Stark Fields from Ions in a Plasma*

A. A. BROYLES
The Rand Corporation, Santa Monica, California
(Received July 5, 1955)

A method is described here for determining the probability of obtaining a given electric field on an ion in a plasma. This quantity is essential for computing the broadening of spectral lines from neighboring ions and therefore for their contributions to the opacity. The Pines-Bohm method of separation into long-range Coulomb interactions has been employed. It is argued that the formulas give the line width for iron at one kilovolt and normal density to ±3%.

I. INTRODUCTION

IT is often essential to know the breadths and shapes of spectral lines to compute the absorption coefficients and Rosseland mean opacities[1] of substances at high temperatures. An understanding of the causes of line broadening also aids the interpretation of the spectra of the sun and stars.[1] Several experiments[1] have been performed for the purpose of observing line shapes under conditions that are controlled enough to check theoretical calculations.

Some of the causes of line broadening[2] are (1) the natural width due to the coupling of the atom with the electromagnetic field, (2) Doppler broadening, (3) electron collisions with the atom, and (4) the Stark effect shift of the atomic levels by fields from neighboring ions.

To compute the line broadening by Stark fields from neighboring ions, it is essential to know the probability $P(\epsilon)d\epsilon$ of finding a given electric field of magnitude ϵ at a radiating ion due to the displacement of neighboring ions. This paper is concerned with the calculation of this probability.

In the past, the Holtsmark distribution[3] has often been used for $P(\epsilon)$. It is obtained by neglecting the Boltzman factor and thus ignores the fact that one ion is hindered from approaching another by the electrostatic repulsion. This is a good approximation when the potential energy of two ions, exerting fields of the largest interesting magnitude on each other, is much less than the thermal energy kT. This potential energy must be at least as great as that for two ions separated by their average spacing. Thus, the Holtsmark approximation is good when the number of ions per cubic centimeter $n \ll 10^{11}T^3/Z^6$ with T in degrees or when $n \ll 10^{23}(kT)^3/Z^6$ with kT in electron volts, where Z is the number of electron charges on the ion. In some cases the upper limit on n may be lowered by as much as a factor 10^{-1}, since the wings of the spectral lines are generated by large Stark displacements of the levels of the ions, and, therefore, by close collisions between ions. These conditions are well satisfied in the reversal layers of normal stars and in laboratory experiments that are known by the author to have been performed to observe line broadening. They are not satisfied, however, in stellar interiors. Here the radiation transport may be computed with the aid of the Rosseland mean opacity which will, in some cases, depend strongly on the widths of the lines. For these cases, then, the Boltzmann factor must be taken into account.

The problem may be considerably simplified by employing the following model. The electrons are assumed to be either attached to an ion or free to move through the plasma. Since the free electrons move so much more rapidly than the ions, they may be treated as a smeared negative charge. For simplicity this smeared charge will be assumed uniform in density and to be undistorted by displacements of the ions. The ions will all be taken to have the same charge.

With this model, a certain set of units appears to be natural. For the unit of length, it is convenient to take the "ion sphere radius." This is the radius of a sphere that, on the average, contains exactly one ion. The unit of field strength then will be taken to be the magnitude of the field exerted on one ion by another ion a unit distance away, and the unit of energy will be the difference in potential energy of two ions when separated by unit distance and when separated by infinite distance. This is equivalent to using the charge on the ions as the unit charge.

Mayer[2] has made two approximations to $P(\epsilon)$ that take account of the Boltzmann factor. The first is useful when θ is much less than one, where θ is the temperature measured in the above energy units. In this case, the ions tend to remain at their equilibrium positions at the center of their respective ion spheres and to feel a restoring force proportional to their displacements. Thus, they move like three-dimensional simple harmonic oscillators (SHO). For very large fields and large θ's, the approximation must break down since an ion will get out of its own ion sphere.

Mayer's second approximation is for large fields and takes account of the fact that the Boltzmann factor and the available volume in configuration space make it more probable to obtain large fields by the close

* This work was sponsored by the U. S. Atomic Energy Commission.

[1] L. H. Aller, *The Atmospheres of the Sun and Stars* (The Ronald Press Company, New York, 1953).

[2] H. Mayer, Los Alamos Scientific Laboratory Report LA-647, 1947 (unpublished).

[3] J. Holtsmark, Ann. Physik **58**, 577 (1919); Physik Z. **20**, 162 (1919); Physik Z. **25**, 73 (1924); S. Chandrasekhar and J. von Neumann, Astrophys. J. **95**, 489 (1942).

approach of a single nearest neighbor than by the cooperative action of several nearby ions. Thus, the contributions to the field at the radiating ion by other than the nearest neighbor are neglected. It is not clear, however, how to normalize this approximation since $P(\epsilon)$ is not represented well in the neighborhood of its maximum.

To determine $P(\epsilon)$ over the entire range of ϵ and to check the normalization used by H. Mayer in his nearest neighbor approximation, it is necessary to consider other methods of treating a plasma than those mentioned above. Another simple approximation may be obtained by visualizing the high-energy ions as being free particles in a box. This gas is made up of ions having sufficient energy to make an approach to another ion close enough to exert the required field on it. The Holtsmark distribution can then be applied to this reduced number of ions to obtain $P(\epsilon)$. The level of the bottom of the box, however, still remains as an undetermined constant and the approximation is only good for large fields.

The work of Pines and Bohm[4] has suggested a very effective way of treating the ions in a plasma. They show how to separate the Coulomb forces into long and short range components and to convert the system of ions in an electron sea into a system of particles interacting with short range forces and a set of waves whose amplitudes become independent coordinates. The wave part of the system is quite easy to treat. The particle part is much more difficult and so formulas have been obtained that give upper and lower limits to $P(\epsilon)$.

All the approximations have been compared for the case of iron at normal density (7.83 g/cm^3) and one kilovolt temperature ($\theta=0.186$) and all but one have been compared at infinite temperature where the Holtsmark distribution gives the correct values of $P(\epsilon)$. An ionic charge of 23 electron charges has been assumed. To compute the Rosseland mean opacity, it is essential to know how far the tail is from the line center when its absorption is equal to the continuous background. For the important lines in iron at one kilovolt, this occurs when $P(\epsilon)$ has values of the order of 10^{-6}. Values of $P(\epsilon)$ smaller than this are of little interest.

The accurate determination of the probability, $Q(\boldsymbol{\varepsilon})d\boldsymbol{\varepsilon}$, of finding an electric field $\boldsymbol{\varepsilon}$ at the radiating ion would require an evaluation of the expression,

$$Q(\boldsymbol{\varepsilon})=\frac{\int\cdots\int \exp\left(-\frac{V}{\theta}\right)\delta\left(\boldsymbol{\varepsilon}+\sum_i \frac{\mathbf{r}_i}{r_i^3}\right)d\mathbf{r}_1\cdots d\mathbf{r}_N}{\int\cdots\int \exp\left(-\frac{V}{\theta}\right)d\mathbf{r}_1\cdots d\mathbf{r}_N}, \quad (1)$$

where θ is the temperature in energy units, V the potential energy of the system, N the number of ions, and $\mathbf{r}_i$ the position vector of the ith ion. The numerator of Q is merely the sum over all configurations having fields in the range $d\boldsymbol{\varepsilon}$ around $\boldsymbol{\varepsilon}$ as selected by the Dirac delta function. If N is very large, the limit of this integral may be taken when N and the volume of the system increase without limit in such a way that the number of particles per unit volume remains constant. The radiating ion is placed at the origin of coordinates since a shift in position of this ion will merely give the same configurations of the other ions with the same probabilities (except for surface effects).

It is convenient to expand the delta function in terms of plane waves so that

$$Q(\boldsymbol{\varepsilon})=\frac{1}{(2\pi)^3}\int T(\mathbf{l})\exp(i\boldsymbol{\varepsilon}\cdot\mathbf{l})d\mathbf{l},$$

$$T(\mathbf{l})=\frac{\int\cdots\int \exp\left(-\frac{V}{\theta}+i\mathbf{l}\cdot\sum_i\frac{\mathbf{r}_i}{r_i^3}\right)d\mathbf{r}_1\cdots d\mathbf{r}_N}{\int\cdots\int \exp\left(-\frac{V}{\theta}\right)d\mathbf{r}_1\cdots d\mathbf{r}_N}. \quad (2)$$

Since $T(\mathbf{l})$ is a function of the magnitude of $\mathbf{l}$ alone, the probability density of finding a field of a given magnitude ϵ is

$$P(\epsilon)=4\pi\epsilon^2Q(\boldsymbol{\varepsilon})=\frac{2\epsilon}{\pi}\int_0^\infty l\sin(l\epsilon)T(\mathbf{l})dl. \quad (3)$$

II. SHO AND NEAREST NEIGHBOR APPROXIMATIONS

For Mayer's[2] SHO approximation, the radiating ion will be attracted toward the center of its ion sphere by the negative smeared electron charge. If r is its displacement from this center, its potential energy is $\frac{1}{2}r^2$, and the electric field on it has magnitude r. For this case, then

$$P(\epsilon)=(2/\pi)^{\frac{1}{2}}(\epsilon^2/\theta^{\frac{3}{2}})\exp(-\epsilon^2/2\theta). \quad (4)$$

For large fields, Mayer makes the assumption that only the nearest neighbor to the ion at the origin makes a contribution to the field there. Thus he is able to relate $P(\epsilon)$ to the probability density,[5] $4\pi r^2ng(r)$, that an ion will lie a certain distance from the origin. The function[5] $g(r)$ is the "radial distribution function." Instead of this probability, the probability of the *nearest neighbor* lying at distance r should be used but the difference is small for large fields. When the nearest neighbor is near the origin, $d\epsilon=-[(2/r^3)+1]dr$. Thus, we find that

$$P(\epsilon)=4\pi r^2ng\Big/\left(\frac{2}{r^3}+1\right), \quad (5)$$

$$\epsilon=r^{-2}-r.$$

The first term in ϵ comes from the interaction between the ions and the second from the shielding of the elec-

[4] D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952).

[5] Hirshfelder, Curtis, and Bird, *Molecular Theory of Gases and Liquids* (John Wiley and Sons, Inc., New York, 1954), pp. 84, 321.

trons. For a system of particles interacting in pairs with short range forces, g may be approximated[5] by $\exp\{-u(r)/\theta\}$ where u is the potential energy between a pair of particles and must go to zero more rapidly than r^{-2} for large r. Because of the shielding effect of the plasma, such a short range interaction appears to be a possibility and the separation of the Coulomb interactions into short and long range components demonstrates that this is essentially the case. For close approach of the two ions u may be approximated by $r^{-1}+\frac{1}{2}r^2+$constant. Since u is not known over its entire range, it is not clear what the constant should be. Since it enters an exponent, $P(\epsilon)$ is quite sensitive to its value. Mayer chose the value $-\frac{3}{2}$ and obtained the formula:

$$P(\epsilon)=[3r^2/(2r^{-3}+1)]\exp[-(r^{-1}+\tfrac{1}{2}r^2-\tfrac{3}{2})/\theta], \qquad \epsilon=r^{-2}-r. \tag{6}$$

The constant $-\frac{3}{2}$ corresponds to including in u the interaction of the nearest neighbor with the central ion and with the electrons in a sphere of unit radius around the origin and with nothing else.

The values of $P(\epsilon)$ given by Eq. (6) are shown in Fig. 1 for iron at normal density and temperatures of one kilovolt and infinity.

III. MODIFIED HOLTSMARK DISTRIBUTION

It is possible to get another simple approximation to $P(\epsilon)$ at large ϵ by modifying the Holtsmark distribution to take account of the Boltzmann factor. Most of the ions moving around in a plasma may well be quite confined by the action of other charges on them. However, there must be a few ions with very high energies that move around quite freely and that only notice the presence of other charges when they approach quite close to another ion. It is also true that only these ions have sufficient energy to approach a second ion closely enough to exert a large field on it. Thus $P(\epsilon)$ may be calculated for large fields from an equivalent system of a reduced number of ions in a box whose bottom has some average potential energy $\bar{V}$. We may again place the radiating ion at the origin and then the potential energy of an ion on close approach will be $r^{-1}+\frac{1}{2}r^2-\bar{V}$. Thus only those ions with energy greater than

$$E=r_0^{-1}+\tfrac{1}{2}r_0^2-\bar{V}, \tag{7}$$

where $\epsilon=r_0^{-2}-r_0$, will contribute to $P(\epsilon)$.

The number of ions per unit volume in the Holtsmark distribution must be reduced by the fraction $S(E/\theta)$ given by the Maxwell Boltzmann law:

$$S(E/\theta)=\frac{4}{\sqrt{\pi}}\int_{\sqrt{(E/\theta)}}^{\infty}\exp(-x^2)x^2dx. \tag{8}$$

The density of particles has been taken into account by the unit of length mentioned in Sec. I. This unit is the ion sphere radius and must be increased by the factor $S^{-\frac{1}{3}}$ when the number of ions is reduced. Thus the field strength unit is proportional to $S^{\frac{2}{3}}$, and $P(\epsilon)$ is proportional to $S^{-\frac{2}{3}}$. If $H(\epsilon)$ is the Holtsmark distribution, the modified distribution is

$$P(\epsilon)\approx H(\epsilon/S^{\frac{2}{3}})/S^{\frac{2}{3}}. \tag{9}$$

As in the case of Mayer's nearest-neighbor approximation, it is not clear what value should be assigned to $\bar{V}$. The points on Fig. 1 correspond to $\bar{V}=\frac{3}{2}$.

IV. SEPARATION INTO SHORT AND LONG RANGE INTERACTIONS

Pines and Bohm[4] have suggested a way of splitting the electrostatic forces into long and short range terms and have found a way of treating the long range interactions quite accurately. A similar procedure will be adopted here.

Suppose we have a plasma of essentially infinite extent, and suppose we consider that portion which lies within a large cube containing N ions. We shall limit this system to one having an electrostatically equipotential boundary. This means that the average field on a test charge as it moves across the cube from one side to the other is zero. The potential for such a system that is electrically neutral is

$$\Phi=4\pi\sum_{\mathbf{k}}{}'\sum_{j}\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_j)}}{k^2}, \tag{10}$$

and the charge density is

$$\rho=\sum_{\mathbf{k}}{}'\rho_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}, \quad \rho_{\mathbf{k}}=\sum_{j}e^{-i\mathbf{k}\cdot\mathbf{r}_j}, \tag{11}$$

where unit volume is taken for the Fourier expansion. These quantities give a potential energy to the system of

$$V=2\pi\sum_{k}{}'\frac{\rho_{\mathbf{k}}\rho_{\mathbf{k}}^*}{k^2}. \tag{12}$$

The primes indicate the omission of $k=0$ from the sums.[4]

The separation into short and long range parts is accomplished by splitting V into two sums, one over k's larger than k_c, the other over k's less than k_c. The sum over large k's may then be evaluated to give the short-range potential energy,

$$V_{\text{s.r.}}=\sum_{i<j}v(R_{ij}), \tag{13}$$

where

$$\mathbf{R}_{ij}=k_c(\mathbf{r}_i-\mathbf{r}_j),$$

$$v(R)=\frac{k_c}{R}\left[1-\frac{2}{\pi}\mathrm{Si}(R)\right],$$

$\mathrm{Si}(x)$ being the sine integral.

The electric field at the origin is the negative gradient

of the potential energy with respect to the coordinates of the particle at the origin. Then

$$\sum_i \frac{\mathbf{r}_i}{r_i^3}=4\pi i \sum_{|\mathbf{k}|<k_c} \frac{\mathbf{k}}{k^2}\rho_{\mathbf{k}}-k_c\sum_{j=1}^{n}\frac{\mathbf{R}_j}{R_j}\frac{dv}{dR_j}, \tag{14}$$

$$\mathbf{R}_j=k_c\mathbf{r}_j.$$

The essential quantities for $Q(\epsilon)$ (Eq. (2)) have now been expressed in terms of the two sets of coordinates made up of the $\rho_{\mathbf{k}}$'s and $\mathbf{r}_j$'s. An approximation to the Jacobian of the transformation from one set of coordinates to the other may be obtained in the following manner. From Eq. (11), it is clear that $\rho_{\mathbf{k}}$ may be thought of as a sum of two dimensional unit vectors at various orientations. While the $\mathbf{r}_j$'s are ranging through the volume of the system for the integrations in $Q(\epsilon)$, the $\rho_{\mathbf{k}}$'s should be points in the complex plane whose density is essentially that given by the solution of a two dimensional random walk problem. Thus, if k_c is chosen so that the number of k's less than k_c is equal to the number of $\mathbf{r}_i$'s, we may write approximately that[6] (to within a constant factor)

$$d\mathbf{r}_1\cdots d\mathbf{r}_n \rightarrow \prod_{|k|<k_c}\exp(-\rho_{\mathbf{k}}\rho_{\mathbf{k}}^*/2n)dx_{\mathbf{k}}dy_{\mathbf{k}}, \tag{15}$$

where $\rho_{\mathbf{k}}=x_{\mathbf{k}}+iy_{\mathbf{k}}$ and n is the number of ions per unit volume.

Unfortunately, the expressions for $\sum_i(\mathbf{r}_i/r_i^3)$ and V [Eqs. (12), (13), and (14)] contain both $\mathbf{r}_j$'s and $\rho_{\mathbf{k}}$'s so that we need both sets of coordinates. Thus the number of coordinates is too large. Actually, however, the number of $\mathbf{r}_j$'s required is small since they appear only in short-range terms and only those particles near to the origin can contribute to the field there. To circumvent this difficulty, we shall alter the system in an unessential way by placing around the origin an impenetrable barrier in the form of a sphere containing a large number N of ions but a very small number compared to the total number of ions in the system. The number of ions inside the impenetrable sphere will be determined so that the average number per unit volume is equal to that for the whole system. This device will serve to keep the number of coordinates from becoming excessive.

The electrostatic field and potential energy are now separated into short and long range terms. The long range terms depend only on the $\rho_{\mathbf{k}}$'s and the short range terms on the $\mathbf{r}_j$'s. With these approximations, the transform $T(\mathbf{l})$ [Eq. (2)] is factorable into a T_p for particles and a T_w for waves. For T_w there still remains a restriction on the $\rho_{\mathbf{k}}$'s arising from the fact that ρ is a real quantity. Thus

$$\rho_{-\mathbf{k}}=\rho_{\mathbf{k}}^*. \tag{16}$$

For this reason we shall take only those $\rho_{\mathbf{k}}$'s to be independent that belong to $\mathbf{k}$'s having a positive Z component and indicate a sum or product over these $\mathbf{k}$'s by a plus sign superscript. This transform can now be reduced to

$$T_w=\prod_{|\mathbf{k}|<k_c}^{+}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{\left[-\frac{4\pi}{\theta}\frac{1}{k^2}\frac{1}{n}\right](x_{\mathbf{k}}^2+y_{\mathbf{k}}^2)+8\pi i\frac{\mathbf{l}\cdot\mathbf{k}}{k^2}y_{\mathbf{k}}\right\}dx_{\mathbf{k}}dy_{\mathbf{k}}}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{\left[-\frac{4\pi}{\theta}\frac{1}{k^2}\frac{1}{n}\right](x_{\mathbf{k}}^2+y_{\mathbf{k}}^2)\right\}dx_{\mathbf{k}}dy_{\mathbf{k}}}=\exp(-\gamma l^2), \tag{17}$$

where

$$\gamma=\frac{k_c}{\pi}[1-(3/k_c^2\theta)^{\frac{1}{2}}\tan^{-1}(k_c^2\theta/3)^{\frac{1}{2}}].$$

In the units mentioned in Sec. I, the number of ions per unit volume n is equal to $(4\pi/3)^{-1}$ and $k_c^3=9\pi/2$. This k_c was determined so that the number of wave degrees of freedom equals the number of particles in the whole system. For the particles, we obtain

$$T_p=\frac{\int\cdots\int\exp\left\{-\sum_{i<j}[v(R_{ij})/\theta]-ik_c\sum_{j=1}^{n}\frac{\mathbf{R}_j\cdot\mathbf{l}}{R_j}\frac{dv}{dR_j}\right\}d\mathbf{R}_1\cdots d\mathbf{R}_N}{\int\cdots\int\exp\{-\sum_{i<j}v(R_{ij})/\theta\}d\mathbf{R}_1\cdots d\mathbf{R}_N}. \tag{18}$$

The approximations up to this point do not appear to be particularly drastic. The formulas above might be expected to give a reasonably good approximation to $P(\epsilon)$. In Eq. (18), however, we run into the age-old difficulty of noncentral interactions. These interactions have been reduced by the above approximations but are not eliminated. Monte Carlo techniques would probably do a good job of evaluating these integrals.

[6] This is the same transformation obtained by Pines and Bohm in their Appendix I.

To reduce them by analytic methods, it is necessary to replace the potential energy by a function involving only central interactions. Since the particle nature seems so important at this stage, approximations similar to that used by Hartree to treat the atom do not appear promising. One thing that can be done is simply to neglect interactions that do not involve the central particle. Another possibility is to neglect short-range contributions to the potential energy and electric field at the origin due to all ions except the ion at the origin and its nearest neighbor.

V. SHORT-RANGE CENTRAL-INTERACTIONS APPROXIMATION

If we neglect interactions that do not involve the central ion, the multiple integrals in T_p [Eq. (18)] then reduce to a product of integrals:

$$T_p=\left[\frac{\int \exp\left\{-\frac{v(R)}{\theta}-ik_c\frac{\mathbf{R}\cdot\mathbf{l}}{R}\frac{dv}{dR}\right\}d\mathbf{R}}{\int \exp\left\{-\frac{v(R)}{\theta}\right\}d\mathbf{R}}\right]^N. \quad (19)$$

The limit of T_p, as N and the volume go to infinity so that their ratio is n, is

$$T_p=\lim\left[\frac{\text{vol}-\int\left\{1-\exp\left[-\frac{v}{\theta}-ik_c\frac{\mathbf{R}\cdot\mathbf{l}}{R}\frac{dv}{dR}\right]\right\}d\mathbf{R}}{\text{vol}-\int\left\{1-\exp\left[-\frac{v}{\theta}\right]\right\}d\mathbf{R}}\right]^N$$

$$=\exp\left\{-\frac{4\pi n}{k_c^3}I(l,\theta)\right\}, \quad (20)$$

$$I(l,\theta)=\int_0^\infty\left\{1-\left[\sin\left(lk_c\frac{dv}{dR}\right)\Big/\left(lk_c\frac{dv}{dR}\right)\right]\right\}\times\exp(-v/\theta)R^2dR.$$

Combining Eqs. (20), (17), and (3) gives

$$P(\epsilon)=\frac{2\epsilon}{\pi}\int_0^\infty l\sin(l\epsilon)\exp\left\{-\gamma l^2-\frac{3}{k_c^3}I(l,\theta)\right\}dl. \quad (21)$$

These formulas have been evaluated to obtain $P(\epsilon)$ for the case of iron at a temperature of one kilovolt and normal density (see Fig. 1).

VI. SHORT-RANGE NEAREST NEIGHBOR APPROXIMATION

If we neglect all terms in the exponent in the integrand of T_p [Eq. (18)] that involve particles other than the one at the origin and its nearest neighbor we obtain

$$T_p\approx\frac{\int_{|\mathbf{R}_1|>0}\int_{|\mathbf{R}_2|>|\mathbf{R}_1|}\cdots\int_{|\mathbf{R}_N|>|\mathbf{R}_1|}\exp\left\{-v(R_1)/\theta-ik_c\mathbf{l}\cdot\frac{\mathbf{R}_1}{R_1}\frac{dv}{dR_1}\right\}d\mathbf{R}_N\cdots d\mathbf{R}_1}{\int_{|\mathbf{R}_1|>0}\int_{|\mathbf{R}_2|>|\mathbf{R}_1|}\cdots\int_{|\mathbf{R}_N|>|\mathbf{R}_1|}\exp\{-v(R_1)/\theta\}d\mathbf{R}_N\cdots d\mathbf{R}_1}. \quad (22)$$

Taking the limit of T_p, as N and the volume go to infinity, and substituting this into Eq. (17) and Eq. (3) leaves

$$P(\epsilon)\approx\epsilon J/[2k_c(\pi\gamma)^{\frac{1}{2}}K],$$

$$J=\int_0^\infty(R^2/v')\exp\{-(R/k_c)^3-(v/\theta)\}\times\{\exp[-(k_cv'-\epsilon)^2/4\gamma]-\exp[-(k_cv'+\epsilon)^2/4\gamma]\}dR, \quad (23)$$

$$K=\int_0^\infty R^2\exp[-(R/k_c)^3-(v/\theta)]dR.$$

Both J and K may be approximated by the method of steepest descents.

For evaluating J [Eq. (23)] at large values of ϵ, only the variation in v' in the exponent of the second term of the integrand is significant. The first term is negligible and the other quantities may be replaced by their values at $R=R_1$ where R_1 is that value of R that makes $k_cv'+\epsilon=0$. Also v' may be replaced by $v_1'+v_1''(R-R_1)$ in the exponent. These approximations together with the steepest descents approximation for K give

$$P(\epsilon)\approx[(-\epsilon R_1^2/(k_c^2R_0^2v_1'v_1''))](g_0''/2\pi)^{\frac{1}{2}}\times\exp\{-[(R_1^3-R_0^3)/k_c^3]-[(v_1-v_0)/\theta]\}, \quad (24)$$

$$g_0''=(6R_0/k_c^3)+(v_0''/\theta)+(2/R_0^2),$$

where R_0 is the location of the saddle point in the integrand of K and the subscripts 1 and 0 indicate arguments of R_1 and R_0.

For small values of ϵ, the rapidly changing part of the integrand of J is the same as the integrand of K. If R_0 is the location of the saddle point of the integrand of K, then

$$P(\epsilon)\approx[\epsilon/(2k_cv_0'\sqrt{\pi\gamma})]\{\exp[-(k_cv_0'-\epsilon)^2/4\gamma]-\exp[-(k_cv_0'+\epsilon)^2/4\gamma]\}. \quad (25)$$

Comparisons between these approximations and the accurate formulas of Eq. (23) are made in Table I.

TABLE I. Comparison of approximations to the short-range nearest neighbor (SRNN) formula with accurate evaluation.

	ϵ	0.3	0.5	1	1.5	2	3	4	5	6	8	10
$\theta=1$ kev	Accurate SRNN Eq. (23)	0.63	1.13	0.66	0.147	0.043	0.0056	0.00082	0.000155	0.0000375	0.0000036	
	Large ϵ approximation Eq. (24)				0.064	0.024	0.0037	0.00067	0.000144	0.0000342	0.0000032	
	Small ϵ approximation Eq. (25)	0.78	1.36	0.61	0.034							
$\theta=\infty$	Accurate SRNN Eq. (23)		0.066	0.20	0.33	0.36	0.225	0.087	0.032	0.018	0.0086	0.0046
	Large ϵ approximation Eq. (24)							0.030	0.019	0.013	0.0068	0.0040
	Small ϵ approximation Eq. (25)		0.084	0.27	0.42	0.45	0.227	0.050				

VII. COMPARISON WITH HOLTSMARK

The Holtsmark distribution is the accurate determination of $P(\epsilon)$ corresponding to the exact evaluation of Eq. (1) for $Q(\epsilon)$ for infinite temperature. For this reason, it is particularly interesting to compare the approximations mentioned above at infinite temperature with the Holtsmark distribution.

Even at infinite temperatures, the "short-range central interaction" approximation [Sec. V, Eq. (21)] requires considerable numerical work. However, it can be shown that it reduces to the Holtsmark distribution in the limit of large fields. In this limit, close approaches of the ions are important and v may be approximated by $1/r$. Setting θ equal to infinity and neglecting γ, we obtain

$$I(l,\infty)=l^{\frac{3}{2}}k_c^{\,3}2(2\pi)^{\frac{1}{2}}/15,$$

$$P(\epsilon)=\frac{2}{\pi\epsilon}\int_0^\infty x\sin x\exp\{-(2(2\pi)^{\frac{1}{2}}/5)(x/\epsilon)^{\frac{3}{2}}\}dx. \tag{26}$$

This is the formula for the Holtsmark distribution as given by Chanrdasekar and von Neumann[3] except for the small difference between $2(2\pi)^{\frac{1}{2}}/5$ and unity arising from a slight difference in units for ϵ. This is actually a convenient derivation of the Holtsmark distribution.

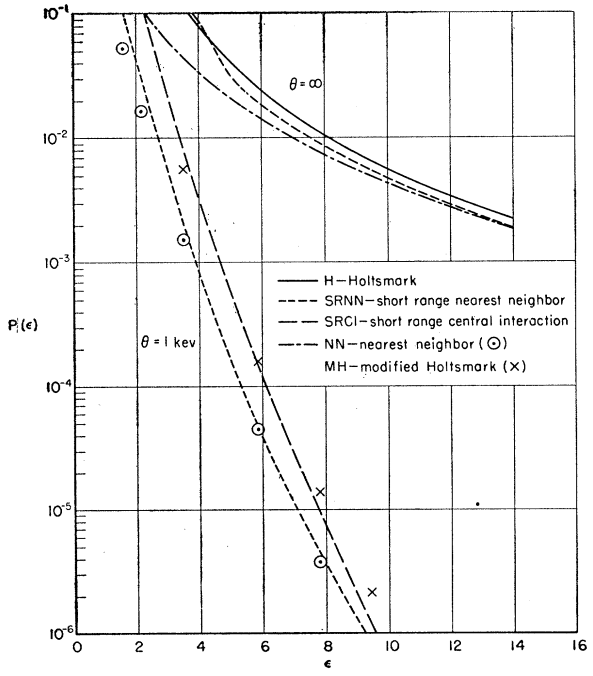

FIG. 1. The probability of obtaining field strengths of magnitude ϵ for $\theta=\infty$ and $\theta=1$ kev for iron at normal density for large values of ϵ. Only approximations good in this region are shown. The correct curve at $\theta=1$ kev is believed to lie between the SRNN and SRCI curves.

Curves are shown in Figs. 1 and 2 for the "short-range nearest neighbor" (SRNN) approximation [Sec. VI, Eq. (23)] at infinite temperature and for the "nearest neighbor" (NN) approximation [Sec. II, Eq. (6)]. The latter approximation does not have long-range and short-range separation for forces and potential energy.

The "modified Holtsmark" approximation goes over into the Holtsmark distribution at infinite temperature.

VIII. DISCUSSION AND CONCLUSIONS

The two approximations involving separation of the Coulomb interaction into short- and long-range components must bracket the correct evaluation of $P(\epsilon)$. The short-range central-interaction approximation (SRCI) in Sec. V gives $P(\epsilon)$ correctly for a system of particles that has no short range interactions between particles other than those involving the ion at the origin. Thus, the Boltzmann factor is too large when two or more particles are near the origin, and the probability of large fields is higher for this system than for the true plasma.

The short-range nearest neighbor approximation (SRNN) (Sec. VI) replaces the plasma by a system where the short-range forces act only between the nearest neighbor and the central ion. Since the short-range contributions to the field at the origin due to other than the nearest neighbor are neglected, the probabilities of large fields should be smaller than that for the true plasma. This is confirmed for the case of infinite temperatures since the SRNN curve in Fig. 1 lies below the Holtsmark for large ϵ.

The nearest neighbor approximation made by Mayer[2]

(Sec. II) is a remarkably good one in view of the uncertainty in the constant appearing in the potential energy. The large field approximation [Eq. (24)] to the SRNN formulas is very close to the NN formula [Eq. (6)] when v is set equal to k_c/R. Mayer's formula is given further justification from the fact that the short-range potential $v(R)$ [Eq. (13)] can be expanded for small R to give

$$v=(1/r)-1.54\ldots+\tfrac{1}{2}r^2+\cdots. \quad (27)$$

The constant 1.54 is very close to the $\frac{3}{2}$ used in Eq. (6) for the NN approximation.

The modified Holtsmark (MH) approximation provides a simple means of computing $P(\epsilon)$ for large fields that is probably good to roughly a factor of two in the parameter ranges considered here. However, it does not seem to be as accurate as the NN approximation [Eq. (6)] and is somewhat more difficult to compute. Nevertheless, since the approximations used in it are different from those used in the (NN) approximation, it can serve as a check on those formulas.

The widths of the spectral lines from the broadening due to electric fields of other ions is bracketed by the SRNN [Eq. (24)] and SRCI [Eq. (21)] approximations so that the widths are determined to $\pm 3\%$ for iron at one kilovolt and normal density. Under these assumptions, the NN approximation [Eq. (6)] provides a simple formula for determining the widths to better than 6% while the MH approximation [Eq. (9)] may serve as a simple check.

The uncertainties stated above do not include any errors arising from the neglect of the supplementary conditions that should be imposed because of the extra coordinates present when the potential is separated into long and short range components. This error is believed to be small because of the arguments presented in Sec. IV below Eq. (15).

The value of $P(\epsilon)$ around its maximum is not well determined but might be expected to lie near the SRNN curve because of the close agreement between it and the SHO curve in this region and its good agreement with the Holtsmark curve at infinite temperature. Accurate evaluation of T_p [Eq. (18)] is required to determine $P(\epsilon)$ more accurately. Monte Carlo methods might accomplish this.

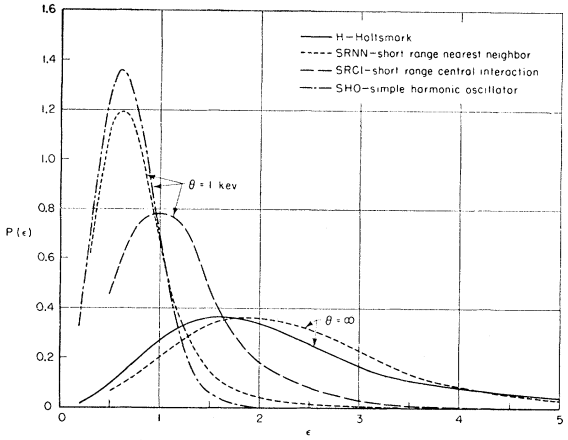

FIG. 2. The probability of obtaining field strengths of magnitude ϵ for $\theta=\infty$ and $\theta=1$ kev for iron at normal density for small values of ϵ. Only approximations good in this region are shown. The correct curve for $\theta=1$ is believed to lie between the SRNN and SRCI curves but nearer to the SRNN curve.

IX. ACKNOWLEDGMENTS

Considerable aid and direction for carrying out the numerical work involved here was given by Dr. Glenn H. Peebles.

The author also wishes to thank Mr. C. Lowery for assisting in the preparation of formulas and the numerical calculations.

The author is also particularly indebted to Dr. R. Ferrell and to his colleagues in the Nuclear Energy Division of The Rand Corporation for many helpful discussions.