time lags, and that τ' is usually very close to the half-width at half-maximum of the coincidence curve, being the half-width of the rectangle that has the same height and area as the curve.

If one writes

$$\tau'^2 = \alpha\sigma^2, \tag{11}$$

the constant α is of the order of magnitude unity for the types of delay curves usually obtained. This can be seen from the examination of a few analytical functions. For example, for a simple exponential $\alpha = 1$, for an isoceles triangle $\alpha = \frac{3}{2}$, for a Gaussian $\alpha = \pi/2$, and for the extreme case of a rectangle $\alpha = 3$. Therefore, in the estimation of statistical error one is often justified in using τ' instead of σ.

The final step is to extend the definitions to the general case in which each channel receives a distribution of pulse shapes. Then a resolving time can be defined in the original manner for every possible kind of coincidence-countable pulse pairs, and τ should be the average of these resolving times over the pulse shape distributions. We again consider first the case of no random time lags, and describe all possible coincidence-countable pulse pairs by a running index "k." Letting C_k be the number of "type k" pulse pairs appearing in unit time, the prompt coincidence curve will be given by

$$\Psi(T) = \sum_k C_k \psi_k(T), \tag{12}$$

where ψ_k is the coincidence curve for a single "type k" pulse pair, i.e., a rectangle of unit height and of the width $2\tau_k$. Without loss in generality, we can assume here that all $\psi_k(T)$ have the same centroid, as failure to meet this condition can be compensated for by introduction of appropriate time lags. Then, since $\sum C_k = C$, the maximum of $\Psi(T)$ will be C, and the average resolving time, τ, will be given by

$$\tau = \frac{\sum C_k \tau_k}{\sum C_k} = \frac{1}{2C}\sum C_k \int_{-\infty}^{+\infty} \psi_k(T)dT = \frac{1}{2C}\int_{-\infty}^{+\infty} \Psi(T)dT, \tag{13}$$

as before.

The internal noise of the coincidence circuit causes fluctuations in τ_k. These will be taken into consideration in Sec. III. Replacing τ_k in Eq. (13) by its average for "type k" pulse pairs, Eq. (13) and the above derivation remain valid for the general case.

The argument already given for the case of uniform pulses again serves to extend Eq. (9) to apply to coincidence curves influenced by random time lags. Thus, 2τ is always given by the area of the reduced coincidence curve $p(T)$ or $n(T)$.

The chance coincidence counting rate will still be given by Eq. (8), provided the quantity $N_A N_B$ only includes coincidence-countable pulse pairs.

The definition of τ' is based on the shape alone of the observed prompt coincidence curve and will also be given in the general case by Eq. (10). In the literature both τ [defined by Eq. (8)] and τ' have occasionally been called "resolving time." We suggest calling τ the "resolving time" and τ' either the "practical resolving time" or, better, just the "half-width of the prompt curve." Since $p_{\max} \leq C$, one always has $\tau' \gg \tau$.

III. EFFECT OF TIME LAGS ON COINCIDENCE CURVES

We will investigate the effect of time lags originating (1) in the detectors and (2) in the rest of the equipment. First we consider prompt events.

(1) The pulses of a "type k" pulse pair both appear with some time delay after the events; the difference of these time delays will be denoted by t_k'. It must be noted that the definition of t_k' is arbitrary since no unique definition for the timing of pulses of different shape can be given. (For example, one can define the time of appearance of a pulse as the center of gravity of a voltage curve or as the time at which the output voltage, or current, rises to a given level, etc.). However, when changing from one definition to another, only the origin of the t_k' axis will change and the variance $v(t_k')$, which is the quantity of greater interest in the investigation of time lags, remains unaltered.

(2) The "type k" pulse pairs for which $t_k'=0$ produce a prompt curve which will be denoted by $P_{k0}(T)$. If there were no statistical fluctuations in the coincidence circuits, P_{k0} would be a rectangle of height C_k, width $2\tau_k$, with its center at (say) T_k. One could then regard $T_k=t_k''$ (where $T=0$ corresponds to equal inserted time delays in the two channels of the coincidence circuit) as the time lag originating in the coincidence equipment (excluding the detectors). One should note that while t_k'' depends on the definition used for t_k', the sum $t_k'+t_k''$ (for prompt events) is independent of that definition.

Now, if $\rho_k'(t_k')$ is the probability density for t_k' and if $t_k=t_k'+t_k''$, the probability density for t_k will be $\rho_k(t_k)=\rho_k'(t_k')$ since t_k'' is a constant. Therefore, when ignoring the fluctuations (noise) of the coincidence circuit,

$$P_k(T)=C_k\int_{-\infty}^{+\infty}\psi_k(T-t_k)\rho_k(t_k)dt_k, \tag{14}$$

where $\psi_k(T)$ is a rectangle of height unity, width $2\tau_k$, with its center at $T=0$.

The prompt curve for all types of pulse pairs is

$$P(T)=\sum_k C_k\int_{-\infty}^{+\infty}\psi_k(T-t_k)\rho_k(t_k)dt_k, \tag{15}$$

its centroid is

$$\mu_1(P)=\frac{\sum C_k 2\tau_k\mu_1(\rho_k)}{\sum C_k 2\tau_k}=\frac{\sum C_k 2\tau_k \bar{t}_k}{\sum C_k 2\tau_k}, \tag{16}$$

and using the notation $\sigma^2=\mu_2-\mu_1^2$, one has

$$\sigma^2(P)-\sigma^2(\Psi)=\frac{\sum C_k 2\tau_k\{v(t_k')+[\bar{t}_k-\mu_1(P)]^2\}}{\sum C_k 2\tau_k}$$

$$=\frac{\sum C_k 2\tau_k\langle[t_k-\mu_1(P)]^2\rangle_{\text{Av}}}{\sum C_k 2\tau_k}, \tag{17}$$

where

$$\Psi(T)=\sum_k C_k\psi_k(T). \tag{18}$$

The right-hand sides of Eqs. (16) and (17) represent the weighted mean and weighted mean square deviation of the time lags, with the weighting factors $2\tau_k$ for t_k.

Now, to take into account also the effect of statistical fluctuations originating in the coincidence circuit, we consider again the "type k" pulse pairs for which $t_k'=0$. Identical pulse pairs now produce different coincidence curves, depending on the state of internal noise of the coincidence circuit. If one distinguishes by a running index l the different states of noise of the circuit, each determined by a set of values of all the noise parameters, then for each such "state l" there will be a $\psi_{kl}(T)$ rectangular curve of height unity and width $2\tau_{kl}$ around $T=0$, a C_{kl} total coincidence rate, and a t_{kl}'' time lag. The prompt curve $P_{k0}(T)$ will therefore be

$$P_{k0}(T)=\sum_l C_{kl}\psi_{kl}(T-t_{kl}''), \tag{19}$$

and

$$P_k(T)=\sum_l C_{kl}\int_{-\infty}^{+\infty}\psi_{kl}[T-(t_k'+t_{kl}'')]\rho_k'(t_k')dt_k'. \tag{20}$$

The centroid is given by

$$\mu_1(P_k)=\frac{\sum_l C_{kl}2\tau_{kl}\{\mu_1(\rho_k')+t_{kl}''\}}{\sum_l C_{kl}2\tau_{kl}}=\frac{\sum_l C_{kl}2\tau_{kl}\bar{t}_{kl}}{\sum_l C_{kl}2\tau_{kl}}, \tag{21}$$

and

$$\sigma^2(P_k)-\sigma^2(\Psi_k)=\frac{\sum_l C_{kl}2\tau_{kl}\{v(t_k')+[\bar{t}_{kl}-\mu_1(P_k)]^2}{\sum_l C_{kl}2\tau_{kl}}, \tag{22}$$

where

$$\Psi_k(T)=\sum_l C_{kl}\psi_{kl}.$$

The prompt curve $P(T)$ in the general case for all types of pulse pairs is

$$P(T)=\sum_k P_k(T)$$

$$=\sum_k\sum_l C_{kl}\int_{-\infty}^{+\infty}\psi_{kl}[T-(t_k'+t_{kl}'')]\rho_k'(t_k')dt_k', \tag{23}$$

the centroid is

$$\mu_1(P)=\frac{\sum_k\sum_l C_{kl}2\tau_{kl}\bar{t}_{kl}}{\sum_k\sum_l C_{kl}2\tau_{kl}}=\frac{\sum_k C_k 2\bar{\tau}_k\bar{t}_k}{\sum_k C_k 2\bar{\tau}_k}, \tag{24}$$

and

$$\sigma^2(P)-\sigma^2(\Psi)=\frac{\sum_k\sum_l C_{kl}2\tau_{kl}\{v(t_k')+[\bar{t}_{kl}-\mu_1(P)]^2\}}{\sum_k\sum_l C_{kl}2\tau_{kl}}, \quad (25)$$

where

$$\Psi(T)=\sum_k\Psi_k(T)=\sum_k\sum_l C_{kl}\psi_{kl}.$$

Equations (24) and (25) show that, also in the general case, when time delays of all origins are taken into account, the coincidence equipment "sees" the weighted average and weighted mean square deviation of the time lags, with the weighting factors equal to the individual resolving times. In addition, one can see that the effect of internal noise in the coincidence circuits is (1) to change the form of $\Psi(T)$ and (2) to introduce additional time lags appearing as fluctuations of t_k''.

The occurrence of these weighted statistics in the operation of the coincidence equipment can readily be understood in the following way: In a multichannel arrangement consisting of a large number of identical coincidence circuits with an inserted time delay ΔT between them, every "type k" input pulse pair would give a coincidence output simultaneously in $2\tau_k/\Delta T$ channels. Therefore in the calculation of the area and of higher moments of the $P(T)$ curve a "type k" pulse pair occurs with the relative frequency $C_k2\tau_k/\Sigma C_k2\tau_k$ instead of $C_k/\Sigma C_k$. What is true for a multichannel equipment is also true for the one-channel equipment since it gives a similar $P(T)$ coincidence curve in the successive measurements.

In Eq. (25), $\sigma^2(\Psi)$ represents the smallest possible width of the prompt curve compatible with the given pulse shape distribution and resolving times τ_{kl} of the equipment (i.e., a prompt curve without time lags, which could be called "prompt-prompt" curve) and the right side of the equation describes the broadening of the prompt curve by time lags. Introducing now the symbol $\hat{t}^2$ for the right sides of Eqs. (17) and (25), one can write

$$\sigma^2(P)=\sigma^2(\Psi)+\hat{t}^2. \quad (26)$$

It should be noted that $\Psi(T)$ can be determined in principle (e.g., by pulse-shape selection, approximated by pulse-height selection) but in most cases it is sufficient simply to write

$$\sigma^2(\Psi)=(1/\alpha)\tau^2, \quad (27)$$

and consequently

$$\sigma^2(P)=(1/\alpha)\tau^2+\hat{t}^2, \quad (28)$$

where α is the same as in Eq. (11), and can be estimated fairly well with some knowledge of the amplitude distribution.

Generalization to the case of the $N(T)$ curve can be obtained simply by adding the first and second moments of the $w(t)$ curve. Therefore,

$$\sigma^2(N)=(1/\alpha)\tau^2+\hat{t}^2+\sigma^2(w). \quad (29)$$

The problem of the variation of $\sigma^2(P)$ with varying conditions in the coincidence circuit is important for Sec. VI. There are methods for varying τ; $v(t_k')$ cannot be reduced below some theoretical limits for scintillation counters and little is known about the time lags $\bar{t}_k=\bar{t}_k'+\bar{t}_k''$ for different pulse shapes (or amplitudes).

The variation of $\hat{t}$ for different values of τ was investigated in a differential coincidence circuit. Using $\gamma-\gamma$ coincidences from Ni^{60} (established as "prompt" within 10^{-11} second[20]) and stilbene scintillators, τ alone was varied and the reduced coincidence curves taken. For τ varying from 7.9×10^{-10} second to 1.65×10^{-10} second, $\hat{t}$, calculated from Eq. (26) with $\alpha=\pi/2$ (i.e., taking Ψ to be Gaussian, since no pulse height selection was used), varied by less than 2 percent around its mean value 10.5×10^{-10} second. Thus for the differential coincidence circuit the functional dependence of $\sigma^2(P)$ on τ as given in Eq. (28) with constant α and constant $\hat{t}$ can be used.

Equations (17) or (25) can be used to obtain an estimate of time fluctuations of scintillation counters. This is important since, to our knowledge, no other methods are at present available to determine these time fluctuations.

For symmetric excitations of the two detectors, as in the above experiment, one can regard $\hat{t}/\sqrt{2}$ as the weighted rms of the time lags related to one channel. This gives 7.5×10^{-10} second for stilbene, 6.5×10^{-10} second for diphenyl acetylene and 3.5×10^{-10} second for diphenyl acetylene when a pulse-clipping method is used (anode pulses shorted by dynode pulses[21]). Since these values are near to the theoretical expectations for the scintillators[22,23] plus time lags in photomultipliers, one can conclude that the spread of time lags attributable to the varying pulse shapes is small ($\sim10^{-10}$ second or less), at least in the differential coincidence circuit. Using Čerenkov counters,[24] $\hat{t}/\sqrt{2}\sim2\times10^{-10}$ second was obtained, this can be attributed to the photomultipliers. Time lags due to the fluctuations of t_k'' have been investigated by the use of pulses branched from one detector (where $t_k'=0$) and were found to be of the order of a few times 10^{-11} second for a differential coincidence circuit, i.e., about one order of magnitude smaller than the time lags $\sqrt{v(t_k')}$ for scintillation counters.

To obtain results more specialized than the above weighted averages, e.g., for scintillation counters, and in particular to obtain the dependency of $v(t_k')$ on the total number of photoelectrons utilized, one has to apply Eqs. (17) or (25) with pulse-height selection in the experiment.

IV. EXPERIMENTAL DETERMINATION OF C

Figure 1 illustrates a possible scheme for obtaining the coincidence curve $N(T)$ and the value of C appro-

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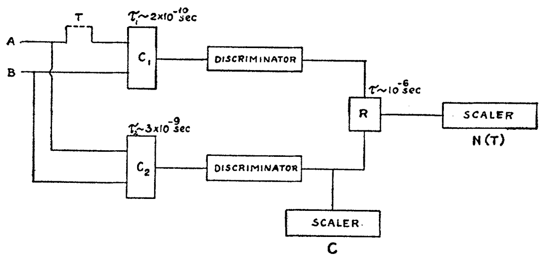

FIG. 1. Block diagram of the equipment for the determination of C.

priate to that curve. In the figure, C_1 represents the fast circuit, with a resolving time τ_1, that yields the coincidence curve; C_2 represents a similar circuit having a resolving time τ_2 at least an order of magnitude greater than both θ and $\hat{t}$. The type of coincidence circuits used for C_1 and C_2 in this measurement is immaterial. In Fig. 1, $\tau_1 \sim 2\times10^{-10}$ second and $\tau_2 \sim 3\times10^{-9}$ second are shown as used in an actual experiment. The level of the discriminator following C_2 is set sufficiently high and that of C_1 sufficiently low (even tolerating noise pulses in C_1), that any pulse pair yielding an output from C_2 great enough to trigger C_2's discriminator is certain to produce an output from C_1's discriminator if the delay time, T, is correctly chosen. R represents a far slower coincidence circuit, the purpose of which is to determine whether a pair of outputs from C_1 and C_2 stem from the same source events. The output counting rate of R as a function of T will be the coincidence curve $N(T)$, and the counting rate of the discriminator following C_2 will be C. In actual experiments,[20,25] we used for C_1 a differential coincident circuit in which one can vary τ independently of the pulse length.

Having determined C, one can use the reduced coincidence curve which is normalized to one pair of events, and present the following advantages:

(1) No correction is needed for the decrease in strength of a radioactive source while making a measurement.

(2) The change of solid angle and thereby counting efficiency introduced by displacing the source, as is done in some time-of-flight measurements, does not affect the reduced coincidence curve.

(3) If instead of utilizing the moments of the coincidence curves for the time measurements one uses a portion of the steep part of the curves (slope method), then if one measures C it is not necessary to normalize the prompt and delayed curves to the same area and therefore one does not need to measure the entire coincidence curves.

(4) If an initially unknown number of prompt coincidences is mixed with the desired pulses whose lifetime is to be measured, then one attempts to find out the ratio of mixing by introducing some changes in the experimental conditions. In such a case it is easier and faster to record changes of C than changes of areas of the coincidence curves.

(5) The C_2 circuit monitors the equipment and diminishes the effect of temporal fluctuations during the experiment.

V. COINCIDENCE EFFICIENCY

The chance coincidence counting rate was given in Eq. (8) as $N_A N_B \cdot 2\tau$, valid only if all pulse pairs are coincidence-countable. Equation (8) can be written in terms of the disintegration rate, ν, of the source by introduction of ϵ_A and ϵ_B, the efficiencies for the counting of singles:

$$N_c = \nu^2 \epsilon_A \epsilon_B \cdot 2\tau. \tag{30}$$

It should be noted that N_c as used in this section contains only chance coincidences originating from the source events. Other types of chance coincidences may be present in a given experiment and N_c should be corrected for them.

In practice, not all pulse pairs are coincidence-countable, and Eq. (30) must be modified to

$$N_c = \nu^2 \epsilon_A \epsilon_B \epsilon_c \cdot 2\tau, \tag{31}$$

where ϵ_c is a proportionality factor for the pulse pairs that are coincidence-countable. Although it is common practice to regard a quantity such as ϵ_c as the efficiency of the coincidence circuit proper, ϵ_c has no fixed value for a given circuit since it depends on the values of ϵ_A and ϵ_B, which in turn are determined by arbitrary discrimination levels. (In fact, it is possible to choose ϵ_A and ϵ_B in such a manner that ϵ_c is greater than unity.) However, the product $\epsilon = \epsilon_A \epsilon_B \epsilon_c$ is well defined, being the fraction of source events yielding coincidence-countable pulse pairs.

For the experimental determination of ϵ, we use the relation

$$C = \nu\epsilon. \tag{32}$$

By Eqs. (9), (31), and (32), ϵ can be expressed in terms of experimentally observable quantities as

$$\epsilon = C\int_{-\infty}^{+\infty} N(T)dT \Big/ N_c = \left[\int_{-\infty}^{+\infty} N(T)dT\right]^2 \Big/ 2\tau N_c. \tag{33}$$

Equation (33) shows that ϵ is independent of time magnitudes such as $\hat{t}$, θ, and τ, and that it is also independent of the source strength ν and can be determined experimentally without knowledge of the source strength. Furthermore, ϵ (being independent of θ) is the same for the prompt and decaying sources. Since ϵ is the ratio of the number of coincidence-countable pulse pairs to the number of source events [Eq. (32)], we call it the coincidence yield of the experiment.

[25] Bay, Henri, and McLernon, Phys. Rev. 97, 1710 (1955).

The ordinates of the reduced coincidence curves $n(T)$ and $p(T)$ give the fraction of coincidence-countable pairs which are in the proper time-delay interval (given by 2τ) around T and which therefore will be counted. It is usual to call $p_{\max}=P_{\max}/C$ the "coincidence efficiency" (as in reference 9). In general, one can call $p(T)$ and $n(T)$ the "coincidence-counting efficiencies" at T, which are related to the number of coincidence-countable pulse pairs. One also can directly relate the counting efficiencies to the number of source events by multiplying $p(T)$ and $n(T)$ by the coincidence yield ϵ.

VI. STATISTICAL ACCURACY OF MEAN-LIFE DETERMINATIONS

In the measurement of an individual point at T_i of a curve $N(T)$, $n_i=n(T_i)$ is the probability of obtaining a coincidence count per countable pulse pair. Let the duration of observation for one point be $[t]$ and let the number of countable pulse pairs occurring during $[t]$ be $\mathcal{C}$ and the number of observed counts be $\mathfrak{N}_i$. Then, $\mathfrak{N}_i$ may be regarded as the number of successes in $\mathcal{C}$ trials, with an *a priori* probability of success n_i. For a given $\mathcal{C}$, $\mathfrak{N}_i$ follows a Bernoulli distribution, its mean being $\mathcal{C}n_i$ and its variance $v(\mathfrak{N}_i)=\mathcal{C}n_i(1-n_i)$.

In the following, we have to distinguish between two different cases.

(1) *C is not measured.*—The number of trials $\mathcal{C}$ during $[t]$ fluctuates from point to point and

$$v(\mathfrak{N}_i)=\langle\mathcal{C}\rangle_{\text{Av}}n_i(1-n_i)+n_i^2v(\mathcal{C}). \tag{34}$$

With a Poisson distribution for $\mathcal{C}$ around its mean value

$$\langle\mathcal{C}\rangle_{\text{Av}}=[t]C,$$

the variance is found to be

$$v(\mathfrak{N}_i)=[t]Cn_i; \tag{35}$$

or, if one regards sampling values as experimental estimates for population values, and uses the same notation for sampling and population values, one gets

$$v(\mathfrak{N}_i)\cong\mathfrak{N}_i, \tag{36}$$

$$v(N_i)\cong N_i/[t], \tag{37}$$

$$v(n_i)\cong n_i/[t]C. \tag{38}$$

If the coincidence curve is obtained from a set of $\mathfrak{N}_i$'s taken at equidistant T_i values, μ_r can be approximated by $\sum T_i^rN_i/\sum N_i$. Taking the same $[t]$ for each point and employing the usual method of treatment of the propagation of errors,

$$\sigma^2(\mu_r)=\frac{\sum(T_i^r-\mu_r)^2N_i}{(\sum N_i)^2[t]}=\frac{\mu_{2r}-\mu_r^2}{\mathfrak{N}}, \tag{39}$$

where $\mathfrak{N}=\sum\mathfrak{N}_i$ is the total number of counts observed along the entire coincidence curve.

The variance of θ in terms of those of the moments used is

$$v(\theta)=\sigma^2[\mu_1(N)]+\sigma^2[\mu_1(P)] \tag{40}$$

for the asymmetric case, and

$$v(\theta)=\frac{1}{16\theta^2}\{\sigma^2[\mu_2(N)]+\sigma^2[\mu_2(P)]\} \tag{41}$$

for the symmetric case.

Simple formulas, particularly useful for order-of-magnitude estimates when planning experiments, result from the use of an approximation. This consists of fitting a Gaussian to the prompt-coincidence curve $P(T)$ and expressing all pertinent moments of $P(T)$ and $N(T)$ in terms of τ'/θ.[26] If one also takes $\mathfrak{N}$ to be the same for the two curves, then

$$\frac{v(\theta)}{\theta^2}=\frac{1}{\mathfrak{N}}\left\{1+\frac{4}{\pi}\left(\frac{\tau'}{\theta}\right)^2\right\} \tag{42}$$

for the asymmetric case, and

$$\frac{v(\theta)}{\theta^2}=\frac{1}{\mathfrak{N}}\left\{\frac{5}{4}+\frac{1}{\pi}\left(\frac{\tau'}{\theta}\right)^2+\frac{1}{\pi^2}\left(\frac{\tau'}{\theta}\right)^4\right\} \tag{43}$$

for the symmetric case.

Valuable information is obtainable from Eq. (39) concerning the optimal use of coincidence equipments among given circumstances. We restrict our discussion to the statistical error of μ_1, in the asymmetric case.

We determine first how to choose the best value of the resolving time for measuring a given lifetime and for a given time of observation. We assume that $\hat{t}$ is unaffected by the variation of τ as was shown to be the case for a differential coincidence circuit (Sec. III).

Using the notations of Eqs. (11), (26), and (28), one can write

$$\sigma^2[\mu_1(P)]=\frac{\mu_2(\Psi)+\hat{t}^2}{\mathfrak{N}}=\frac{\alpha^{-1}\tau^2+\hat{t}^2}{\mathfrak{N}}. \tag{44}$$

For $\mathfrak{N}$ we substitute

$$\mathfrak{N}=[t]C\sum n_i=[t]C2\tau/\Delta T, \tag{45}$$

where ΔT is the distance between two adjacent points. The measured points m are distributed along the length T_m-T_1 which is proportional to the width of the coincidence curve. Thus, writing

$$T_m-T_1=2\kappa\{\mu_2(P)-\mu_1^2(P)\}^{\frac{1}{2}}, \tag{46}$$

we have

$$\sigma^2[\mu_1(P)]=\frac{\kappa}{[T]C}\frac{(\alpha^{-1}\tau^2+\hat{t}^2)^{\frac{3}{2}}}{\tau}, \tag{47}$$

where $[T]=m[t]$ is the total time of observation for the entire coincidence curve. The value of κ will be

[26] Kanner, Bay, and Henri, Phys. Rev. **90**, 371 (1953).

discussed later. Equation (47) yields the optimal τ, denoted by τ_0, for which $\sigma^2[\mu_1(P)]$ is a minimum. We have

$$\tau_0^2=\tfrac{1}{2}\alpha\hat{t}^2, \tag{48}$$

and

$$\sigma^2[\mu_1(P)]=\left[\frac{3}{\alpha}\right]^{\frac{1}{2}}\frac{3}{2}\frac{\kappa}{[T]}. \tag{49}$$

When $\Psi(T)$ is a Gaussian, $\alpha=\pi/2$. When it is a rectangle, $\alpha=3$. Since $\Psi(T)$ is most probably between these two extreme cases, it is safe to write Eq. (48) as

$$\tau_0^2\sim\hat{t}^2 \tag{50}$$

for the $P(T)$ curve and

$$\tau_0^2\sim\hat{t}^2+\theta^2 \tag{51}$$

for the $N(T)$ curve. Thus the best choice for τ is the weighted rms of the time delays present in the measurement. This choice of τ corresponds to a $p_{\max}$ being ~ 60 to 70 percent of C [depending somewhat on the shape of the $\Psi(T)$ curve].

Since both $P(T)$ and $N(T)$ are involved in the measurement of a lifetime, the resulting optimal choice for τ will be determined by $\hat{t}$, θ, and the intensities of the prompt and decaying sources.

With τ_0, and $\alpha\sim 2$, one can write

$$\sigma^2[\mu_1(P)]\sim\frac{2\kappa}{[T]C}\hat{t}^2, \tag{52}$$

and

$$\sigma^2[\mu_1(N)]\sim\frac{2\kappa}{[T]C}(\hat{t}^2+\theta^2). \tag{53}$$

The factor κ is the range of the measured interval relative to the width of the curve. It is of practical interest to have κ as small as possible, i.e., to avoid long measurements with small counting efficiencies. Utilizing some geometrical properties present in individual cases, one finds means to shorten T_m-T_1. In practice one seldom needs $\kappa>3$ (or T_m-T_1 three times the half-width of the curve between half-maxima).

Larger reduction in κ can be achieved in the case of a pure parent-daughter exponential decay (if one is also sure that no mixture of other, e.g., prompt, coincidences is present). Using the differential equation

$$N(T)-P(T)=-\theta(dN/dT), \tag{54}$$

derived by Newton,[15] and its integrated form

$$\int_{T_1}^{T_m}N(T)dT-\int_{T_1}^{T_m}P(T)dT=\theta[N_1-N_m], \tag{55}$$

where $N_1=N(T_1)$ and $N_m=N(T_m)$, and choosing T_1, and T_m such that $N_1=N_m$, one finds that the areas of the two curves within the section T_m-T_1 are equal, i.e., the two curves can be normalized without measuring the outer parts. Calculating the first moments $\mu_1(N)$ and $\mu_1(P)$ for the measured sections, one obtains:

$$\theta=[\mu_1(N)-\mu_1(P)]\frac{1}{1-N_m(T_m-T_1)/A}, \tag{56}$$

where A represents the area of $N(T)$ or $P(T)$ within the measured section. With $N_m(T_m-T_1)/A$ small compared to unity, the variance of θ will be

$$v(\theta)\sim\frac{\mu_2(N)-\mu_1^2(N)}{\mathfrak{N}_N}+\frac{\mu_2(P)-\mu_1^2(P)}{\mathfrak{N}_P}+\frac{\theta^2(T_m-T_1)^2}{A^2}\frac{N_m^2}{\mathfrak{N}_1+\mathfrak{N}_m}. \tag{57}$$

By proper choice of T_1 and T_m the small additional error due to the third term in Eq. (57) can be overcompensated by the gain in $\mathfrak{N}_N$ and $\mathfrak{N}_P$.

Instead of measuring the first moments of the measured sections, one can utilize Eqs. (54) and (55) directly to calculate θ.[15] Dividing the measured sections into two intervals T_0-T_1 and T_m-T_0, where T_0 is the abscissa of the intersection of $N(T)$ and $P(T)$, one obtains

$$\theta=\frac{A_2-A_1-(a_2-a_1)\eta}{2N_0-(N_1+N_m)}, \tag{58}$$

where A_1 and A_2 are the areas under the $N(T)$ curve for the two intervals, a_1 and a_2 are the corresponding areas under the $P(T)$ curve, and $\eta=(A_1+A_2)/(a_1+a_2)$ is the normalizing factor. Disregarding the statistical error of η, one obtains, for $N_m\ll N_0$,

$$v(\theta)=\frac{(A_1+A_2)^2}{4N_0^2\mathfrak{N}_n}+\frac{(A_1+A_2)^2}{4N_0^2\mathfrak{N}_p}+\theta^2\left(\frac{N_m}{N_0}\right)^2\frac{1}{\mathfrak{N}_1+\mathfrak{N}_m}+\frac{\theta^2}{\mathfrak{N}_0}. \tag{59}$$

The first three terms are of the same order as in Eq. (57) since $(A_1+A_2)/2N_0$ represents the width of the $N(T)$ curve and $A\sim(T_m-T_1)N_0/2$. Except for $\theta\ll\tau'$ the additional term is not small ($\mathfrak{N}_0\ll\mathfrak{N}$), and therefore the total error is greater than that obtained with the moment method.

We next examine the following question: Is it possible to improve the accuracy by utilizing the total time of observation $[T]$ in a better way, using unequal counting times at the various delays T_i?

Rewriting Eq. (39) in the following form:

$$\sigma^2[\mu_1(N)]=\frac{\sum(T_i-\mu_1)^2N_i/[t_i]}{(\sum N_i)^2}, \tag{60}$$

where $[t_i]$ is now the time of observation for the point T_i, and minimizing the expression on the right in

Eq. (60) with the condition

$$\sum [t_i]=[T]=\text{constant}, \tag{61}$$

results in the time schedule

$$[t_i]=[T]\frac{|T_i-\mu_1|\sqrt{N_i}}{\sum|T_i-\mu_1|\sqrt{N_i}}, \tag{62}$$

and

$$\sigma^2[\mu_1(N)]=\frac{1}{[T]}\left(\frac{\sum|T_i-\mu_1|\sqrt{N_i}}{\sum N_i}\right)^2. \tag{63}$$

A comparison of Eqs. (63) and (52) for a Gaussian and for an experimental coincidence curve gives in both cases, when utilizing the time schedule of Eq. (62), a rather small improvement ($\sim$10 percent).

(2) *C is measured.*—The ordinate of the reduced coincidence curve is calculated as $n_i=\mathfrak{N}_i/\mathfrak{C}$ and its variance is

$$v(n_i)\cong n_i(1-n_i)/\mathfrak{C}. \tag{64}$$

In the successive measurements of individual points of a coincidence curve the observations are made either for a constant $\mathfrak{C}$ ("preset count") or for a constant $[t]$ ("preset time"). Since n_i is always calculated from correlated values of $\mathfrak{N}_i$ and $\mathfrak{C}$, Eq. (64) is valid for all separate points and in the calculation of errors $\mathfrak{C}$ can be replaced by its average value $[t]C$. Thus

$$v(n_i)\cong n_i(1-n_i)/([t]C). \tag{65}$$

A comparison of Eqs. (65) and (38) shows that the statistical error is smaller when C is measured.

The expression for $\sigma^2[\mu_1(n)]$ is:

$$\sigma^2[\mu_1(n)]=\frac{\sum (T_i-\mu_1)^2 n_i(1-n_i)}{[t]C(\sum n_i)^2}=\frac{1}{[t]C\sum n_i}\left\{\mu_2-\mu_1^2-\frac{\sum (T_i-\mu_1)^2 n_i^2}{\sum n_i}\right\}. \tag{66}$$

The first term in the bracket is the same as in Eq. (39) and the term to be subtracted from it is always positive. Calculation for a Gaussian curve and for actual coincidence curves showed that the second term is $\sim$30 percent of the first term when $\tau=\tau_0$.

Other advantages of the measurement of C have been pointed out in Sec. IV.

It is interesting to compare the errors given in Eqs. (52), (63), and (66) with the least possible error obtainable during $[T]$ in the presence of time lags characterized by $\hat{t}$ and in the presence of the time delay function $w(t)$. The least possible error would be obtained with a multichannel coincidence system consisting of channels having a negligibly small τ, and of a sufficiently large number to cover, without overlap, the entire range of the coincidence curve. In such a system none of the $[T]C$ coincidences would be missed being counted, the variance of $\mu_1(P)$ would be

$$\sigma^2[\mu_1(P)]=\hat{t}^2/([T]C), \tag{67}$$

and the variance of $\mu_1(N)$ would be

$$\sigma^2[\mu_1(N)]=(\hat{t}^2+\theta^2)/([T]C). \tag{68}$$

With $\kappa=3$, the factor appearing in front of $\hat{t}^2/([T]C)$ in Eq. (52) is $\sim$6, i.e., the one-channel method yields, in about six times larger a time of observation, the same accuracy as the (infinite) many-channel system. By the use of unequal times [Eq. (63)] this factor is reduced to $\sim$5.5, and by the measurement of C to $\sim$4 [Eq. (66)]. It can be further reduced by methods diminishing κ. Therefore, related to the same time of observation, the standard error obtained with the one-channel equipment is approximately twice $(2\kappa)^{\frac{1}{2}}$ the least theoretical standard error.

Slope Method

One can utilize the slope of a coincidence curve to measure time delays when:

(a) $w(t)$ is a simple exponential function. In this case θ can be determined [Eq. (54)] as[15]:

$$\theta=\frac{N(T)-P(T)}{dN/dT}. \tag{69}$$

(b) $w(t)$ is of a general form but its entire range is not longer than the linear part of the prompt curve. It has been shown[27] in this case, that

$$\mu_1(w)\cong\frac{N(T)-P(T)}{dP/dT}. \tag{70}$$

As was pointed out in Sec. IV, the slope method is simple only if one avoids the normalization procedure, i.e., if one measures C (denoted by C_n and C_p for the two curves). We will therefore relate the slope s to the reduced coincidence curves. We approximate s by

$$s=(n_1-n_2)/(T_1-T_2), \tag{71}$$

where T_1 and T_2 are two separate points chosen around T, and obtain for the variance of θ

$$v(\theta)=\frac{1}{s^2}\left\{\frac{n(1-n)}{[t_n]C_n}+\frac{p(1-p)}{[t_p]C_p}\right\}+\frac{\theta^2}{s^2(T_1-T_2)^2}\left\{\frac{n_1(1-n_1)}{[t_1]C_n}+\frac{n_2(1-n_2)}{[t_2]C_n}\right\}. \tag{72}$$

The second term in Eq. (72) shows that the method is feasible only for very small θ's. Namely, T_1-T_2 must be small compared to $\hat{t}$ to avoid a large systematic error in the determination of s. On the other hand, large $\theta^2/(T_1-T_2)^2$ introduces large statistical errors. For

[27] Bay, Meijer, and Papp, Phys. Rev. 82, 754 (1951).

$\theta \ll \hat{t}$ the second term in Eq. (72) can be neglected and we calculate the first term for the highest slope $s_{\max}$ of the $n(T)$ curve.

Since $n(T)$ is a threefold convolution integral of $\psi(T)$, $\rho(t)$ (we take here the time lag function to be uniform for all pulse pairs), and $w(t)$, it follows that $s_{\max}$ cannot exceed $(d\psi/dT)_{\max}$, $\rho_{\max}$, or $w_{\max}$.

For the ideal case where $(d\psi/dT)_{\max}=\infty$ [rectangular $\psi(T)$, with uniform τ for all pulse pairs], when τ is large enough to cover the entire range of $\rho(t)$ and when $\theta \ll \hat{t}$, we have $s_{\max} \cong \rho_{\max}$ and

$$n(T) \sim p(T) \sim \int_{-\infty}^{T} \rho(t)dt. \tag{73}$$

Writing

$$\rho_{\max}=1/(2\alpha^{\frac{1}{2}}\hat{t}), \tag{74}$$

we obtain for each of the coincidence curves

$$v(n/s) \sim \alpha \hat{t}^2/([T]C), \tag{75}$$

where the measured ordinates of n or p are taken as $\frac{1}{2}$, corresponding to the nodal point of the $\rho(t)$ curve.

A comparison of Eq. (75) with Eq. (67) shows the important fact that the ultimate determining factor for the accuracy of very short time measurements is also $\hat{t}$ when using the slope method.

In practice $(d\psi/dT)_{\max}$ is finite, $s_{\max}<1/(2\alpha^{\frac{1}{2}}\hat{t})$, and the actual statistical error may be similar to that obtained by the moment method, Eq. (52) corrected as in Eq. (66).

The convenience of the slope method makes it very useful for the *detection* of very short time delays.[25]

Tail Method

For large T's where $N(T) \gg P(T)$, the tail of the coincidence curve approaches an exponential for both the asymmetric and the symmetric parent-daughter cases. θ can be determined by fitting a straight line by least squares through a semilog plot of the experimental points. This leads to the slope[28]:

$$\lambda=\frac{1}{\theta}=\frac{\sum N_i(T_i-\bar{T})(y_i-\bar{y})}{\sum N_i(T_i-\bar{T})^2}, \tag{76}$$

where $\bar{T}=\sum N_iT_i/\sum N_i$, $y_i=\log N_i$, $\bar{y}=\sum N_iy_i/\sum N_i$, and the points of the plot are weighted by N_i to satisfy the condition

$$N_iv(y_i)=1/[t]=\text{constant}, \tag{77}$$

if $[t]$ is the same for all points.

The variance of θ is

$$v(\theta)=\frac{\theta^4}{[t]}\frac{1}{\sum N_i(T_i-\bar{T})^2}, \tag{78}$$

or, when measuring at equidistant T_i's,

$$v(\theta)=\frac{\theta^2}{\mathfrak{N}}\frac{\theta^2}{\mu_2-\mu_1^2}, \tag{79}$$

where μ_2 and μ_1 are the normalized moments of the measured section. For $T_m-T_1 \gg \theta$, $v(\theta)$ approaches $\theta^2/\mathfrak{N}$, which is just the variance of the center of gravity of the measured section. Thus the moment method with the use of Eqs. (56) and (57) having here $P(T)=0$, choosing a coordinate system such that $T_1=0$, and omitting $\mathfrak{N}_1$ in Eq. (57), leads to similar statistical accuracy.

The determination of θ from the tail has the advantage of involving only one coincidence curve. There is therefore no need to be concerned here with pulse shape distributions [Sec. I, condition (b)]. Of course in this case $\mathfrak{N}$, being dependent on τ, τ', θ, and T_m-T_1, is generally a rather small fraction of $C[T]$. This restricts the practical use of the method to $\theta \gg \tau'$.

Summarizing the results of statistical error calculations for the different methods, one can conclude that the moment methods can be used generally. Other methods, applicable only with some restrictions, can lead to similar statistical accuracy as the moment method.

Effect of Chance Coincidences

The chance coincidence counting rate is measured at $T=\infty$ and its mean value subtracted from all the measured counting rates along the coincidence curves. Again we consider two cases:

(1) *C is not measured.*—N_i is obtained by measuring $\mathfrak{N}_i'=\mathfrak{N}_i+\mathfrak{N}_c$ for the time $[t]$ and calculating

$$N_i=\frac{\mathfrak{N}_i+\mathfrak{N}_c}{[t]}-N_c, \tag{80}$$

where $\mathfrak{N}_c$ is the number of chance coincidences occurring during $[t]$ and N_c is the chance-coincidence background, determined separately at $T=\infty$, during the time $[t_c]$. The variance of N_i is

$$v(N_i)=\frac{N_i}{[t]}+\frac{N_c}{[t]}+\frac{N_c}{[t_c]}, \tag{81}$$

and since the last term of Eq. (81) is common for all points of the coincidence curve, the variance of μ_r is

$$\sigma^2(\mu_r)=\frac{\mu_{2r}-\mu_r^2}{\mathfrak{N}}+\frac{N_c}{[t](\sum N_i)^2}\sum (T_i^r-\mu_r)^2 + \frac{N_c}{[t_c](\sum N_i)^2}(\sum T_i^r-\mu_r)^2. \tag{82}$$

[28] For example see R. H. Bacon, Am. J. Phys. 21, 428 (1953).

For μ_1, Eq. (82) gives

$$\sigma^2(\mu_1)=\frac{1}{\mathfrak{N}}\left\{(\mu_2-\mu_1^2)+\frac{N_c}{\langle N_i\rangle_{\text{Av}}}\left[\frac{1}{3}\left(\frac{T_m-T_1}{2}\right)^2+\left(\frac{T_m+T_1}{2}-\mu_1\right)^2\right]\right\}+\left(\frac{T_m+T_1}{2}-\mu_1\right)^2\frac{N_c}{\langle N_i\rangle_{\text{Av}}^2}\frac{1}{[t_c]}. \quad (83)$$

A similar expression is valid for the $P(T)$ curve. The first term of the right-hand side of Eq. (83) shows that the additional error caused by the statistical fluctuation of the number of chance coincidences during the measurement of the individual points, is proportional to $N_c/\langle N_i\rangle_{\text{Av}}$. The second term depends on the time $[t_c]$ spent in the determination of N_c. It is interesting to note that the second term can be suppressed by taking the measured interval T_m-T_1 symmetrically around μ_1. In this case even a statistically poor determination of N_c is sufficient, and one can write

$$\sigma^2(\mu_1)=\frac{\mu_2-\mu_1^2}{\mathfrak{N}}\left(1+\frac{\kappa^2}{3}\frac{N_c}{\langle N_i\rangle_{\text{Av}}}\right), \quad (84)$$

where κ is defined in Eq. (46).

Since $\langle N_i\rangle_{\text{Av}}\sim\frac{1}{2}N_{\max}=\frac{1}{2}C\tau/\tau'$ and $\tau/\tau'=1/\sqrt{3}$ for $\tau=\tau_0$, and $\kappa=3$, one can estimate that, for $N_c/C=1\%$, the variance of μ_1 increases by $\sim5\%$ or the standard error by $\sim2.5\%$.

If only chance coincidences originating from source events are present, then N_c/C can be calculated and one obtains

$$N_c/C=\nu\cdot 2\tau, \quad (85)$$

which can be used as an estimate when designing experiments.

(2) *C is measured.*—Here n_i is calculated as follows:

$$n_i=\frac{(1/[t])(\mathfrak{N}_i+\mathfrak{N}_c')-N_c'}{(1/[t])(\mathfrak{C}+\mathfrak{N}_c'+\mathfrak{N}_c'')-N_c'}, \quad (86)$$

where $\mathfrak{N}_c'$ is the number of chance coincidences appearing in the circuit with the resolving time τ_1, within the time $[t]$, and $\mathfrak{N}_c'+\mathfrak{N}_c''=\mathfrak{N}_c$ is the number of chance coincidences appearing with $\mathfrak{C}$ in the other circuit (resolving time τ_2).

The corresponding chance coincidence counting rates are measured at $T=\infty$ during the respective times of observation $[t_c']$ and $[t_c]$. The $\mathfrak{N}_c'$ counts appear in both circuits and $N_c'/N_c=\tau_1/\tau_2$. A simple calculation gives

$$v(n_i)=\frac{n_i(1-n_i)}{[t]C}+\frac{N_c}{C^2}\left\{(1-n_i)^2\frac{\tau_1}{\tau_2}\left(\frac{1}{[t]}+\frac{1}{[t_c']}\right)+n_i^2\left(1-\frac{\tau_1}{\tau_2}\right)\left(\frac{1}{[t]}+\frac{1}{[t_c]}\right)\right\}. \quad (87)$$

Comparing Eq. (87) with Eq. (81) rewritten in the form:

$$v(n_i)=v\left(\frac{N_i}{C}\right)=\frac{n_i}{[t]C}+\frac{N_c'}{C^2}\left(\frac{1}{[t]}+\frac{1}{[t_c']}\right), \quad (88)$$

one can see that for all practical cases where $N_c<C$ and $[t_c]\sim[t_c']$ the variance shown in Eq. (87) is less than that given by Eq. (88).

VII. APPENDIX A

To illustrate the application of Eqs. (1) and (4) to more complicated cases, we will now treat the case of parent-daughter-granddaughter decay and will then make the obvious generalization to larger radioactive families.

Let us denote the mean life of the daughter by θ_1 and that of the granddaughter by θ_2. Without loss in generality, we can restrict the problem to the case in which only the radiation of the parent and granddaughter decays can excite the detectors.

The $w(t)$ function for the time interval between the parent and granddaughter decays will be the superposition of the two exponential decay functions with the respective mean lives θ_1, and θ_2. The $w(t)$ function and hence its moments will be symmetric in θ_1, and θ_2. The moments, μ_r, will be given by

$$\begin{aligned}
\mu_1&=\theta_1+\theta_2=S_1,\\
\mu_2&=2[\theta_1^2+\theta_1\theta_2+\theta_2^2]=2[S_1^2-S_2],\\
\mu_3&=6[\theta_1^3+\theta_1^2\theta_2+\theta_1\theta_2^2+\theta_2^3]=6[S_1^3-2S_1S_2],\\
\mu_4&=24[\theta_1^4+\theta_1^3\theta_2+\theta_1^2\theta_2^2+\theta_1\theta_2^3+\theta_2^4]\\
&=24[S_1^4-3S_1^2S_2+S_2^2],
\end{aligned} \quad (A1)$$

etc.;

where $S_1=\theta_1+\theta_2$, $S_2=\theta_1\theta_2$ are the elementary symmetric functions of the θ's. Now, for an asymmetric experiment, the first moment of $w(t)$ will yield S_1, the second moment and the known value of S_1 will yield S_2, and θ_1 and θ_2 will be the roots of the second-degree equation:

$$\theta^2-S_1\theta+S_2=0. \quad (A2)$$

The generalization to the analysis of an asymmetric experiment performed on a family of N members is straightforward. We denote the N elementary symmetric functions of the θ's by S_{kN} $(k=1\cdots N)$. Since $\mu_n(w)$ is a symmetric function of the nth degree in the θ's, μ_n can always be expressed as a function of the S_{kN} such that

$$\mu_n(w)=\Phi_n(S_{1N},\cdots,S_{nN}), \quad (A3)$$

where Φ_n does not contain any S_{kN} for $k>n$. Thus the first N moments of w utilized in ascending order will yield the S_{kN}'s by simple substitution, since the right hand side of Eqs. (A3) will be a triangular array in the S_{kN}'s. By a well-known theorem of algebra, θ_1, θ_2,

$\cdots, \theta_N$ will be the roots of the Nth-degree equation

$$\sum_{k=0}^{n} (-1)^k S_{kN}\theta^{N-k}=0. \quad \text{(A4)}$$

The $w(t)$ function obtained in a symmetric experiment differs in that all the odd moments are zero. The even moments, however, will be the same as those obtained in an asymmetric experiment. Thus, for the parent-daughter-granddaughter case, S_1 and S_2 can be determined by inserting μ_2 and μ_4 in Eqs. (A1). While the previous triangular array is no longer available, Eq. (A2) is still valid and yields θ_1, and θ_2.

For the case of a symmetric experiment performed on a family of N members, we use the first N even moments $\mu_2, \mu_4, \cdots, \mu_{2N}$ from which we determine S_{kN}; $k=1, \cdots, N$. Then Eq. (A4) can be solved for $\theta_1, \theta_2, \cdots, \theta_N$ as before.

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Inadmissible Auxiliary Conditions in Quantized Linear Systems

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Methods of applying supplementary linear conditions to quantized linear systems are reviewed; it is seen that such a procedure is self-consistent for Fermi-Dirac systems in general, and for Einstein-Bose systems possessing positive definite constants of the motion. Possible modifications are discussed for those cases in which the operator conditions to be added appear intrinsically inconsistent with the commutation relations. Alterations of the commutation relations which maintain the invariance properties of the system are found to be generally inapplicable. Increasing the number of side conditions achieves the desired objective, but at the expense of reducing drastically the class of constants of motion. Finally the introduction of a restricted set of field variables is explored. It is shown that the new variables, in terms of which all pertinent field quantities may be expressed, permit a consistent formulation to be realized; the case of a massless spin s boson field is treated in this manner.

INTRODUCTION

It is often necessary to augment the equations of motion of a physical system by relations of a more peripheral nature. The effect, and frequently the intent, is to reduce some symmetry of the problem and so negate the attendant decoupling of properties which appear correlated in nature, such as internal and external angular momentum for the vector meson without Lorentz condition. Whereas in the classical situation the additional equations may simply be tacked on as dynamical conditions, this option is not available, in an operator sense, when the original system is presented together with appropriate commutation relations; instead one may judiciously modify the commutation relations, construct a new Lagrangian which implies all relations, or perhaps use more specialized methods[1] in the case of linear systems. However, perverse situations arise in which none of these approaches can be reconciled with the maintaining of important invariance properties and the associated constants of the motion; the usual cure is to apply the truculent side condition only to the quantum states of the system. Another possible approach, with admittedly limited applicability, is to formulate the theory in terms of variables of presumably more direct physical significance, such as field strengths in electromagnetic theory, and thereby, in some mysterious way, banish the associated difficulties. We shall examine, in detail, considerations leading to the latter method, applying it to linear systems, in which its inherent imperfections are well camouflaged.

A quantization formalism for linear systems previously employed[1–3] by the writer will be of considerable aid. Briefly, the recipe runs as follows. A real linear system is determined by

$$\textstyle\sum_s M_{rs}\psi_s(x)=0, \quad (1)$$

or $M\psi=0$, where $M=(M_{rs})$ is a real matric differential operator and x refers to the full set of space-time coordinates. Then if the basic commutator (or anticommutator) $\psi_\alpha(x)\cdot\psi_\beta(x')\equiv\psi_\alpha(x)\psi_\beta(x')+J\psi_\beta(x')\psi_\alpha(x)$, with $J=\pm1$, is a c-number, the full set of pertinent commutation relations may be written succinctly as

$$\mathcal{T}(\omega,\psi)\cdot F\psi=F\omega; \quad (2)$$

here ω is any c-number solution of $M\omega=0$, F is any linear operator, $\mathcal{T}(\omega,\psi)$ is bilinear in ω and ψ and coordinate independent. Further, if U, real, is an invariant transformation in the sense that $M(U\psi)=0$, whenever $M\psi=0$, and if U satisfies $\mathcal{T}(\omega,U\psi)+\mathcal{T}(U\omega,\psi)=0$ as well,

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