Quantum Theory of Fluctuations*

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An operator is constructed for the spectral density of a fluctuating dynamical variable. The formulation is based on an analysis of method of measurement. An essential feature of the operator is the replacement of classical dynamical variables by time-dependent Heisenberg operators. The expectation value of the spectral density includes fluctuations of statistical origin, as well as fluctuations of quantum mechanical origin. It can be computed for any physical system when the initial state has been specified. Two applications are given: Nyquist's law is rigorously deduced for a physical system that is similar to the one discussed by Callen and Welton, and the shot effect is calculated for free uncorrelated electrons. A quantum correction is obtained whose origin lies in the wavelike character of the electron.

I. INTRODUCTION

FLUCTUATION phenomena are common and important features of all branches of the physical sciences. Since the origin of most fluctuations is of a statistical nature, it is perhaps not surprising that classical methods dominate the literature on the subject. Callen and Welton[1] (C-W) have given a quantum mechanical theory of thermal (Nyquist) noise. More recently, Weber has made some additional applications of their method.[2] The C-W method did not give a well-defined procedure for calculating what is usually measured, namely the spectral density. Instead, the total fluctuation was obtained in the form of an integral. The integrand was conjectured to be the spectral density.

According to the usual theory of measurement in quantum mechanics, the procedure for predicting the results of a measurement is to formulate an operator corresponding to the quantity measured and then to calculate the expectation value of the operator with respect to the state of the system.

The measurements usually considered in quantum mechanics involve observations at a definite time, given a complete or partial knowledge of the state of the system at a previous time. It was pointed out recently by Schrödinger[3] that the vast majority of measurements actually performed in the laboratory have an entirely different character. Measurements of the spectral power density of fluctuations, for example, do not fall into the usual pattern because the observation takes place over a very long time. While operators corresponding to such measurements have not been previously used, to our knowledge,[4] it is possible to construct them by consistent application of the accepted postulates of the theory of measurement in quantum mechanics, as will be shown presently. Thus it is not necessary, at least for the present purpose, to "recast the conceptual scheme of quantum mechanics."[3]

II. DEFINITION OF THE SPECTRAL DENSITY OPERATOR

We shall give a definition of the operator in question for two definite measurements which are only slightly idealized versions of actual observations (Fig. 1).

We assume that the fluctuating variable $A(t)$ produces by some means an electric voltage proportional to it which we will also designate by $A(t)$. The voltage is fed into a band-pass filter which transmits only the frequency components between $\nu_0-\Delta\nu/2$ and $\nu_0+\Delta\nu/2$, so that only the part $\delta A(t)$ is transmitted. This signal heats a thermocouple whose temperature variations are determined by the time integral of $[\delta A(t)]^2$. We now distinguish two cases:

(1) The thermocouple is connected to a ballistic galvanometer which indicates the magnitude

$$\int_{-\infty}^{+\infty}[\delta A(t)]^2 dt.$$

This arrangement is meaningful only if $A(t)$ is constant outside a finite time interval, or, more generally, if the

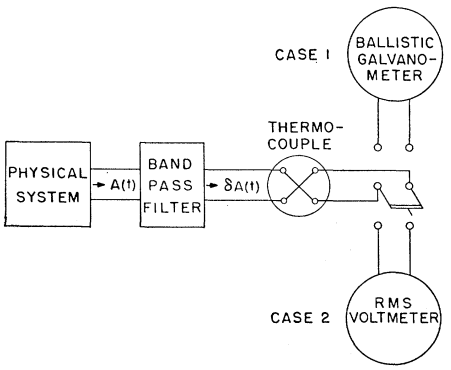

FIG. 1. Experimental arrangement for measurement of spectral density.

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[1] H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951).

[2] J. Weber, Phys. Rev. **96**, 556 (1954); **94**, 214 (1954); **90**, 977 (1953).

[3] E. Schrödinger, Nuovo cimento **1**, 5 (1955).

[4] We disregard the discussion of measurements in relativistic field theory where the time interval of the measurements is assumed to be very small, and is ultimately shrunk to zero. See, for example, W. Heitler, *The Quantum Theory of Radiation* (Clarendon Press, Oxford, 1954).

integrand decreases rapidly enough so that the integral exists. In other words, the fluctuation begins and terminates.

If the Fourier transform of $A(t)$ is

$$A(\nu)=\int_{-\infty}^{+\infty}A(t)\exp(-2\pi i\nu t)dt, \tag{1}$$

then the part of the signal transmitted through the filter is

$$\delta A(t)=\int_{\nu_0-\Delta\nu/2}^{\nu_0+\Delta\nu/2}2\,\mathrm{Re}[A(\nu)\exp(2\pi i\nu t)]d\nu, \tag{2}$$

and the ballistic galvanometer reading is

$$\int_{-\infty}^{\infty}[\delta A(t)]^2dt=\int_{\nu_0-\Delta\nu/2}^{\nu_0+\Delta\nu/2}G_s(\nu)d\nu, \tag{3}$$

where

$$G_s(\nu)=2|A(\nu)|^2$$

$$=2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}A(t)A(t')\exp[-2\pi i\nu(t-t')]dtdt'. \tag{4}$$

(2) The fluctuation does not decrease for $t\rightarrow\pm\infty$. In this case, the measurement consists in observing the time-average of the thermal emf produced by the thermocouple, if the physical system is connected to the measuring device between the times $-T/2$ and $+T/2$.

One may assume a sequence of measurements with increasing T, in which the observed value tends to a limit. In this case, the voltage flowing through the filter is given by

$$\delta A(t)=\int_{\nu_0-\Delta\nu/2}^{\nu_0+\Delta\nu/2}2\,\mathrm{Re}[A_T(\nu)\exp(2\pi i\nu t)]d\nu, \tag{2a}$$

with

$$A_T(\nu)=\int_{-T/2}^{T/2}A(t)\exp(-2\pi i\nu t)dt, \tag{1a}$$

and the deflection of the meter is determined by the average value of $[\delta A(t)]^2$:

$$\frac{1}{T}\int_{-\infty}^{+\infty}[\delta A(t)]^2dt=\int_{\nu_0-\Delta\nu/2}^{\nu_0+\Delta\nu/2}G_T(\nu)d\nu, \tag{3a}$$

where

$$G_T(\nu)=\frac{2}{T}\int_{-T/2}^{T/2}\int A(t)A(t')\exp[-2\pi i\nu(t-t')]dtdt'. \tag{4a}$$

If the sequence of measurements has a limit, as assumed, then the observed quantity is

$$G(\nu)=\lim_{T\rightarrow\infty}\frac{2}{T}\int_{-T/2}^{T/2}\int A(t)A(t')\exp[-2\pi i\nu(t-t')]dtdt'. \tag{5}$$

We have now given a classical definition of the observed quantities in the two cases. The accepted procedure for the construction of the corresponding quantum mechanical operators[5] consists in replacing the classical variable A by the corresponding operator and to "Hermitize" the resultant expression. The only peculiarity of the present case consists in the fact that these operators must obviously be Heisenberg operators. The resultant operators for the spectral density in the two cases are

$$G_s(\nu)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[A(t)A(t')+A(t')A(t)]\times\exp[-2\pi i\nu(t-t')]dtdt' \tag{6}$$

for case 1, and

$$G(\nu)=\lim_{T\rightarrow\infty}\frac{1}{T}\int_{-T/2}^{+T/2}\int[A(t)A(t')+A(t')A(t)]\times\exp[-2\pi i\nu(t-t')]dtdt' \tag{7}$$

for case 2.

The expectation value

$$\langle G(\nu)\rangle=(\psi_0,G(\nu)\psi_0) \tag{8}$$

can be computed if the initial state ψ_0 is specified. If, however, the initial state is not precisely known, then all information concerning it can be expressed by the density matrix ρ, and the expectation value of $G(\nu)$ or $G_s(\nu)$ is[6]

$$\langle G(\nu)\rangle=\mathrm{Tr}(G(\nu)\rho). \tag{9}$$

We can now express precisely what the term "fluctuating variable" means. A variable A is fluctuating with respect to an initial state ψ_0, if the expectation value of $G(\nu)$ or of $G_s(\nu)$ exists.

It is sometimes more convenient to consider the correlation function, which has the classical expression

$$F(\tau)=\lim_{T\rightarrow\infty}\frac{1}{T}\int_{-T/2}^{T/2}A(t)A(t+\tau)dt. \tag{10}$$

The same procedure leads to the operator

$$F(\tau)=\lim_{T\rightarrow\infty}\frac{1}{2T}\int_{-T/2}^{T/2}[A(t)A(t+\tau)+A(t+\tau)A(t)]dt. \tag{11}$$

Since only time or frequency integrations are involved, the Wiener-Khintchine theorem applies to $G(\nu)$ and $F(\tau)$. That is, the operators $G(\nu)$ and $F(\tau)$ are Fourier transforms, as are $\langle G(\nu)\rangle$ and $\langle F(\tau)\rangle$.

While we have defined an operator for the correlation function $F(\tau)$, it must be admitted that it is not directly

[5] E.g., H. A. Kramers, *Grundlagen der Quantentheorie* (Akademische Verlagsgesellschaft, Leipzig, 1938).

[6] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Dover Publications, New York, 1943).

derived from an analysis of a precise experiment. According to the experimental procedure chosen, other operators may be appropriate.

III. NYQUIST NOISE

As an application, we shall calculate the spectral density of current fluctuations for the physical system considered by Callen and Welton.[1] The system is characterized by boundary conditions (such as reflecting walls) that lead to zero current for every eigenstate. A detailed specification of the Hamiltonian is not required. The initial state is not completely known. Instead it is known that, at $t=0$, the system is in thermodynamic equilibrium at temperature T. This fact is expressed by the density matrix which in energy representation has the form

$$\rho_{ns;ns}=\exp(-E_n/kT)/[\textstyle\sum_{ns}\exp(-E_n/kT)], \quad (12)$$

while all nondiagonal elements vanish. The eigenvalues E_n may be degenerate, as provided for in the notation which includes a degeneracy parameter s. From Eq. (9) the expectation value of the spectral density operator is

$$\langle G(\nu)\rangle=\textstyle\sum_{ns}\rho_{ns;ns}G_{ns;ns}. \quad (13)$$

The Schrödinger operator for the x component of the electric current, averaged over the system's volume, is

$$\dot{Q}=(i/\hbar)(HQ-QH), \quad (14)$$

where $Q=\sum_n e_n x_n/L$, e_n is the charge of the nth particle, x_n is its x-coordinate, and L is the length of the system in the x direction. The Heisenberg operator for the current is $\dot{Q}(t)=\exp(iHt/\hbar)\dot{Q}\exp(-iHt/\hbar)$. The matrix element $G_{ns;ns}$ of the operator $G(\nu)$ [Eq. (7) with $A=\dot{Q}$] in the energy representation is

$$G_{ns;ns}=\lim_{T\to\infty}\frac{2}{T}\int_{-T/2}^{T/2}\int\{\sum_{n's'}\omega_{nn'}{}^2|Q_{n's';ns}|^2 \times\cos[\omega_{nn'}(t-t')]\exp[-i\omega(t-t')]\}dtdt',$$

where $\hbar\omega_{nn'}=E_n-E_{n'}$, $\omega=2\pi\nu$, and $Q_{n's';ns}$ is the matrix element of the operator Q in Eq. (14). After carrying out the time integrations and recognizing that

$$\lim_{T\to\infty}\frac{2}{\pi}\frac{\sin^2[(\omega_{nn'}\pm\omega)T/2]}{(\omega_{nn'}\pm\omega)^2T}=\delta(\omega_{nn'}\pm\omega),$$

we obtain

$$G_{ns;ns}=2\pi\sum_{n's'}\omega_{nn'}{}^2|Q_{n's';ns}|^2 \times[\delta(\omega_{nn'}+\omega)+\delta(\omega_{nn'}-\omega)]. \quad (15)$$

Substituting Eq. (15) into Eq. (13), one obtains a final expression for $\langle G(\nu)\rangle$. When the eigenvalues E_n are discrete, $\langle G(\nu)\rangle$ is singular if the frequency ν is equal to the jump frequency of a spontaneous transition. In the case of quasi-continuous eigenvalues, the summation over n' in Eq. (15) can be replaced by an integration, so that the matrix elements $G_{ns;ns}$ are nonsingular.

A more explicit expression for $\langle G(\nu)\rangle$ requires detailed knowledge of the eigenstates. Instead of continuing in this direction, we consider other measurements for which the expectation value involves the same matrix elements. For example, the rate at which the system absorbs energy when subjected to a uniform sinusoidal electric field involves the matrix elements of Q, since the perturbation term is $V(t)Q$, where $V(t)=V_0\sin\omega t$ is the voltage applied across the length of the system in the x-direction. We assume the system to be initially in an eigenstate ϕ_{ns} and calculate the state at time t,

$$\psi=\sum_{n's'}C_{n's'}(t)\phi_{n's'}\exp\left[-\frac{iE_{n'}}{\hbar}t\right]. \quad (16)$$

According to first-order time-dependent perturbation theory,

$$|C_{n's'}|^2=\frac{V_0{}^2}{\hbar^2}|Q_{n's';ns}|^2 \times\left\{\frac{\sin^2[(\omega_{nn'}+\omega)t/2]}{(\omega_{nn'}+\omega)^2}+\frac{\sin^2[(\omega_{nn'}-\omega)t/2]}{(\omega_{nn'}-\omega)^2} +\frac{1}{2}\frac{1+\cos2\omega t-\cos(\omega_{nn'}+\omega)t-\cos(\omega_{nn'}-\omega)t}{(\omega_{nn'}-\omega)(\omega_{nn'}+\omega)}\right\}. \quad (17)$$

The transition probability $w(t)=(1/t)\sum_{n's'}|C_{n's'}(t)|^2$ has a useful form in the limit of large t. That is,

$$w=\lim_{t\to\infty}w(t)=\frac{\pi V_0{}^2}{2\hbar^2}\sum_{n's'}|Q_{n's';ns}|^2 \times[\delta(\omega_{nn'}+\omega)+\delta(\omega_{nn'}-\omega)]. \quad (18)$$

The limit $t\to\infty$ would appear to have dubious validity in a calculation involving first-order perturbation theory. Some comfort may be derived from the thought that the perturbation term may be made vanishingly small, and from the fact that the procedure leads to physically significant results.

If the system is initially in the state ϕ_{ns}, the power absorbed is[7]

$$\frac{\pi V_0{}^2}{2\hbar}\omega\sum_{n's'}|Q_{n's';ns}|^2[\delta(\omega_{nn'}+\omega)-\delta(\omega_{nn'}-\omega)].$$

The power absorbed if the system is initially in a state of thermodynamic equilibrium, at temperature T, is obtained by averaging over the eigenstates weighted

[7] In the transition for which $\omega_{nn'}=\omega$, the system loses energy $\hbar\omega$; for $\omega_{nn'}=-\omega$ the system gains energy $\hbar\omega$.

according to the Boltzmann factor. That is,

$$P=\frac{\pi V_0^2}{2\hbar}\omega\sum_{ns;n's'}\rho_{ns;ns}|Q_{n's';ns}|^2 \times[\delta(\omega_{nn'}+\omega)-\delta(\omega_{nn'}-\omega)]. \quad (19)$$

The power absorbed can be expressed in terms of the experimentally determined complex impedance $Z=R+iX$. That is,

$$P=\tfrac{1}{2}V_0^2R/|Z|^2. \quad (20)$$

It is now clear that the expressions for $R/|Z|^2$ and $\langle G(\nu)\rangle$ are related. We consider the ratio:

$$\frac{\langle G(\nu)\rangle}{R/|Z|^2}=\frac{2\hbar}{\omega}\frac{\sum_{ns;n's'}[\exp(-E_n/kT)]\omega_{nn'}^2|Q_{n's';ns}|^2[\delta(\omega_{nn'}+\omega)+\delta(\omega_{nn'}-\omega)]}{\sum_{ns;n's'}[\exp(-E_n/kT)]|Q_{n's';ns}|^2[\delta(\omega_{nn'}+\omega)-\delta(\omega_{nn'}-\omega)]}.$$

The only terms which contribute to the sums are those for which $\omega_{nn'}=\pm\omega$, or $E_n=E_{n'}\pm\hbar\omega$. For example, consider E_1 and E_2 such that $E_1=E_2+\hbar\omega$. The contribution to the numerator is

$$[\exp(-E_1/kT)]\omega_{12}^2|Q_{21}|^2\delta(\omega_{12}-\omega)+[\exp(-E_2/kT)]\omega_{21}^2|Q_{12}|^2\delta(\omega_{21}+\omega)=[\exp(-E_1/kT)]\omega^2|Q_{21}|^2\delta(\omega_{12}-\omega)\times[1+\exp(-\hbar\omega/kT)].$$

Similarly the contribution to the denominator is

$$\exp[-E_1/kT]|Q_{21}|^2\delta(\omega_{12}-\omega)[1-\exp(-\hbar\omega/kT)].$$

Except for different factors, the sums in the numerator and denominator are identical. Thus

$$\frac{\langle G(\nu)\rangle}{R/|Z|^2}=2\hbar\omega[1+\exp(-\hbar\omega/kT)]/[1-\exp(-\hbar\omega/kT)],$$

or

$$\langle G(\nu)\rangle=\frac{4R}{|Z|^2}\left[\frac{h\nu}{2}+\frac{h\nu}{\exp(h\nu/kT)-1}\right]. \quad (21)$$

Equation (21) is Nyquist's law in the form obtained with quantum statistics.[1]

The calculation establishes a rigorous relationship between the exact result for $\langle G(\nu)\rangle$ and the approximate (first-order perturbation theory) result for $R/|Z|^2$. Since Nyquist's law is established experimentally, we may consider that the present calculations plus the experimental data establish the validity of the technique for calculating $R/|Z|^2$, although the latter calculation has not been actually carried out.

Equation (21) was derived by Callen and Welton. The present method differs from their derivation in several respects. The spectral density is computed directly, whereas in the C-W paper the total fluctuation was calculated and the deduction of the spectral density was not rigorous. Secondly, the C-W calculation was limited to cases where the energy levels of the physical system are quasi-continuous. This restriction has been removed, providing it is understood that the quantity $R/|Z|^2$ is evaluated for a vanishingly small perturbation. We have considered a degenerate physical system instead of a nondegenerate one as in the C-W paper. The computed values of $\langle G(\nu)\rangle$ and P will certainly be altered by degeneracy, but in the same way so that the Nyquist theorem is unaltered.

IV. SHOT EFFECT

When a series of charged particles arrives at a detector, current fluctuations are observed due to the discreteness of the charges. The current consists of a series of pulses at uncorrelated times if the electrons are independent, as is usually supposed. Any one pulse has a white spectral density and the effect of many electrons can be obtained by simple addition if the electrons are independent. Quantum corrections should be expected, due to the Pauli principle which introduces correlation of electrons, and due to the wave-like character of the electron. We shall treat only the latter correction in the present paper. For simplicity, we consider only one electron. In the usual calculation of shot effect, one determines the spectral density, such that

$$\int_0^\infty G(\nu)d\nu=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}J_x^2(t)dt,$$

where J_x is the current. For only one electron, $G(\nu)$ would vanish. In order to obtain finite results, we use the definition of $G_s(\nu)$ for case 1, [Eqs. (4) and (6)]. The classical expression for $G(\nu)$ for many electrons is $G(\nu)=2eJ_x$. For one electron, we obtain the finite result $G_s(\nu)=2e^2$. For N independent electrons, $G_s(\nu)=2e^2N$, and $G(\nu)=\lim_{T\to\infty}[G_s(\nu)/T]=2eJ_x$ since $J_x=\lim_{T\to\infty}(eN/T)$.

1. Classical Calculation

We consider fluctuations in the current arriving at a detector located in the region $-x_0/2\leq x\leq x_0/2$. An electron is observed only if it is in the detector region so that the current from a single electron may be defined as

$$J_x=-\frac{ep_x}{mx_0}f(x,x_0), \quad (22)$$

where p_x is the x-component of the momentum of the electron, $-e$, m are, respectively, charge and mass of the electron, and $f(x,x_0)$ is a function which has the

property that

$$f(x,x_0)=1 \quad \text{for} \quad -x_0/2\leq x\leq x_0/2$$
$$=0 \quad \text{if } x \text{ is outside this interval.}$$

An explicit form for $f(x,x_0)$ can be given in terms of the Dirichlet integral. That is

$$f(x,x_0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\rho x_0/2)}{\rho}\exp(i\rho x)d\rho. \tag{23}$$

The definition of J_x can be idealized for a detector of infinitesimal extension in the x-direction by taking the limit as $x_0\rightarrow 0$. In this case Eq. (22) takes the simpler form

$$J_x=(-e/m)p_x\delta(x), \tag{24}$$

where

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(i\rho x)d\rho.$$

We assume that no force acts on the electron, so that $p_x=$const and $x(t)=x(0)+(p_xt/m)$. Substituting

$$J_x(t)=\frac{-ep_x}{2\pi m}\int_{-\infty}^{\infty}\exp\left\{i\rho\left[x(0)+\frac{p_x}{m}t\right]\right\}d\rho$$

into Eq. (4), the classical expression for $G_s(\nu)$, and carrying out the indicated integrations, one obtains

$$G_s(\nu)=2e^2. \tag{25}$$

We shall compare this result with that obtained by quantum-mechanical treatment.

2. Quantum-Mechanical Calculation

The Heisenberg operator corresponding to $G_s(\nu)$ is, by Eq. (6),

$$G_s(\nu)=\int_{-\infty}^{\infty}\int[J_x(t)J_x(t')+J_x(t')J_x(t)]\times\exp[-i\omega(t-t')]dtdt', \tag{26}$$

where now $J_x(t)$ is the Heisenberg time-dependent operator which is constructed from the classical expression for the dynamical variable given by Eq. (24). That is,[5]

$$J_x(t)=-(e/2m)\exp(iHt/\hbar)\times[p_x\delta(x)+\delta(x)p_x]\exp(-iHt/\hbar), \tag{27}$$

where p_x is the Schrödinger operator for the x component of momentum. We consider a single free electron, for which $H=(1/2m)(p_x^2+p_y^2+p_z^2)$. The expectation value of the spectral density is

$$\langle G_s(\nu)\rangle=\int\psi_0^*G_s(\nu)\psi_0 d\tau. \tag{28}$$

The wave function ψ_0 at $t=0$ is assumed to be a wave packet of the form

$$\psi_0=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}c(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})d^3k. \tag{29}$$

The integrations are carried out in the order t, t', τ. Making use of the relations

$$H\exp(i\mathbf{k}\cdot\mathbf{r})=E(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}),$$
$$E(\mathbf{k})=(\hbar^2/2m)(k_x^2+k_y^2+k_z^2),$$

we obtain for a typical t integration:

$$\int_{-\infty}^{\infty}\exp\left\{\frac{i}{\hbar}[E(\mathbf{k})-E(\mathbf{k}')+\hbar\omega]t\right\}dt=2\pi\hbar\delta[E(\mathbf{k})-E(\mathbf{k}')+\hbar\omega].$$

The resultant expression for $\langle G_s(\nu)\rangle$ is

$$\langle G_s(\nu)\rangle=(2\pi\hbar)^2\int\int_{-\infty}^{\infty}\int d^3kd^3k'd^3k''c^*(\mathbf{k}')c(\mathbf{k})(J_x)_{\mathbf{k}',\mathbf{k}''}\times(J_x)_{\mathbf{k}'',\mathbf{k}}\delta[E(\mathbf{k})-E(\mathbf{k}')]\{\delta[E(\mathbf{k})-E(\mathbf{k}'')-\hbar\omega]+\delta[E(\mathbf{k})-E(\mathbf{k}'')+\hbar\omega]\}. \tag{30}$$

The form of this expression is similar to the usual result from perturbation theory for an electron that makes elastic transitions from $\mathbf{k}$ to $\mathbf{k}'$ through an intermediate state $\mathbf{k}''$, whose energy differs from the initial state by $\pm\hbar\omega$. The matrix elements $(J_x)_{\mathbf{k},\mathbf{k}'}$ have the specific form

$$(J_x)_{\mathbf{k},\mathbf{k}'}=\int\phi_{\mathbf{k}}^*J_x\phi_{\mathbf{k}'}d\tau,$$

where

$$\phi_{\mathbf{k}}=(2\pi)^{-\frac{3}{2}}\exp(i\mathbf{k}\cdot\mathbf{r}),$$

and

$$J_x=(-e/2m)[p_x\delta(x)+\delta(x)p_x]$$

is the Schrödinger current operator. After carrying out the indicated integrations, $(J_x)_{\mathbf{k},\mathbf{k}'}$ takes the form

$$(J_x)_{\mathbf{k},\mathbf{k}'}=(-e\hbar/4\pi m)(k_x+k_x')\times\delta(k_y-k_y')\delta(k_z-k_z'). \tag{31}$$

Further reduction of Eq. (30) is accomplished by carrying out first the integrations with respect to k_y, k_y'; k_z, k_z' and finally the integrations with respect to k_x, k_x', making use of relations such as[8]

$$\delta\left[\frac{\hbar^2}{2m}(k_x^2-k_x''^2)\pm\hbar\omega\right]=\frac{m}{\hbar^2a^{\mp}}[\delta(k_x-a^{\mp})+\delta(k_x+a^{\mp})],$$

where

$$a^{\pm}=\{k_x''^2\pm(2m\omega/\hbar)\}^{\frac{1}{2}}.$$

[8] P. A. M. Dirac, *Quantum Mechanics* (Clarendon Press, Oxford, 1935).

The final expression for the expectation value of the spectral density is

$$\langle G_s(\nu)\rangle=\frac{e^2}{2}\int_{-\infty}^{\infty}\int dk_y dk_z\Bigg\{\int_{\alpha}^{\infty}\frac{k_x dk_x}{(k_x^2-\alpha^2)^{\frac{1}{2}}}$$
$$\times\Bigg[[|c(k_x)|^2+|c(-k_x)|^2]\left(2-\frac{\alpha^2}{k_x^2}\right)$$
$$-[c^*(-k_x)c(k_x)+c^*(k_x)c(-k_x)]\frac{\alpha^2}{k_x^2}\Bigg]$$
$$+\int_0^{\infty}\frac{k_x dk_x}{(k_x^2+\alpha^2)^{\frac{1}{2}}}\Bigg[[|c(k_x)|^2+|c(-k_x)|^2]\left(2+\frac{\alpha^2}{k_x^2}\right)$$
$$+[c^*(-k_x)c(k_x)+c^*(k_x)c(-k_x)]\frac{\alpha^2}{k_x^2}\Bigg]\Bigg\}. \quad (32)$$

We have written α^2 for $2m\omega/\hbar$ and omitted the designation of the functional dependence of c on k_y and k_z in this expression.

The specific form of the spectral density depends on the form of the wave packet as prescribed by $c(k_x,k_y,k_z)$. The expression given by Eq. (32) for $\langle G_s(\nu)\rangle$ is not particularly simple, and in order to appreciate its implications, it is necessary to consider a specific case. We shall consider the limiting case of an electron with a definite momentum and therefore assume that $|c(\mathbf{k})|^2=\delta(\mathbf{k}-\mathbf{k}^0)$. Introducing the circular frequency ω_0 that corresponds to the electron's energy of motion in the x direction, the result of carrying out the integrations indicated in Eq. (32) may be expressed as follows:

$$\langle G_s(\nu)\rangle=\frac{e^2}{2}\left\{\frac{2-(\omega/\omega_0)}{[1-(\omega/\omega_0)]^{\frac{1}{2}}}+\frac{2+(\omega/\omega_0)}{[1+(\omega/\omega_0)]^{\frac{1}{2}}}\right\}\quad(\text{for }\omega<\omega_0)$$
$$=\frac{e^2}{2}\frac{2+(\omega/\omega_0)}{[1+(\omega/\omega_0)]^{\frac{1}{2}}}\quad(\text{for }\omega>\omega_0). \quad (33)$$

$\omega_0=E(k_x^0)/\hbar$, where $E(k_x^0)=\hbar^2(k_x^0)^2/2m$. In the classical limit when $\hbar\to 0$, $\omega_0\to\infty$ and $\langle G_s(\nu)\rangle=2e^2$ which is the same as the result in Eq. (25) obtained by classical mechanics. For finite ω_0, a singularity appears at $\omega=\omega_0$, which is similar to what one might expect for the spectral density of the electromagnetic field of a photon. The result expressed by Eq. (33) can, therefore, be interpreted in the following way. The electron exhibits a compromise between "wave-like" and "particle-like" behavior. At low frequencies, $\omega\ll\omega_0$, the behavior is "particle-like" and the classical result obtains. At frequencies of the order of ω_0 the behavior is "wave-like," exhibiting a resonance such as would be expected for a photon of circular frequency ω_0.

3. Shot Effect for a Finite Detector

The current operator for this case is

$$J_x=(-e/2m)[p_x f(x,x_0)+f(x,x_0)p_x], \quad (34)$$

where

$$f(x,x_0)=\frac{1}{\pi x_0}\int_{-\infty}^{\infty}\frac{\sin(\rho x_0/2)}{\rho}\exp(i\rho x)d\rho,$$

and x_0 is the width of the detector region.

The calculation follows the same pattern as the previous case. If we define

$$\gamma^2\equiv 1+\frac{\omega}{\omega_0},\quad \beta^2\equiv 1-\frac{\omega}{\omega_0},$$

then the result for an electron with definite momentum is

$$\langle G_s(\nu)\rangle=\frac{e^2}{4\gamma}\left\{\left(\frac{1+\gamma}{1-\gamma}\right)^2\frac{\sin^2[(1-\gamma)\omega_0\tau]}{(\omega_0\tau)^2}\right.$$
$$\left.+\left(\frac{1-\gamma}{1+\gamma}\right)^2\frac{\sin^2[(1+\gamma)\omega_0\tau]}{(\omega_0\tau)^2}\right\}$$
$$+\frac{e^2}{4\beta}\left\{\left(\frac{1+\beta}{1-\beta}\right)^2\frac{\sin^2[(1-\beta)\omega_0\tau]}{(\omega_0\tau)^2}\right.$$
$$\left.+\left(\frac{1-\beta}{1+\beta}\right)^2\frac{\sin^2[(1+\beta)\omega_0\tau]}{(\omega_0\tau)^2}\right\} \quad (35)$$

for $\omega<\omega_0$, and

$$\langle G_s(\nu)\rangle=\frac{e^2}{4\gamma}\left\{\left(\frac{1+\gamma}{1-\gamma}\right)^2\frac{\sin^2[(1-\gamma)\omega_0\tau]}{(\omega_0\tau)^2}\right.$$
$$\left.+\left(\frac{1-\gamma}{1+\gamma}\right)^2\frac{\sin^2[(1+\gamma)\omega_0\tau]}{(\omega_0\tau)^2}\right\}$$

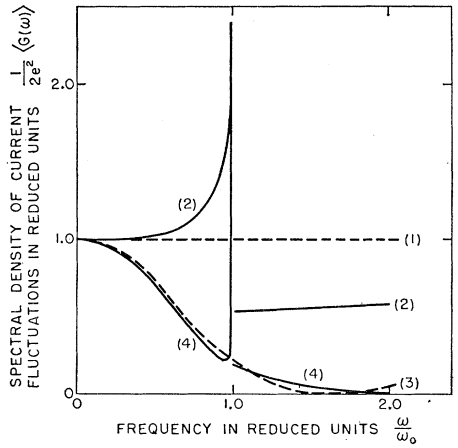

FIG. 2. Quantum corrections for the shot effect. (1) Classical: zero-width detector. (2) Quantum theory: zero-width detector. (3) Classical: finite detector, $\omega_0\tau=4$. (4) Quantum theory: finite detector, $\omega_0\tau=4$.

for $\omega>\omega_0$. In Eq. (35), ω_0 has the same meaning as before and $\tau=x_0k_x^0/2\omega_0$ is the transit time. We now consider a number of limiting cases in order to appreciate the significance of Eq. (35). When $x_0\rightarrow 0$, and, therefore, $\tau\rightarrow 0$, Eq. (35) reduces to the previous result of Eq. (33). In the classical limit $\hbar\rightarrow 0$ and $\omega_0\rightarrow\infty$ in which case, Eq. (35) has the limit

$$\langle G_s(\nu)\rangle=2e^2\sin^2(\tfrac{1}{2}\omega\tau)/(\tfrac{1}{2}\omega\tau)^2. \quad (36)$$

Equation (36) is the familiar classical result for this case.

In the case of $x_0=0$, the previous result of Eq. (33) for the spectral density was singular at $\omega=\omega_0$. The present result for finite x_0 also has this property. Also, for $x_0=0$, $\lim_{\nu\rightarrow\infty}\langle G_s(\nu)\rangle=\infty$. In the case of finite x_0, $\lim_{\nu\rightarrow\infty}\langle G_s(\nu)\rangle=0$. In Fig. 2, some numerical calculations of the spectral density for various cases are shown.

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Transmission of Positrons and Electrons*†

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The transmissions of monoenergetic beams of positrons and electrons with energies up to 960 kev have been measured in aluminum, brass, silver, tin, lead, and gold. The absorber forms the window of a 2π counter whose counting efficiency is better than 99% down to a few hundred electron volts. Particles from a radioactive source, focused into a beam by a 90-degree magnetic analyzer impinge perpendicularly on the absorber window of the 2π counter. The total transmission is therefore measured independently of forward angle of emergence or of partial energy loss. Positrons are found to be transmitted to a greater extent than electrons except at low energies in aluminum. These results are correlated with previous backscattering experiments and are in qualitative agreement with theoretical calculations of Rohrlich and Carlson. The shapes of the transmission curves are compared semiquantitatively with predictions of the Spencer theory of electron penetration.

I. INTRODUCTION

THE author has previously reported an excess of electron backscattering over positron backscattering.[1–3] The question naturally arises as to whether, in view of their excess backscattering, electrons are transmitted to a lesser degree than positrons. As will be shown in this paper, a lower transmission of electrons is usually, but not always, observed.

The problem of the penetration of positrons and electrons in thick foils has not been calculated at the present time. Bothe[4] has made estimates of the backscattering of electrons using nonrelativistic single-scattering cross sections. Miller[5] used Bothe's results, substituting the relativistic cross sections obtained by Bartlett and Watson[6] and Massey[7] for electrons and positrons, respectively, to show that an excess of electron backscattering over positron backscattering is to be expected. Rohrlich and Carlson[8] have published the results of theoretical calculations of range and stopping power of positrons and electrons, providing a qualitative interpretation of the experimental results to be reported here.

Recently, a theory of electron penetration in infinite media has been developed by Spencer.[9] This theory does not quite apply to the present experiments because

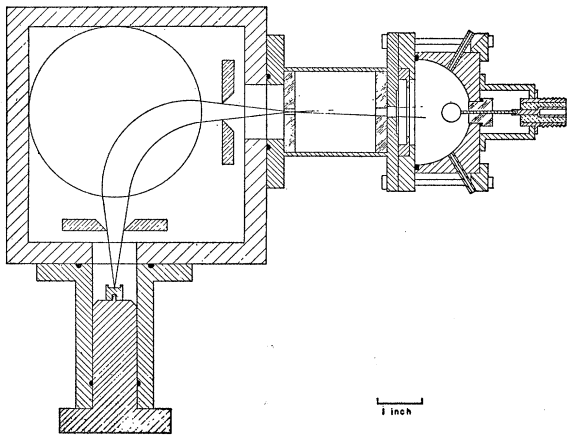

FIG. 1. Scale drawing of transmission geometry. The Lucite lining is not shown.

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