ON A MOLECULAR THEORY OF FERROMAGNETIC SUBSTANCES.

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Index to Paragraphs.

§ 1. Historical.

According to the Ampère-Weber theory of magnetism, the molecules of a ferromagnetic substance are all small magnets, the axes of which in an unmagnetized state, are turned uniformly in all directions, so that as a whole no magnetic polarity is observed. These magnetic molecules are believed to exert directive force upon one another. If an external force acts on the substance, the molecules tend to turn their axes in the direction of the field in opposition to the mutual directive force. With the increase of the field, the axes of the magnetic molecules are turned more and more in the direction of the field; if all the molecules are turned in this direction, magnetic saturation is reached.

The theory was afterward improved by Sir J. A. Ewing in a most satisfactory manner by taking into account the magnetic force due to the neighboring molecules.[1] He assumed that molecular magnets in every microscopic crystal are arranged in a cubic space-lattice, corresponding to the crystalline system of iron, which is the regular system. In each minute crystal, all magnets naturally assume one of three orientations of stable equilibrium, which are parallel to the sides of the space-lattice; but as the directions of the axes of these crystals are distributed uniformly in all directions, their external action is as a whole zero. If an external field acts on the substance, all the elementary magnets in each crystal will tend, as a whole, to turn with their axes in the direction of the field, but they are partially prevented from doing so

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[1] Phil. Mag. (5), 30 (1891), 205. See also Magnetic Induction in Iron and Other Metals.

by action of the mutual force, tending to draw these magnets back to their original stable orientation. With the increase of field, the molecules will more and more turn in the direction of the field and consequently the intensity of magnetization becomes greater, tending to an asymptotic value. Though the theory is very simple in its content, it explains many observed facts quite satisfactorily, at least qualitatively.

R. Gans[1] tried to treat Ewing's model of molecular magnets mathematically; but his theory differs essentially from that of Ewing in assuming the distribution of the molecular magnets in the substance to be quite arbitrary and in considering the magnetic action of its neighbors on each molecule to be constant. In Ewing's model, the mutual action is not a constant, but a function of the angle of the rotation of molecules. In fact the conclusions from his theory are only a rough approximation to the observed facts.

On the base of Langevin's theory of magnetism for paramagnetic gas, Prof. P. Weiss developed a theory of ferromagnetism,[2] by introducing an assumption that every molecule of the ferromagnetic substance, though it is not acted on by any external field, undergoes the action of a uniform molecular field of an enormous strength amounting to several ten millions of gauss. It is however very difficult to conceive the origin of such a molecular field and also to explain the fundamental phenomenon regarding the induced magnetism by means of his theory. As is well known, ferromagnetic elements can easily be magnetized with a field of 100 gauss to a value of seventy or eighty per cent. of its saturation value. If a molecular field of such an enormous strength really acts on each molecule, how is such an easy magnetization of the substance in any direction possible? Hence it seems now to the present writers very probable that the existence of the molecular field as conceived by Weiss does not correspond with the facts.

In what follows, we shall treat mathematically Ewing's theory of magnetism exactly in the same form as put forward by himself and show how the conclusions arrived at agree with the facts actually observed.

§ 2. CALCULATIONS OF INTERNAL FORCE DUE TO A GROUP OF ELEMENTARY MAGNETS.

According to Ewing's model, it is assumed that in every minute crystal, or "elementary complex" as we shall call it, composing a mass of iron, elementary magnets are all arranged in a space-lattice consisting

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[1] Gött. Nachr. (1910), 197; (1911), 118. R. Gans a. P. Hertz, Zeitsch. für Math. u. Phys., 61 (1913), 13.

[2] Arch. des Sci., No. 6, 31 (1911), 401.

of squares, but that the axes of these elementary crystals are distributed quite arbitrarily in all directions. If no external force acts on the substance, the axes of the elementary magnets in each complex take positions of stable equilibrium, that is, towards either side of the space-lattice. If an external force acts on the substance, the elementary magnets in each complex are assumed to turn, as a whole, in the direction of the field against the mutual force. The magnetization of the mass of iron is then the sum of the magnetizations of all these complexes in the direction of the magnetizing field. In order therefore to find the intensity of magnetization, it is first of all necessary to deduce the law of magnetization for each complex.

Suppose we have a group of elementary magnets arranged in the space-lattice and with their magnetic axes all parallel to one of the orientations of stable equilibrium and an external field acts in the plane of the lattice, as shown in the annexed figure. The action on each of these magnets by its neighbors is then the sum of their magnetic actions, but we may with a fair approximation suppose that the actions of eight only of the surrounding magnets are effective and those of the rest negligible. On this supposition it is easy to calculate the magnetic force acting on one of these magnets.

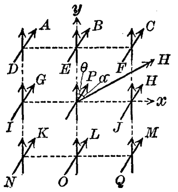

Fig. 1.

Let $2a$ be the sides of the space lattice, $2r$ and m be the length and the pole strength of the elementary magnets respectively. We take one side of the space-lattice as the axis of y and the other side as that of x, the initial direction of the elementary magnets being supposed to coincide with the direction of the y axis.

A pole of each magnet is acted on by 16 poles of the neighboring magnets, and, the action of four pairs of poles neutralizing each other by symmetry, there remain only the following eight forces:

Forces between E and P, J and P, F and P, O and P, Q and P, N and P, D and P, I and P.

Now

$$\begin{aligned}
EP^2 &= 4(a^2 + r^2 - 2ar\cos\theta),\\
JP^2 &= 4(a^2 + r^2 - 2ar\sin\theta),\\
FP^2 &= 4\{2a^2 + r^2 - 2ar(\cos\theta + \sin\theta)\},\\
OP^2 &= 4(a^2 + r^2 + 2ar\cos\theta),\\
QP^2 &= 4\{2a^2 + r^2 + 2ar(\cos\theta - \sin\theta)\},\\
NP^2 &= 4\{2a^2 + r^2 + 2ar(\cos\theta + \sin\theta)\},\\
IP^2 &= 4(a^2 + r^2 + 2ar\sin\theta),\\
DP^2 &= 4\{2a^2 + r^2 - 2ar(\cos\theta - \sin\theta)\}.
\end{aligned}$$

Among the eight forces, those tending to increase θ are:

$$\frac{m^2}{JP^2} = \frac{m^2}{4(a^2 + r^2 - 2ar\sin\theta)},$$

$$\frac{m^2}{OP^2} = \frac{m^2}{4(a^2 + r^2 + 2ar\cos\theta)},$$

$$\frac{m^2}{QP^2} = \frac{m^2}{4\{2a^2 + r^2 + 2ar(\cos\theta - \sin\theta)\}},$$

$$\frac{m^2}{FP^2} = \frac{m^2}{4\{2a^2 + r^2 - 2ar(\cos\theta + \sin\theta)\}},$$

those tending to decrease θ are:

$$\frac{m^2}{EP^2} = \frac{m^2}{4(a^2 + r^2 - 2ar\cos\theta)},$$

$$\frac{m^2}{IP^2} = \frac{m^2}{4(a^2 + r^2 + 2ar\sin\theta)},$$

$$\frac{m^2}{DP^2} = \frac{m^2}{4\{2a^2 + r^2 - 2ar(\cos\theta - \sin\theta)\}},$$

$$\frac{m^2}{NP^2} = \frac{m^2}{4\{2a^2 + r^2 + 2ar(\cos\theta + \sin\theta)\}}.$$

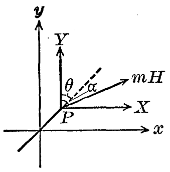

Fig. 2.

Denoting by X and Y the sum of the components of these forces in the directions of x and y respectively, we have for equilibrium

$$mH\sin(\alpha - \theta) = -Y\sin\theta + X\cos\theta.$$

If we calculate X and Y from the eight forces above given and put in the last equation, we get

$$H\sin(\alpha - \theta) =$$

$$-\frac{m}{2r^2}\Bigg[\frac{k}{(1 + k^2)^{\frac{3}{2}}}\frac{\sin\theta}{(1 - p^2\cos^2\theta)^{\frac{3}{2}}}\{(1 - p\cos\theta)^{\frac{3}{2}} - (1 + p\cos\theta)^{\frac{3}{2}}\}$$

$$+\frac{k}{(1 + k^2)^{\frac{3}{2}}}\frac{\cos\theta}{(1 - p^2\sin^2\theta)^{\frac{3}{2}}}\{(1 + p\sin\theta)^{\frac{3}{2}} - (1 - p\sin\theta)^{\frac{3}{2}}\}$$

$$+\frac{k}{(1 + 2k^2)^{\frac{3}{2}}}\frac{\cos\theta - \sin\theta}{\{1 - q^2(\cos\theta + \sin\theta)^2\}^{\frac{3}{2}}}\{[1 + q(\cos\theta + \sin\theta)]^{\frac{3}{2}}$$

$$-[1 - q(\cos\theta + \sin\theta)]^{\frac{3}{2}}\} + \frac{k}{(1 + 2k^2)^{\frac{3}{2}}}\frac{\cos\theta + \sin\theta}{\{1 - q^2(\cos\theta\sin\theta)^2\}^{\frac{3}{2}}}$$

$$\{[1 + q(\cos\theta - \sin\theta)]^{\frac{3}{2}} - [1 - q(\cos\theta - \sin\theta)]^{\frac{3}{2}}\}\Bigg],$$

where $k = a/r$, $p = 2k/(1 + k^2)$ and $q = 2k/(1 + 2k^2)$. Since $a > r$, $\therefore 0 < p < 1$ and $0 < q < 2$. Expanding the right-hand member of the above equation in powers of p and q, we get

$$
\begin{aligned}
&H\,(\sin\,(\alpha - \theta) \\
&\quad = \frac{mk}{r^2}\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\sin 4\,\theta\left\{\frac{p^3}{4(1 + k^2)^{\frac{3}{2}}}\left(\frac{1}{3!} + \frac{9}{2}\frac{p^2}{5!} + \frac{9}{2}\cdot\frac{11}{2}\frac{p^4}{7!} + \cdots\right)\right. \\
&\quad - \frac{q^3}{(1 + 2k^2)^{2\frac{3}{2}}}\left(\frac{1}{3!} + \frac{9}{2}\frac{q^2}{5!} + \frac{9}{2}\cdot\frac{11}{2}\cdot\frac{q^4}{7!} + \cdots\right) - \frac{9}{2}\cdot\frac{11}{2} \\
&\quad\quad \left[\frac{p^7}{4(1 + k^2)^{\frac{3}{2}}}\left(\frac{1}{7!} + \frac{1}{2}\frac{13}{2}\frac{p^2}{9!} + \cdots\right) + \frac{q^7}{(1 + 2k^2)^{\frac{3}{2}}}\right. \\
&\quad\quad\quad \left.\left.\left(\frac{1}{7!} + \frac{1}{2}\frac{13}{2}\frac{q^2}{9!} + \cdots\right)\right]\sin^2 2\theta + \cdots\right\}.
\end{aligned}
\tag{1}
$$

The right-hand side of the above equation is a function of θ only, provided r, m, a are given. Let us denote it by $F(\theta)$. If we put $F(\theta) = (m/r^2)f(\theta)$, $f(\theta)$ contains k only as a parameter. $F(\theta)$ or $f(\theta)$ is evidently a periodic function of θ, having $\pi/2$ as its period.

The fact, that the internal restoring force $F(\theta)$ has a period of $\pi/2$ follows at once from the following physical considerations: If all the magnets in a complex, starting from a given orientation, turn through a right angle, the mutual action between the molecules must remain unchanged on account of the property of the square space-lattice, and hence $F(\theta)$ must be a periodic function of $\pi/2$. In an orientation of stable equilibrium of these magnets, there is no deflecting force acting on any magnet due to the surrounding ones, that is, $F(\theta) = 0$ for $\theta = 0$. As the magnets deflect from this position, $F(\theta)$ increases. It is evident that for $\theta = \pi/4$, $F(\theta)$ must again vanish through the symmetrical orientation of molecules. Hence as θ increases from 0 to $\pi/4$, $F(\theta)$ must pass through a maximum. From $\theta + \pi/4$ upward, the axes of the magnets tend to place themselves in the next orientation of stable equilibrium, that is, in the orientation for $\theta = \pi/2$. Hence $F(\theta)$ changes its sign in passing through $\pi/4$. As θ increases from $\pi/4$ to $\pi/2$, $F(\theta)$, which is now negative, at first decreases, attains a minimum and then increases, vanishing at $\theta = \pi/2$. The same change of $F(\theta)$ is repeated in the other quadrants.

It is also to be remarked that if r be very small in comparison with a, that is, powers of p and q higher than the third are negligibly small, $F(\theta)$ vanish for all values of θ, that is, no resisting couple acts, if the axes of the magnets be deflected from their orientations of stable equilibrium.

In the above calculation, we have only taken account of the mutual force due to eight surrounding magnets. If we take the next 16, 24, 32, . . . magnets in the outer squares into consideration step by step, the expression for $F(\theta)$ rapidy converges to a definite value; because for each outer square, the number of magnets increases by 8, while the force exerted by each pole in different squares, decreases by the inverse square of the distance. For example, if 24 magnets in the first two squares be taken, the amplitude in $F(\theta)$, assuming $k = 2$, increases only by 3.6 per cent. as compared with the case above discussed. Moreover, by taking all the magnets in the complex into consideration, the periodicity of $F(\theta)$ cannot evidently vary for the reason as explained above. The only change consists in the variation of the coefficients of $\sin 4\theta$ and $\sin^2 2\theta$. More generally, if we consider the distribution of magnets in a cubical space-lattice and the effect of the magnets situated in two adjacent planes on the magnet under consideration, it is easily found by calculation that the correction due to this effect amounts only to 4.8 per cent. as compared with the case before mentioned. Hence we can conclude that in the most general case, $F(\theta)$ is a periodic function of θ, having $\pi/2$ as its period and k as a parameter.

The expression for $F(\theta)$ may generally be written as

$$F(\theta) = A \sin 4\theta,$$

where

$$A = \frac{3\cdot5\cdot7}{2^3}\,\frac{m}{r^2}\left(\frac{2^3}{3\cdot5\cdot7}\,\varphi(k) - \varphi'(k)\sin^2 2\theta + \cdots\right)$$

and φ, φ' are the functions of k only, k being always greater than 1. For $k = 1$, the amplitude of $F(\theta)$ is infinitely large; as k increases from 1, the amplitude rapidly decreases, the value of φ' becomes very small in comparison with that of φ, and the form of the curve approaches to the sine as given by the first term of the above series. Fig. 3 shows this manner of approaching the sine curve; here curves 1, 2, 3 are those corresponding to $k = 1.3$, 1.6, 2.0 and curve 4 represented by a broken line is a sine curve. Their amplitudes

$$A_1 : A_2 : A_3 = 4.431 : 0.976 : 0.256$$

are all reduced to the same magnitude as that of the sine for the sake of comparison. Thus we see that for a value of k greater than 2, the form of the curve $F(\theta)$ is very nearly equal to $\sin 4\theta$. In the case of iron, nickel and cobalt, which are easily magnetizable, this restriction seems to be quite reasonable. Hence, under this limitation, we may use, for the first approximation, A sin

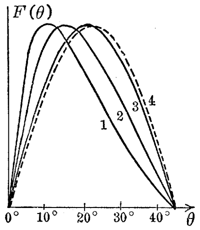

Fig. 3.

4θ with a constant amplitude instead of $F(\theta)$, and proceed to develop the theory of magnetization. Relation (I) takes then the following form

$$H \sin(\alpha - \theta) = A \sin 4\theta \tag{2}$$

and

$$A = \frac{m}{r^2}\varphi(k),$$

where φ is a function of k only. As shown in Fig. 4, φ rapidly decreases with increasing k.

If a special investigation be necessary for the case of a closer molecular distance, we must use the exact relation (1). But, as we shall see presently, we have always used a graphical solution for relation (2) and consequently the substitution of relation (1) for the last one does not cause much complication in our calculations.

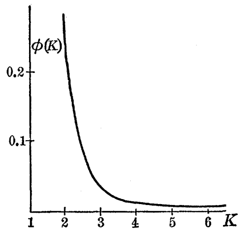

Fig. 4.

§ 3. Magnetization of a Single Complex.

Suppose in a complex an external field H acts in a plane parallel to the face of the elementary cube and in a direction making an angle α with one of the directions of stable equilibrium of the molecular magnets arranged in the space-lattice; the magnets will then be in equilibrium after turning through a common angle θ from their initial direction. It is required to find the component of magnetization I in the direction of the applied field. We have obviously

$$I = 2mrn \cos(\alpha - \theta) = I_0 \cos(\alpha - \theta),$$

where n is the number of elementary magnets and I_0 the saturation value of the intensity of magnetization. Denoting $I/I_0 = i$, we have from the above relation

$$i = \cos(\alpha - \theta), \tag{3}$$

in which α, θ, H are related by an equation

$$H \sin(\alpha - \theta) = A \sin 4\theta;$$

if we denote $H/A = h$, we get

$$h \sin(\alpha - \theta) = \sin 4\theta. \tag{4}$$

A contains m, r, a and depends on the properties of particular substance; so also I_0. But if we use the reduced i and h instead of the actual intensity of magnetization and field, relations (3) and (4) apply for all ferromagnetic substances belonging to the regular system. If h and α be given, equation (4) gives the value of θ and therefore equation (3) the value of i. Hence equations (3) and (4) may be considered as the laws of magnetization.

Since equation (4) does not change, if we put for α and θ the values $\alpha + \pi/2$ and $\theta + \pi/2$, it follows that the force required acting in the direction α to deflect the system of magnets through an angle θ is equal to that acting in a direction $\alpha + \pi/2$ and deflecting these magnets by $\theta + \pi/2$. Hence the curve of magnetization by a force acting in a direction α partly coincides with the curve corresponding to a system of magnets, whose initial direction makes with the field an angle $\alpha + \pi/2$.

If h and α be given, θ can be found from equation (4), which is of the eighth degree in $\sin\theta$ or $\cos\theta$; hence we can not solve it analytically. However, as θ is given as the intersections of the two curves

$$y = \sin 4\theta \quad \text{and} \quad y = h \sin(\alpha - \theta),$$

we can easily find it by a graphical method. In Fig. 5, curve I represents $y = \sin 4\theta$, and curves a, b, c, d, those of $y = h \sin(\alpha - \theta)$ for $\alpha = 30°$, 70°, 120° and 160° respectively, h being taken as 0.6. By giving different values to h, the curve of magnetization can be obtained.

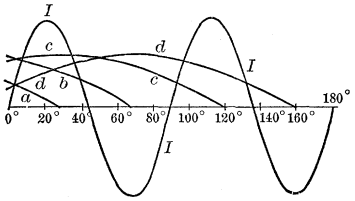

Fig. 5.

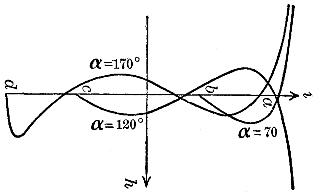

Fig. 6.

In Fig. 6, four curves representing the relation between i and h are given, in which for the angle α were taken angles of 30°, 70°, 120° and 170° respectively. They give the intensity of magnetization in the direction of the respective field, when the magnitude of the latter is so varied that it is always in equilibrium with the internal resisting force $\sin 4\theta$. In the curve for $\alpha = 30°$, the initial point a corresponds to the value of $\cos 30°$; as h increases, θ becomes greater, but always less than α, and therefore $i = \cos(30° - \theta)$ steadily increases, tending asymptotically to the value of $i = 1$ with $h = \infty$. In the curve for $\alpha = 70°$, the point b corresponds to the value of $\cos 70°$; as h increases from 0, θ and therefore $\sin 4\theta$ also increases. Since however the latter quantity reaches a maximum at $\theta = \pi/8$, h must be diminished from a certain value of θ upward, if the magnetization is to be effected statically or reversibly. With $\theta = \pi/4$, the resisting force $\sin 4\theta$ vanishes and therefore h must be diminished to zero; with a further increase of θ, $\sin 4\theta$ changes sign and therefore h must be applied in an opposite direction, if the magnetization is to be made reversibly. If θ approaches to 70°, h becomes $-\infty$ in the limit and the magnetization tends asymptotically to unity. The curve for $\alpha = 120°$, which begins at the point c on the negative side of i passes through a maximum and a minimum of h, and coincides with the curve for $\alpha = 30°$, as the value of i increases. The curve for $\alpha = 170°$, beginning at a point d on the negative side of i, passes through two maxima and one minimum of h with the increase of i and approaches asymptotically to the line $i = 1$.

In the ordinary case of magnetization, the field is continuously increased, and therefore the magnetization is only partly reversible. But it is easy to see in what manner the magnetization in the direction of the field increases by applying a continuously increasing field.

Case 1, $0 < \alpha < (\pi/4)$. The component magnetization i in the direction of the field increases with h and becomes 1 for $h = \infty$. If the field is gradually reduced, i takes its original value, and there is no hysteresis.

Case 2, $(\pi/4) < \alpha < (\pi/2)$. i increases with h continuously up to the maximum resisting force; here it undergoes an abrupt change and takes a value corresponding to the rotation of $\pi/2$ of the initial orientation of molecular magnets. With a further increase of the field, i continuously increases in a manner, as if the initial orientation were $\alpha - (\pi/2)$. If the field is reduced, i takes a value quite different from its initial, as shown in Fig. 7; that is, there gives rise a hysteresis phenomenon.

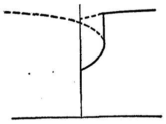

Fig. 7.

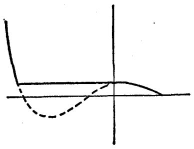

Fig. 8.

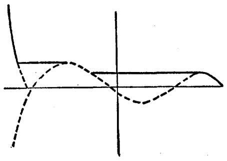

Fig. 9.

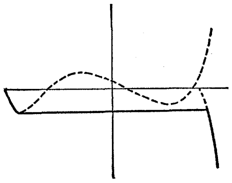

Fig. 10.

Case 3, $(\pi/2) < \alpha\ (3/4)\pi$. i increases with h, at first continuously, and then abruptly, when the resisting force reaches a maximum. After this, the curve of magnetization follows the course corresponding to the case with the initial orientation of $\alpha - (\pi/2)$ (Fig. 8). With the reduction of the field, hysteresis phenomenon is also observed.

Case 4, $(3/4)\pi < \alpha < \pi$. The curve of initial magnetization is the same as in the above cases. If the first maximum of the resisting force is less than the second maximum, its next magnetization is the same as in the case with the initial orientation of $\alpha - (\pi/2)$ (Fig. 9); if the first maximum is greater than the second, the magnetization is the same as that for the initial orientation of $\alpha - \pi$ (Fig. 10). The subsequent magnetization takes place continuously. By reducing the field, the corresponding hysteresis is observed.

The relation between the initial orientation and the maximum resisting force h_m can be found in the following way:

From

$$h = \frac{\sin 4\theta}{\sin(\alpha - \theta)},$$

we have

$$\frac{dh}{d\theta} = \frac{5 \sin(\alpha + 3\theta) + 3 \sin(\alpha - 5\theta)}{2 \sin^2(\alpha - \theta)}.$$

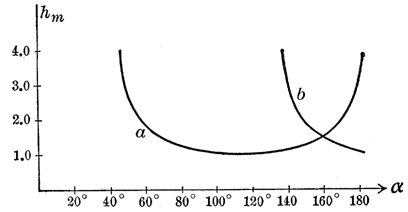

Fig. 11.

If the value of θ corresponding to the maximum force be denoted by θ_0 we have

$$5 \sin(\alpha + 3\theta_0) = 3 \sin(5\theta - \alpha) \qquad (5)$$

and

$$h_m = \frac{\sin 4\theta_0}{\sin(\alpha - \theta_0)}.$$

The existence of such values of θ_0 can be understood from Fig. 6. The calculated values of h_m for different values of α are given in the following table and in Fig. 11.

α.	h_m.	α.	h_m.	α.	h_m.	α.	h_m.
45°	4.000	100°	1.025	70°	1.405	160°	1.541
50°	2.625	120°	1.008	80°	1.205	170°	2.018
60°	1.750	140°	1.137	90°	1.088	180°	4.000

Curve a in Fig. 11 refers to the first maximum; in the interval between 135° and 180°, the second maximum is also possible. However, as h_m corresponding to α for the first maximum is equal to that corresponding to $\alpha + (\pi/2)$ for the second maximum, curve b for the second maximum has the same form as curve a being only displaced through $\pi/2$.

§ 4. Deduction of the Curve of Magnetization.

Hitherto we have exclusively considered the magnetization of a single complex; but we are now able to study the magnetization of a mass of ferromagnetic substance, such as iron, which consists of a great number of such elementary complexes with their magnetic axes uniformly distributed in all directions. Now, the faces of the elementary cubes or the complexes are in actual cases directed uniformly in all directions; but for the sake of the simplicity of calculation, it is here assumed that the complexes have one of their faces all parallel to a common plane, other faces being distributed quite arbitrarily, and the magnetic field acts parallel to this plane. The problem is then reduced to the two-dimensional. The magnetization of this simple case does not obviously differ from that of the actual case in its character.

Let N be the number of elementary complexes; if there is no magnetic force acting on these complexes, the number of complexes, whose magnetic axes make, with a certain direction, an angle lying between α and $\alpha + d\alpha$, is equal to

$$dN = \frac{N}{2\pi} d\alpha.$$

If M be the magnetic moment of a complex, whose magnetic axis makes initially an angle α with the direction of the field, then the component of magnetization in the direction of the field is $M \cos (\alpha - \theta)$. Considering M to be the same for all complexes, the total magnetization due to these complexes is

$$I = \int_{-\pi}^{+\pi} \frac{MN}{2\pi} \cos (\alpha - \theta) dd = \frac{I_0}{\pi} \int_0^{\pi} \cos (\alpha - \theta) d\alpha,$$

where $I_0 = NM$ is the saturation value of the magnetization. Hence we have for i

$$i = \frac{1}{\pi} \int_0^{\pi} \cos (\alpha - \theta) d\alpha. \tag{6}$$

The relation connecting α and θ must however be different from that for a single complex. Here besides $F(\theta)$, we must also consider the magnetic force due to surrounding complexes. If no field acts on the substance, the resultant effect of the surrounding complexes is obviously zero; but in its magnetized state, this is not the case. To calculate this force exactly is almost impossible; but it is not difficult to calculate approximately its mean effect. Since the total action of a complex on a magnet within it is the same as the sum of the effects of neighboring magnets, those of the distant ones being very small, we may consider the form of the complex under consideration to be a sphere, without causing sensible error in the value of $F(\theta)$. The magnetic effect of other complexes on the magnet under consideration may approximately be replaced by that due to a uniform distribution of magnetization with a mean intensity in the space in which other complexes are found. As the boundary of the said complex is assumed to be a sphere, this force is $(4/3)\pi I$ acting in the direction of the external field and does not generally coincide in direction with that of the axis of the magnet under consideration; and hence it exerts a couple tending to turn the magnet in the direction of the field. Hence instead of relation (2), we must use the following formula:

$$\left(H + \frac{4}{3}\right)\pi I \sin(\alpha - \theta) = A \sin 4\theta.$$

But for a given value of H, I is a constant, so that for a while we may regard $H + (4/3)\pi I$ as an external field and proceed to calculate I for different assigned values of $H + (4/3)\pi I$. After finding I, the actual field may be found by simply subtracting $(4\pi/3)I$ from the assigned field. Hence the same relation as (2) may also be used in the present case, that is,

$$h = \frac{\sin 4\theta}{\sin(\alpha - \theta)}. \tag{7}$$

If h be given, equation (7) gives θ in terms of α, and if this value of θ be substituted in equation (6), this gives the intensity of magnetization i in terms of h, and thus the problem is formally solved. But in actual calculation, some complications are involved, and we must separately consider cases corresponding to several graded values of h.

First let us consider the case when h is very small; then θ is also small, and therefore $\sin 4\theta = 4\theta$. From (7), we get

$$h(\sin\alpha - \theta\cos\alpha) = 4\theta;$$

$$\therefore \quad \theta = \frac{h\sin\alpha}{4 + h\cos\alpha}.$$

Equation (6) gives

$$i = \frac{1}{\pi}\int_0^\pi (\cos\alpha + \theta\sin\alpha)d\alpha = \frac{1}{\pi}\int_0^\pi \left\{\cos\alpha + \frac{h\sin^2\alpha}{4 + h\cos\alpha}\right\}d\alpha$$

$$= \frac{1}{\pi}\int_0^\pi \cos\alpha d\alpha + \frac{h}{4\pi}\int_0^\pi \sin^2\alpha\cdot\left(1 + \frac{h}{4}\cos\alpha\right)^{-1}d\alpha.$$

The first integral vanishes; and if the second term be expanded and intergrated, we have

$$i = \frac{h}{4}\left(\frac{1}{2} + \frac{h^2}{8.4^2} + \frac{\pi}{16}\frac{h^4}{4^4} + \cdots\right)$$

$$= 0.125h + 0.00195h^3 + 0.00007h^5 + \cdots. \tag{8}$$

As it ought to be, i is an odd function of h. If h be sufficiently small, the terms of any higher order of h than the third can be neglected, and i and h are linearly related to each other. This fact was verified by experiments of Bauer,[1] Lord Rayleigh[2] and others. In this case, the magnetization is perfectly reversible, that is, there is no hysteresis, a fact which agrees with the result of the experiments.

Secondly, we consider the case, where h is large. To change the integration variable from α to θ, we differentiate equation (7),

$$h \cos (\alpha - \theta) \left(\frac{d\alpha}{d\theta} - 1 \right) = 4 \cos 4\theta$$

$$\therefore \quad \frac{d\alpha}{d\theta} = \frac{4 \cos 4\theta}{h \cos (\alpha - \theta)} + 1.$$

And also

$$\cos (\alpha - \theta) = \pm \frac{1}{h} \sqrt{h^2 - \sin^2 4\theta}$$

$$\therefore \quad i = \frac{1}{\pi} \int \left\{ \frac{4 \cos 4\theta}{h} \pm \frac{1}{h} \sqrt{h^2 - \sin^2 4\theta} \right\} d\theta. \qquad (9)$$

According to the magnitude of h, all the complexes, during magnetization, do not necessarily change their angle of deflection continuously; in fact, some of these complexes made an abrupt rotation of $\pi/2$ or π. Hence in evaluating the above integral, it is necessary to divide the limits of integration into several parts. If h be given, we can find from Fig. 11 the value of α having h as h_m; the values of θ for these values of α may then be found from equation (7). We have generally three values of α and θ, let us call them by α_1, α_2, α_3 and θ_1, θ_2, θ_3. Then we have

$$\int_0^{\pi} = \int_0^{\alpha_1} + \int_{\alpha_1}^{\alpha_2} + \int_{\alpha_2}^{\alpha_3} + \int_{\alpha_3}^{\pi}.$$

In the first and fourth integrals, the elementary magnets in the complexes belonging to these integrals remain stable, since the field is less in their cases than the critical value. The magnets in the complexes belonging to the second integral all lie beyond the position of stable equilibrium, and therefore the magnetization is the same, as if the initial orientation of the complexes were $\alpha - (\pi/2)$. Hence the limit of the second integral must be changed from α_1 and α_2 to $\alpha_1 - (\pi/2)$ and $\alpha_2 - (\pi/2)$. In the third integral, the magnets in the complexes lie beyond the first and second positions of stable equilibrium, and therefore the magnetization is the same, as if the initial orientation were $2 - \pi$.

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[1] Bauer, Inaug. Diss. Zürich (1879). Wied. Ann., II. (1880), 399.

[2] Phil. Mag., March (1887). See also Ewing's "Magnetic Induction," 124.

Hence the limits of the third integral are to be changed from α_2 and α_3 to $\alpha_2 - \pi$ and $\alpha_3 - \pi$. If the integration variable be then changed from α to θ, we have

$$\int_0^{\pi} = \int_{\theta_1}^{\theta'_1} + \int_{\theta_2}^{\theta'_2} + \int_{\theta_3}^{\theta'_3} + \int_{\theta_4}^{\theta'_4}.$$

Now from equation (9), we have

$$\begin{aligned} i &= \frac{1}{\pi h} \left| \sin 4\theta \right|_{\theta}^{\theta'} \pm \frac{1}{\pi} \int_{\theta}^{\theta'} \sqrt{1 - \frac{1}{h^2} \sin^2 4\theta}\, d\theta \\ &= \frac{1}{\pi h} (\sin 4\theta' - \sin 4\theta_0) \pm \frac{1}{\pi} \left\{ \int_{\theta}^{\theta'} \sqrt{1 - k^2 \sin^2 4\theta}\, d\theta \right. \\ &\qquad\qquad \left. - \int_0^{\theta} \sqrt{1 - k^2 \sin^2 4\theta} d\theta \right\}, \end{aligned}$$

where $k^2 = 1/h^2$. Hence if E be an elliptic integral of the second kind, we have

$$i = \frac{1}{\pi h} (\sin 4\theta' - \sin 4\theta) \pm \frac{1}{4\pi} \{E(k, 4\theta') - E(k, 4\theta)\}. \tag{10}$$

By expanding E in a power series of k, we have

$$\begin{aligned} i = \frac{1}{\pi h} (\sin 4\theta' - \sin 4\theta) \pm \frac{1}{4\pi} \Bigg| \Bigg\{ & 4\theta - \left(\frac{1}{2^2} 4\theta + \frac{1}{2^3} \sin 8\theta \right) k^2 \\ & - \left\{ \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \frac{1}{3} \cdot 4\theta + \frac{1 \cdot 3}{2 \cdot 4} \frac{1}{2} \left(\frac{3}{2.4} + \frac{1}{4} \sin^2 4\theta \right) \sin 8\theta \right\} k^4 \\ & - \left\{ \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{1}{5} \cdot 4\theta + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(\frac{5 \cdot 3}{6 \cdot 4 \cdot 2} + \frac{5}{6 \cdot 4} \sin^2 4\theta + \frac{1}{6} \sin^4 4\theta \right) \frac{1}{2} \sin 8\theta \right\} k^2 \\ & - \cdots \Bigg|_{\theta}^{\theta'}. \end{aligned} \tag{11}$$

The double sign of the second term must be so chosen that upper and lower signs correspond to $\alpha - \theta > (\pi/2)$ and $\alpha - \theta < (\pi/2)$ respectively, with the condition that if an abrupt turning of the molecules through $\pi/2$ takes place, α and θ are measured from the new position of equilibrium.

In the following tables and in Fig. 12, the result of our calculation according to the above relations is given. Up to $h = 0.5$, i was calculated by relation (8), while for higher fields, it was obtained by means of relation (11). Thus, we found at first three values of α corresponding to different values of h:

h.	α_1.	α_2.	α_3.	h.	α_1.	α_2.	α_3.
1.5	62° 0′	157° 0′	159° 30′	2.5	50° 30′	140° 0′	174° 0′
2.0	54° 38′	145° 0′	169° 0′	3.0	47° 0′	138° 30′	178° 0′

From equation (7) and these values of α, we found the following values of θ for the limits of integrations:

h.	θ_1.	θ_1'.	θ_2.	θ_2'.	θ_3.	θ_3'.	θ_4.	θ_4'.
1.5	0°	21°	– 8°	21°	– 8°	– 6°	16°	0°
2.0	0°	26°	– 13°	26°	– 13°	– 4°	14°	0°
2.5	0°	34°	– 17° 20′	34°	– 17° 20′	– 2° 30′	12° 30′	0°
3.0	0°	37°	– 21° 30′	37°	– 21° 30′	– 1° 30′	8°	0°

We have then for each of the integrals the following numbers:

h.	$\int_{\theta_1}^{\theta_1'}$.	$\int_{\theta_2}^{\theta_2'}$.	$\int_{\theta_3}^{\theta_3'}$.	$\int_{\theta_4}^{\theta_4'}$.	Sum.
1.0	....	....	....	......	0.183
1.5	0.301	0.436	0.036	– 0.096	0.677
2.0	0.286	0.448	0.127	– 0.045	0.816
2.5	0.289	0.475	0.172	– 0.031	0.875
3.0	0.241	0.483	0.196	– 0.011	0.909

Thus the form of the curve of magnetization agrees precisely with that experimentally found. This curve starts from the origin at a definite angle, and increases at first linearly with the field. With a further increase of field, the magnetization increases more and more rapidly; in a certain field, its rate attains a maximum and then gradually decreases. The curve of magnetization passes therefore through an inflexion point, and gradually approaches to an asymptotic value 1, as the field is increased. The curve is the normal curve of magnetization with the reduced intensity of magnetization and field; it is common for all the ferromagnetic substances belonging to the regular system. The curve of magnetization for a particular substance can be obtained by multiplying I_0 and A, characteristic constants for the substance, to i_0 and h respectively.

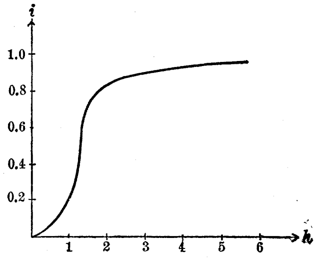

Fig. 12.

If the curve of magnetization be plotted against the actual field as explained at the beginning of the present paragraph, the characteristic form of the curve will not materially change.

§ 5. Residual Magnetism and Hysteresis Phenomenon.

If a mass of iron is once magnetized to saturation, and then the field reduced to zero, there remains a residual magnetism. The amount of this residual magnetism can easily be found in the following way: The complexes, whose magnetic directions lie initially between 0 and $\pi/4$ will return to their original position with $h = 0$; the complexes, whose magnetic directions were initially $\pi/4 > \alpha > \pi/2$, or $\pi/2 > \alpha > (3/4)\pi$, take a new position of equilibrium differing from the initial by $\pi/2$ with $h = 0$. Lastly the complexes, whose magnetic directions were initially $(3/4)\pi > \alpha > \pi$, will come to a new position differing by π from the initial with $h = 0$. Hence, if the field be reduced to zero, the magnetic directions of all the complexes are distributed uniformly within an angle making $\pi/4$ on both sides of the field. The residual magnetism may therefore be found thus:

$$R = 2\int_0^{\pi/4} M \cos\theta dN. \quad dN = \frac{2N}{\pi}d\theta$$

$$= \frac{4I_0}{\pi}\int_0^{\pi/4}\cos\theta d\theta = \frac{4I_0}{\pi\sqrt{2}}.$$

Hence the reduced residual magnetism r is

$$r = \frac{R}{I_0} = 0.8927. \tag{12}$$

This is the same value as obtained by Ewing.[1] Thus there remains a residual magnetism of about 90 per cent. The experiments with very long iron wires confirm the correctness of this conclusion.

According to the above consideration, the process of reducing the field from ∞ to 0 is reversible, that is, the magnetization during the reduction of the field from ∞ to 0 exactly coincides with the magnetization from $h = 0$ to ∞, the initial magnetization being r. This curve of magnetization can easily be found: because the initial orientation of the complexes is known to be uniformly distributed within an angle subtended by the lines inclined at $\pi/4$ to the field. If h be small,

$$i = \frac{4}{\pi}\int_0^{\pi/4}\cos(\alpha - \theta)d\alpha \quad \text{and} \quad h = \frac{\sin 4\theta}{\sin(\alpha - \theta)},$$

$$\therefore\ i = \frac{1}{\pi}\int_0^{\pi/4}(h + 4\cos\alpha)(1 + \frac{h}{4}\cos\alpha)^{-1}d\alpha$$

$$= 0.8927 + 0.047h - 0.083h^2 + \cdots. \tag{13}$$

For a large value of h, we find from equation (7) the value of θ corresponding to the limit of integration. By means of equation (11), the value of i will be found on simple substitutions.

Starting from the residual magnetism, the magnetization by a gradually increasing negative field can be calculated in a similar way.

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[1] Magnetic Induction (1900), 325.

This case is equivalent to the magnetization by a positive field of a group of complexes, whose initial magnetic directions are uniform and given by ± $(3/4)\pi > \alpha > \pi$. For small values of h, we have

$$i = -\frac{4}{\pi}\int_{3/4\pi}^{\pi}\cos(\alpha-\theta)d\alpha = -\frac{1}{\pi}\int_{3/4\pi}^{\pi}(h+4\cos\alpha)\left(1+\frac{h}{4}\cos\alpha\right)^{-1}d\alpha$$

$$= +0.8927 - 0.047h - 0.083h^2 - \cdots. \quad (14)$$

For large values of h, we find i from equations (7) and (11), as in the former case. The results of calculation are included in the following table:

h.	i.	h.	i.	h.	i.	h.	i.
$+\infty$	1.000	− 1.0	0.815	2.0	0.944	− 3.0	− 0.847
3.5	0.973	− 1.5	0.015	1.5	0.932	− 5.0	− 0.981
3.0	0.962	− 2.0	− 0.584	1.0	0.922	$-\infty$	− 1.000
2.5	0.956	− 2.5	− 0.786	0.0	0.893		

In this way, we can obtain a well-known hysteresis loop, when the field is varied between $+\infty$ and $-\infty$, as shown in Fig. 13. It possesses all the characteristics shown by iron, nickel and cobalt, and is far nearer the experimental curve than the rectangular hysteresis loop obtained by Gans.

The hysteresis loop accompanying a cyclic change of magnetic field between $+h$ and $-h$ can be calculated in a similar manner. For this purpose, the residual magnetism obtained by reducing the field from h to 0 will be at first calculated. Then, the curve of magnetization having this residual magnetism as the initial will be calculated, it must coincide with the curve of demagnetization obtained by reducing the field from h to 0. Next, the curve of magnetization from 0 to $-h$, having the state of residual magnetism as the initial, will be calculated, and so on. In this way, we have obtained a complete cycle of magnetization.

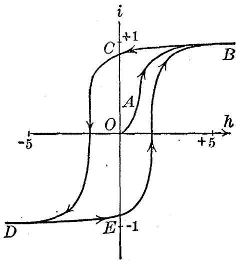

Fig. 13.

The residual magnetism, when the field h is reduced to zero, is easily known; because for a given value of h, we can find from Fig. 11 the values of α having h as the maximum resisting force, and therefore it can be completely known how many complexes, which had initially a uniform distribution of their axes, will return to their original position by reducing the field to zero and how many of them will rotate through one or two right angles from their initial positions. Hence the residual magnetism can be calculated by the following expression:

$$r = \frac{1}{\pi}\left\{ \int_0^{\alpha_1} \cos \alpha d\alpha + \int_{\alpha_1}^{\alpha_2} \cos\left(\alpha - \frac{\pi}{2}\right) d\alpha + \int_{\alpha_2}^{\alpha_3} \cos(\alpha - \pi) d\alpha + \int_{\alpha_3}^{\pi} \cos \alpha d\alpha \right\}.$$

Since the orientation of the magnetic axes of these complexes in the residual state of magnetization is thus completely known, a further magnetization with positive and negative fields can be calculated in the same way as the case above discussed. In this way, we calculated three curves of hysteresis for different values of h, which are shown graphically in Fig. 14. The curves are found to agree with the results of experiments.

In our theory, the hysteresis phenomenon takes place only when the molecular magnets in the complexes turn abruptly; otherwise the process of magnetization should be reversible. Thus, as we have seen, the initial magnetization up to about $h = 1$ and also the demagnetization and the second magnetization between 0 and h, ought to be reversible. In actual cases, however, we also find a small but distinct hysteresis in weak fields. This discrepancy between theory and experiment may probably be due to two causes, which are not considered in the above theory.

In the important paper[1] on the modulus of rigidity of rocks, Prof. S. Kuskabé has shown that by cyclically changing the twist between $+\tau$ and $-\tau$, all rocks investigated by him show a distinct hysteresis, though Hooke's law is fairly well satisfied. The form of his hysteresis loop is quite similar to that observed in iron in weak fields. As an example, we reproduce here his hysteresis curve of twist for marble (Fig. 15). He explained the phenomenon quite satisfactorily by his theory based on the experimental fact that by applying couple, the twist of the specimen, after its instantaneous increase of a definite amount, gradually increases with time, asymptotically tending to its final value, that is, the twist shows a time-effect.

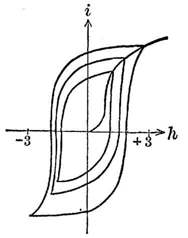

Fig. 14.

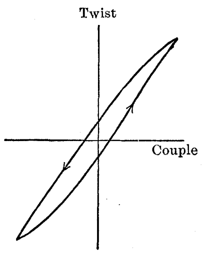

Fig. 15.

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[1] Journ. Coll. Sci., 19, Art. 6 (1903).

Now the magnetization has also a time-effect called the magnetic viscosity, though it is not so conspicuous as it is in the case of rcoks. Namely, the magnetization does not instantaneously increase to its final value by applying a magnetic field, but it requires some time for arriving at its maximum value. This effect is specially conspicuous in weak fields, and may therefore be considered as the first, but less important cause of the hysteresis observable in weak fields.

The second, but principal cause of the hysteresis in weak fields is probably the irregular distribution of the axes of the elementary magnets situated on the bounding surfaces of different complexes. In our theory, we have assumed that if there acts no external fields, all the elementary magnets in each complex assume the same direction of stable equilibrium. But in actual cases, the elementary magnets on the bounding surfaces of the complexes may place themselves in quite different directions, as do those in the interior through the action of the magnets in the neighboring complexes; and thus there results an irregular distribution of elementary magnets on the bounding surfaces. Hence some of the elementary magnets may initially be found in positions, which are not far from those of unstable equilibrium. If a weak magnetizing force acts on such magnets, it may cause the abrupt rotation of the magnets and therefore a hysteresis phenomenon results even in a weak field.

§ 6. Calculation of the Hysteresis-loss by Magnetization.

According to our theory, the hysteresis-loss takes place only when the rotation of the molecular magnets caused by the external field becomes discontinuous. That is, if the reduced field h be less than 1, there is no sudden rotation of molecules, and hence no hysteresis-loss by magnetization; if however h be greater than 1, some of the molecules make abrupt rotations and give rise to the hysteresis phenomenon. The number of such molecules will increase with the strength of the field and attain to an asymptotic value at $h = 4$. A further increase of magnetizing field does not cause any more abrupt rotation of molecules.

In the curve of magnetization, O AB in Fig. 13, the hysteresis-loss takes place only in a portion ($h = 1$ to 4) of the magnetization curve. During the demagnetization from $h = \infty$ to 0, no abrupt rotation of molecules occurs, and therefore we have no hysteresis. But the magnetization in the opposite direction from $h = 0$ to -4 involves a loss of energy. Similarly, in portion DE of the magnetization curve, there is no loss of energy, but in portion EB, we have a loss of energy equal in amount to that in portion CD.

According to the general theory of magnetism, which assumes no hypothesis as to the molecular magnets, the total loss of energy during a complete cycle is equal to the area of the hysteresis-loop. By our theory, the hysteresis-loss is the kinetic energy obtained by the molecules during their abrupt rotations, and hence it is very interesting to investigate, whether in a cyclic process of magnetization, the kinetic energy thus obtained is equivalent to the area enclosed by the hysteresis-loop. As the following calculation will show, the result completely agrees with the above theory; moreover, in the process of magnetization, we can distinguish the energy dissipated during the magnetization from the total energy.

We shall at first consider the energy loss of a single complex during magnetization. If h increases from 0 to h_m, which is the critical field for the abrupt turning, the molecular magnets in the complex will turn reversibly towards the field; at $h = h_m$, an abrupt turning of the molecules occurs, and their axes take new orientations corresponding to the initial position differing by $\pi/2$ or π from the original. During the abrupt turning, the molecules will acquire a kinetic energy, which is nothing but the heat energy produced; the quantity of this energy must be equal to the sum of the work done on the molecules.

The couple N acting on a molecular magnet, whose magnetic moment is M, is

$$N = M\{H_m \sin(\alpha - \theta) - A \sin 4\theta\},$$

where

$$M = 2mr.$$

If θ_0 and θ_1 be the angles of deflection of a molecule from its initial position, which correspond to the positions just before and after the abrupt turning, we have

$$W = \Sigma \int_{\theta_0}^{\theta_1} N d\theta = \Sigma M \int_{\theta_0}^{\theta_1} \{H_m \sin(\alpha - \theta) - A \sin 4\theta\} d\theta,$$

where the summation is to be extended to all the molecules n in the complex. Since M, α, θ are the same for all the molecules, the above equation may be written as

$$W = nM \int_{\theta_0}^{\theta_1} \{H_m \sin(\alpha - \theta) - A \sin 4\theta\} d\theta,$$

or

$$w_s = \frac{W}{nMA} = \int_{\theta_0}^{\theta_1} \{h_m \sin(\alpha - \theta) - \sin 4\theta\} d\theta, \qquad (1)$$

where w_s is the reduced hysteresis-loss by magnetization. The latter does not involve any quantity depending on the nature of a substance; it is therefore applicable for all substances belonging to the cubic system.

Now the reduced intensity of magnetization for a single complex is

$$i_s = \cos(\alpha - \theta);$$

for a small variation in the magnitude of h, we have

$$di_s = \sin(\alpha - \theta)d\theta.$$

Hence

$$w_s = \int_{I_0}^{I_1} h_m di - \int_{\theta_0}^{\theta_1} \sin 4\theta d\theta = \int_{I_0}^{I_1} h_m di + \int_0^{\theta_0} \sin 4\theta d\theta - \int_0^{\theta_1} \sin 4\theta d\theta.$$

But the last two terms are the integrals along the reversible courses of magnetization, in which case we have the relation

$$h \sin(\alpha - \theta) = \sin 4\theta,$$

$$\therefore \quad h \sin(\alpha - \theta)d\theta = h\, di_s = \sin 4\theta d\theta.$$

Moreover, the molecular magnets at θ_1, have the same potential energy with regard to the axis $\theta = 0$, or that perpendicular to it. Hence

$$\int_0^{\theta_1} \sin 4\theta d\theta = \int_{\pi/2}^{(\pi/2)-\theta_1} \sin 4\theta d\theta_1 = \int_{I_1'}^{I_1} ddi_s$$

and therefore we get finally

$$w_s = \int_{I_0'}^{I_1} h_m di_s + \int_{I_0'}^{I_0} hdi_s - \int_{I_1'}^{I_1} hdi_s. \tag{2}$$

Referring to the annexed figure (Fig. 16), in which $ABB'C$ is the curve of magnetization and $B'A'$ the course taken by the magnetization curve, when the field is reduced to zero, we see that the first integral represents the area $DBB'D'$ and the second the area ABD and the third the area $A'B'D'$, so that w_s is equal to the area $ABB'A'$.

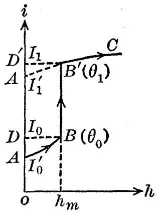

Fig. 16.

Next, consider the case of the mass of a ferromagnetic substance consisting of an immense number of minute complexes, whose magnetic axes are uniformly distributed in all directions. From the above result, we see that if I_1 and I_2 be the intensities of magnetization of a complex corresponding to the magnetizing and demagnetizing stages for the same strength of field, we have

$$w_s = \int_0^{h_m} (I_2 - I_1)dh = \int_0^{h} (I_2 - I_1)dh,$$

where h may take any value whatever, as w_s vanishes for larger values of h than h_m. Hence, for the hysteresis-loss w of a mass of the ferromagnetic substance, we must summarize the above expression for all the complexes constituting the substance.

$$\therefore \quad w = \Sigma w_s = \int_0^h \Sigma(I_2 - I_1)dh;$$

but ΣI_1 and ΣI_2 are respectively the reduced intensities of magnetization corresponding to the ascending and descending branches of the magnetization curve. Hence putting

$$\Sigma I_1 = i, \quad \text{and} \quad \Sigma I_2 = i_2,$$

$$w = \int_0^h (i_2 - i_1)dh. \qquad (3)$$

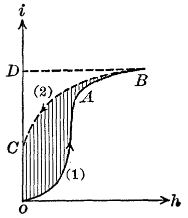

Fig. 17.

Referring to Fig. 17, w represents the area $OABC$ enclosed by the magnetizing and demagnetizing branches (1) and (2) of the magnetization curve. The area $OABD$ is known to be the total energy of magnetization, and therefore the area CBD, which is the difference between the areas OBD and OBC, corresponds to the net energy of magnetization.

In the same way, it can be shown that in a cyclic process of magnetization, as shown in Fig. 13, the hysteresis-loss during the magnetization CD is given by the area CDE and the loss during the magnetization EB by the area BCE, and that the total loss during the cyclic magnetization is equal to the area enclosed by the hysteresis-loop.

In our theory, we have assumed that if no external field acts on them, all the elementary magnets in each complex assume the same direction of stable equilibrium. But in actual cases, the elementary magnets on the bounding surfaces of the complexes may place themselves in quite different directions as do those in the interior through the action of the magnets in the neighboring complexes, and thus there results an irregular distribution of elementary magnets on the bounding surfaces of the complexes. Hence, some of the elementary magnets may initially be found in portions corresponding to $h = h_m$. If a weak magnetizing force acts on such magnets, it may cause the abrupt rotation of the magnets, and therefore the hysteresis phenomena result even in weak fields. The small hysteresis usually observable in portions OA, CB and DE in Fig. 13 are explained in this way.

1. We shall next calculate the value of the reduced hysteresis-loss for different magnetizing fields. Now

$$w = \frac{1}{\pi}\int_0^{\pi}\int_{\theta_0}^{\theta_1} \{h_m \sin(\alpha - \theta) - \sin 4\theta\} d\theta d\alpha$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[h_m\left\{\cos\left(n\frac{\pi}{2} - \alpha - \theta_1\right) - \cos 4\theta_0\right\}\right. \qquad (4)$$

$$+\frac{1}{4}\{\cos 4\theta_1 - \cos 4\theta_0\}\Bigg] d\alpha,$$

where $n = 1$ or 2 and h_m, α, θ_0, θ_1 are related by the equations of conditions:

$$\begin{aligned} h_m \sin(\alpha - \theta_0) &= \sin 4\theta_0, \\ h_m \sin\left(n\frac{\pi}{2} - \alpha - \theta_1\right) &= \sin 4\theta_1, \\ 5\sin(3\theta_0 + \alpha) &= 3\sin(5\theta_0 - \alpha). \end{aligned} \tag{5}$$

If we eliminate from these four equations α, θ_0, θ_1 the required relation between w and h_m will be obtained; it is, however, very difficult to find actually an analytical expression for w; but the problem can be solved graphically without any difficulty. Since w is the reduced hysteresis-loss applicable for all substances crystallizing in a cubic system, it is sufficient to find its value once for all in some convenient way; from this value, the actual hysteresis-loss for a given substance can be obtained simply by multiplying it by the product I_0A, depending on the properties of the substance.

The curve representing the relation between h_m and α is given in Fig. 11; hence if h_m be given, the corresponding value of α can be known. If from the first and third equation of condition, α be eliminated, the relation

$$\cos 4\theta_0 = \pm\frac{\sqrt{h_m^2 - 1}}{15}$$

is obtained, which gives θ_0 in terms of h_m. The double sign can be determined without ambiguity. Knowing h_m, α, n, θ_1 can be obtained from the second equation of condition (5). Thus, from the given value of h_m, all quantities under the integral sign in expression (4) can be evaluated.

Now, from the first and third equations of conditions, we get

$$\frac{d\alpha}{dh_m} = -\frac{1}{h_m}\tan(\alpha - \theta_0);$$

hence for w, we obtain the expression

$$w = \frac{1}{\pi}\int_0^{\pi} \varphi\, d\alpha = \frac{1}{\pi}\int_0^{h_m} \varphi\frac{d\alpha}{dh_m} dh_m \tag{6}$$

where

$$\varphi = h_m\left\{\cos\left(n\frac{\pi}{2} - \alpha - \theta_1\right) - \cos(\alpha - \theta_0)\right\} + \frac{1}{4}\{\cos 4\theta_1 - \cos 4\theta_0\};$$

φ can be graphically evaluated, provided h_m is given. Then draw the curve and evaluate the area bounded by the curve and the abscissa;

we thus obtain w in terms of h_m. The following table contains our result of calculation:

h_m.	$\phi \frac{da}{dh_m}$.	h_m.	$\phi \frac{da}{dh_m}$.	h_m.	$\phi \frac{da}{dh_m}$.	h_m.	$\phi \frac{da}{dh_m}$.
1.1	3.143	2.0	1.081	3.0	0.512	4.0	0.000
1.5	1.743	2.5	0.735	3.5	0.310		

$\varphi(d\alpha/dh_m)$, h_m curve is given in Fig. 18; and the result of its mechanical integration is given in the following table and in Fig. 19.

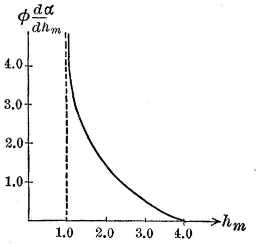

Fig. 18.

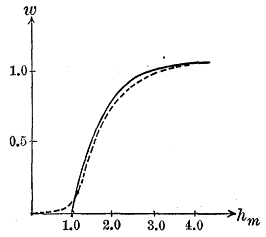

Fig. 19.

h_m.	w.	h_m.	w.	h_m.	w.	h_m.	w.
1.0	0.000	1.5	0.472	2.5	0.886	3.5	0.997
1.1	0.075	2.0	0.721	3.0	0.975	4.0	1.024

Similarly the hysteresis-loss by magnetization in the opposite direction, of a substance which has previously been magnetized in one direction, can be calculated. The results for $\varphi(d\alpha/dh_m)$ and w are given in the following table and in Figs. 20 and 21.

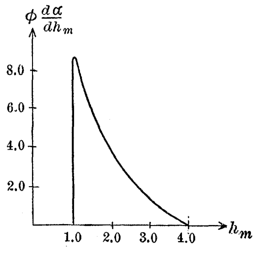

Fig. 20.

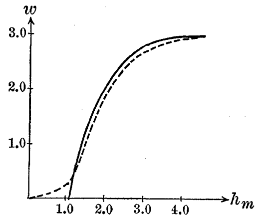

Fig. 21.

h_m.	w.	h_m.	w.	h_m.	w.	h_m.	w.
1.0	0.000	1.5	0.998	2.5	2.445	3.5	2.865
1.3	0.665	2.0	1.945	3.0	2.820	4.0	2.995

Here the initial increase of $\varphi(d\alpha/dh_m)$ and w is comparatively less abrupt than in the former case. The double value $2w$ is equal to the loss during a cyclic process of magnetization. The dotted curves in Figs. 19 and 21 are the supposed ones, in which an irregular distribution of the molecular magnets on the bounding surfaces of different complexes already referred to is taken into consideration. The dotted curve in Fig. 21 resembles in its character with the curve given by the Steinmetz formula, that is,

$$w = \eta B^{1.6},$$

where η is a constant depending on the nature of a substance and B the magnetic induction.

§ 7. Effect of Temperature on Magnetization.

In the above theory, we have taken no account of the thermal motion of the molecules, and therefore the results so far obtained hold good only in the absolute zero, where no thermal agitations exist. In this paragraph, we shall consider the effect of temperature on magnetization, the established facts of which may be summarized in the following words: In a very weak field, the magnetization increases with the rise of temperature, at first slowly and then very rapidly, and after reaching a sharp maximum, it falls very rapidly at the critical temperature. With the increase of magnetizing field, this effect of increasing magnetization becomes continuously less. In a field of several gausses, the magnetization remains constant up to the critical range, and then falls very rapidly. With further increase of field, the magnetization begins gradually to decrease from a temperature which is lower as the field is stronger. Above a field of some hundreds of gausses, the magnetization begins gradually to decrease from room temperature.

It is commonly admitted that the diminution of magnetization at high temperatures is due to the rotational vibration of molecules, the amount of diminution increasing with the amplitude of vibration, and that when the rotational vibration is changed into a continuous revolution, magnetization completely disappears. Such an explanation assumes no change either in the molecules or in their mutual configuration; what is assumed is simply the change of the amplitude of the rotational vibration during the heating. It is however questionable whether this is sufficient to explain the so-called magnetic or A_2 transformation.[1] We shall at first show that simple revolution of molecules about their own centers are not sufficient to account for the disappearance of magnetism at the critical point.

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[1] K. Honda, Sci. Rep., 4 (1915), 169.

Consider the case when the external field is very strong and the mutual action between the molecular magnets can be neglected. All the molecular magnets are then directed nearly in the direction of the field. Owing to their thermal energy, they make translational and rotational vibrations about their mean positions. If 2β be the complete amplitude of the rotational vibration of a molecule, its equation of rotational motion will be

$$K\frac{d^2\beta}{dt^2} = -2rH\sin\beta,$$

or

$$\frac{d^2\beta}{dt^2} = -n^2\sin\beta, \quad n^2 = \frac{2Hr}{K}, \tag{1}$$

where K is the moment of inertia of the molecule about the center of mass, H the external field and $2r$ the pole distance of the molecule.

Suppose at first $\beta < \pi$ and integrate the above equation; we get

$$\frac{d\beta}{dt} = \pm 2n\sqrt{\sin^2\frac{1}{2}\beta_0 - \sin^2\frac{1}{2}\beta},$$

where β_0 is the maximum amplitude of the vibration.

Putting

$$\sin \tfrac{1}{2}\beta = \sin \tfrac{1}{2}\beta_0 \sin\varphi,$$

and changing the variable from β to φ, we get

$$nt = \int_0^{\phi}\frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}} = F(\varphi, k), \tag{2}$$

where $k = \sin\tfrac{1}{2}\beta_0$ and $F(\varphi, k)$ is the elliptic integral of the first kind. Hence, if T be the period of oscillation,

$$n\frac{T}{4} = F\left(\frac{\pi}{2}, k\right) = K(k)$$

or

$$T = \frac{4K(k)}{n}.$$

Now

$$\cos\tfrac{1}{2}\beta = dn\cdot nt;$$

$$\therefore \quad \cos\beta = 2\cos^2\tfrac{1}{2}\beta - 1 = 2dn^2nt - 1.$$

Hence if I_m and I be the intensity of magnetization as affected by the thermal motion and that at absolute zero respectively, we have

$$I_m = \frac{1}{T}\int_0^T I\cos\beta dt$$

$$= \frac{4I}{T}\int_0^{T/4} \cos \beta dt = \frac{In}{K(k)}\int_0^{K(k)/n} (2dn^2nt - 1)dt$$

$$= \frac{I}{K(k)}\{2E(amK, k) - K(k)\},$$

where E is the elliptic integral of the second kind.

Hence

$$\frac{I_m}{I} = \frac{2E(amK, k)}{K(k)} - 1 = \frac{2E(k)}{K(k} - 1. \tag{3}$$

The calculation of the ratio presents no difficulty. In the following table and Fig. 22, the values of I_m/I for different values of β_0 are given:

β_0.	κ.	I_m/I.	β_0.	κ.	I_m/I.
0°	0.000	1.000	100°	0.766	0.352
20°	0.174	0.970	120°	0.866	0.126
40°	0.342	0 882	140°	0.940	– 0.108
60°	0.500	0.742	160°	0.985	– 0.340
80°	0.643	0 562	180°	1.000	– 1.000

Next, suppose $\beta > \pi$; then the vibration changes into the revolution, but its angular velocity is not uniform. As before, we have

$$\frac{d^2\beta}{dt^2} = - n^2 \sin \beta$$

or

$$\frac{1}{2}\left(\frac{d\beta}{dt}\right)^2 = C + n^2 \cos \beta.$$

If for $\beta = 0$,

$$\frac{1}{2}\left(\frac{d\beta}{dt}\right)^2 = 2n^2v^2, \quad \text{where} \quad v^2 > 1,$$

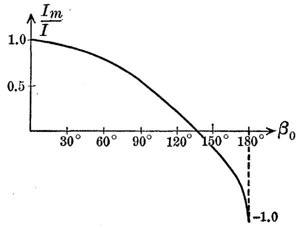

Fig. 22.

then

$$\frac{1}{2}\left(\frac{d\beta}{dt}\right)^2 = 2n^2(v^2 - \sin^2 1 \beta);$$

putting $\beta = 2\varphi$ and $1/v^2 = k$, we have

$$\left(\frac{d\varphi}{dt}\right)^2 = \frac{n^2}{k^2}(1 - k^2 \sin^2 \varphi),$$

or

$$\frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = \frac{n}{k} dt;$$

$$\therefore \quad \frac{n}{k} t = \int_0^{\phi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}. \tag{4}$$

For $t = 0$ and T, let $\varphi = 0$ and π respectively, we have

$$\frac{n}{k}T = \int_0^{\pi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$$

or

$$= 2\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = 2K(k);$$

$$\therefore \quad T = \frac{2k}{n} K(k).$$

Now

$$\tfrac{1}{2}\beta = \varphi = am\frac{n}{k}t:$$

$$\therefore \quad \cos\beta = 2\cos^2 \tfrac{1}{2}\beta - 1 = 2Cn^2\frac{n}{k}\iota - 1;$$

$$\therefore \quad I_m = \frac{1}{T}\int_0^{T} I\cos\beta\, dt = \frac{In}{2kK(k)}\int_0^{(2k/n)K(k)} \left(2Cn^2\frac{n}{k}t - 1\right) dt.$$

If we put $x = (n/k)t$, $dx = (n/k)dt$.

If $t = 0$, then $x = 0$; if $t = (2k/n)K(k)$, then $x = 2K(k)$

$$\therefore \quad \frac{I_m}{I} = \frac{1}{2K(k)}\int_0^{2K(k)} (2Cn^2 x - 1)dx$$

$$= \frac{1}{2K(k)}\left\{\frac{2}{k^2}[E(am\cdot 2K(k)\cdot k) - k'^2\cdot 2K(k)] - 2K(k)\right\}$$

$$= \frac{E(am\cdot 2K(k)\cdot k)}{k^2K(k)} - \left(2\frac{k'^2}{k^2} + 1\right),$$

where k' is the modulus complementary to k; but

$$am\cdot 2K(k) = \pi;$$

$$\therefore \quad \frac{I_m}{I} = \frac{2}{k^2}\frac{E(k)}{K(k)} - \left(2\frac{k'^2}{k^2} + 1\right). \qquad (5)$$

Since $k = 1/v^2$ and v may take any value from 1 to ∞, k^2 can vary from 1 to 0. It is evident that so long as the angular velocity of the molecules at $\beta = 0$ is not infinitely large, this velocity is not uniform, so that I_m does not theoretically vanish unless $v^2 = \infty$. This result is also evident from the above relation.

In the following table, the values of I_m/I corresponding to the different values of k are given:

κ.	v^2.	I_m/I.	κ.	v^2.	I_m/I.
1.000	1.000	− 1.000	0.643	2.410	− 0.090
0.985	1.030	− 0.385	0.500	4.000	− 0.030
0.940	1.130	− 0.252	0.342	8.550	− 0.027
0.866	1.1335	− 0.168	0.174	33.450	− 0.020
0.776	1.690	− 0.122	0 000	∞	− 0.000

The relation between I_m/I and v^2 is also shown in the following figure. As β or v^2 increases from 0, the magnetization diminishes at first slowly and then somewhat rapidly; in passing through $\beta_0 \doteqdot 131°$, it vanishes and changes its sign. With a further increase of β_0 or v^2, the magnetization increases negatively and at $\beta_0 = \pi$, I_m/I becomes -1. Afterwards, the magnetization rapidly decreases in absolute value, tending asymptotically to the value zero, as v^2 approaches to ∞.

Now we find experimentally no evidence that the magnetization becomes negative at high temperatures, though the field is very strong. What is then the cause of the discrepancy between the theory and the experiments? The cause is obviously to be sought for the fact that in the above theory, we have assumed no transformation either in the molecules or in their mutual configuration. It is certainly true that the above effect plays a part in changing the magnetization at high temperatures. Probably in a value of the amplitude β_0, which is far less han 131°, a gradual A_2 transformation will begin to proceed in the substance, and consequently the substance is changed into the paramagnetic state as conceived by P. Langevin.[1]

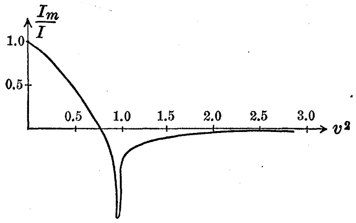

Fig. 23.

A few years ago, one of the present writers published a theory of magnetism,[2] which is based on the Langevin theory of paramagnetic gases; the theory connects the ferromagnetic and paramagnetic substances and coincides with the Ewing theory for the former substance. It may be summarized in the following words: The form of the molecules of a ferromagnetic sybstance is nearly spherical and consequently the effect of thermal impacts in rotating the molecules is very small in comparison with the mutual action; while in the case of paramagnetic substance, the molecules have an elongated or flattened form, so that here the effect of mutual action is very small compared with the rotating

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[1] P. Langevin, Ann. de chem. et phys. (8), 5, (1905), 70.

[2] K. Honda, Sci. Rep., 3 (1914), 171.

effect of thermal impact. The transformation of a ferromagnetic substance to a paramagnetic at high temperatures is by this theory explained as a consequence of the gradual deformation of the spherical molecules with the rise of temperature. The heat evolved or absorbed during this transformation is considered to be the energy of transformation and that imparted to the molecules to cause their rotational vibrations.

The above theory accords with the result of the present investigation. On the other hand, the simple theory of the revolution of molecules is not solely sufficient to account for the disappearance of magnetism at high temperatures.

Next we shall consider the effect of temperature on magnetization in the light of our theory of molecular magnetism. In weak fields, the temperature affects the magnetization in two opposite ways; that is, the first effect, which exists in all fields, is to diminish the magnetization on account of the rotational vibrations of the molecules, and the second, which is noticeable only in weak fields, is to increase the magnetization by virtue of the abrupt turning of the molecules towards the field due to heat motion. The observed change of magnetization at high temperatures is the sum of these two effects. We shall firstly consider the first effect from the standpoint of our theory of magnetism.

If the thermal agitation be zero, molecular magnets in each complex will take a common direction determined by the external and internal fields. Suppose this direction to make an angle θ_0 with the field. In virtue of the thermal energy, they will in an actual case execute translational and rotational vibrations about their mean positions. The amplitude of their rotational vibrations will actually differ from one magnet to another; but as the first approximation, we may consider their mean value to be β_0. Since, in each complex, the molecules exert their mutual action on each other, the rotational vibration of molecules with the same phase takes place more easily than in the case of those with arbitrary phases. Hence in a stationary state, we may, as the first approximation, suppose that all the magnets in each elementary complex oscillate with a common phase, but that the phase of the oscillation differs from one complex to another.

Consider at first the case, where the external field is very small as compared with the internal; neglecting the couple due to the former field, the equation of motion becomes

$$K\frac{d^2\beta}{dt^2} = -2Ar\sin 4(\theta_0 + \beta). \qquad (6)$$

As θ_0 is very small in weak fields, we may neglect it in comparison with

β. Hence, putting $n^2 = (2Ar/K)$, we get

$$\frac{d^2\beta}{dt^2} = -n^2 \sin 4\beta.$$

Now, let $\sin 2\beta = \sin 2\beta_0 \cdot \sin \varphi = k \sin \varphi$, and change the variable from β to φ, we get, after integration,

$$nt = \int_0^{\phi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = F(\varphi, k).$$

Let for $t = 0$, $\beta = 0$ $\therefore$ $\varphi = 0$; for $t = I/4$, $\beta = \beta_0$ $\therefore$ $\varphi = \pi/2$. Hence

$$n\frac{I}{4} = F\left(\frac{\pi}{2}, k\right) = K(k)$$

or

$$T = \frac{4K(k)}{n}.$$

Now

$$\sin \varphi = sn \cdot nt,$$

$$\sin 2\beta = \sin 2\beta_0 \sin \varphi = k \cdot dn\ nt;$$

$$\therefore \quad \cos 2\beta = dn \cdot nt.$$

Hence the mean effect of a molecule making initially an angle α with the field in the direction of the latter will be given by

$$M_m = \frac{1}{T}\int_0^T M \cos(\alpha - \beta)dt$$

$$= \frac{Mn}{4K(k)}\left\{ \cos\alpha \int_0^T \cos\beta dt + \sin\alpha \int_0^T \sin\beta dt \right\},$$

where M_m and M are the magnetic moment of a molecule as affected by the thermal motion and that at absolute zero respectively. But,

$$\cos\beta = \sqrt{\frac{1 + \cos 2\beta}{2}} = \sqrt{\frac{1 + dn\ nt}{2}},$$

$$\sin\beta = \sqrt{\frac{1 - \cos 2\beta}{2}} = \sqrt{\frac{1 - dn\ nt}{2}}.$$

Since $dn\ nt$ is an even function and its period $2K$, we have, putting $x = nt$

$$M_m = \frac{M\cos\alpha}{\sqrt{2}K(x)}\int_0^K \sqrt{1 + dnx}\, dx$$

or

$$\frac{M_m}{M\cos\alpha} = \frac{\sqrt{2}}{\sqrt{1 + k'}} \frac{K(G(k/(1 + k')))}{K(k)}. \tag{7}$$

$M \cos \alpha$ is the magnetic moment in the direction of the field. Thus the ratio $M_m/M \cos \alpha$ for each molecule is a constant depending on β_0. Hence if I_m and I represent the intensities of magnetization with and without the thermal motions respectively, we have

$$\frac{I_m}{I} = \frac{\sqrt{1 + k'}}{\sqrt{2}} \frac{K(k/(1 + k'))}{K(k)} . \tag{8}$$

The ratio gradually decreases with the increase of β_0 or of k; for $\beta_0 = \pi/4$, it becomes $1/\sqrt{2} = 0.7071 \cdots$. If β_0 increases beyond $\pi/4$, the vibration changes into a revolution and the mean effect of magnetization vanishes; because in the present case, the external field is neglected and the motion governed by the internal resisting force $A \sin 4\theta$ with a period of $\pi/4$.

In the following table and in Fig. 24, the values of the ratio for different values of β_0 are given to show how the magnetization diminishes with increasing β_0.

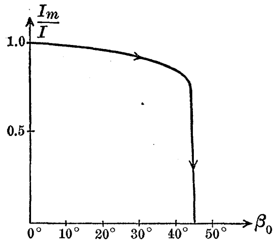


Fig. 24.

β_0.	I_m/I.	β_0.	I_m/I.	β_0.	I_m/I.	β_0.	I_m/I.
0°	1.000	15°	0.966	30°	0.920	45°	0.707
5°	0.992	20°	0.955	35°	0.895		
10°	0.985	25°	0.938	40°	0.861		

Thus the ratio gradually diminishes with increasing β_0 up to $\beta_0 = \pi/4$, where it suddenly vanishes. As we have already remarked, the diminution of magnetization with the increase of β would also be accelerated by the A_2 transformation, so that the fall of the curve with increasing β must actually take place at a smaller value of β than $\pi/4$.

We shall next consider the second effect of temperature, which increases the magnetization in weak fields. If the thermal motion be absent, that is, at the absolute zero, the orientation of the equilibrium of a complex, whose magnetic axis making initially an angle α with the direction of the field, is given by

$$h \sin (\alpha - \theta_0) = \sin 4\theta_0;$$

hence if h be given, the relation between α and θ_0 can easily be found by the graphical method. If for a complex (α), $\theta_0 + \beta_0 > \pi/4$, then the complex will oscillate about its mean orientation θ_0; on the other hand, if $\theta_0 + \beta_0 > \pi/4$, the complex will undergo an abrupt turning and take a position, as if the initial orientation were $\alpha - (\pi/2)$, causing thereby an increase of magnetization. Hence, even in weak fields, where at absolute zero, there is no complex which abruptly turns in the direction of the field, the complexes will more and more begin to make an abrupt turning with the rise of temperature.

If there is no thermal motion, the reduced intensity of magnetization is given by

$$i = \frac{1}{\pi} \int_0^{\pi} \cos (\alpha - \theta_0) d\alpha,$$

where α and θ_0 are connected with each other by the foregoing relation. This relation for $h = 0.5$ is shown graphically in Fig. 25. If β_0 be given, we can find from the above figure the limits or the range of α, for which the complexes make an abrupt turning toward the direction of field. Let α_1 and α_2 be such limits, then i is given by

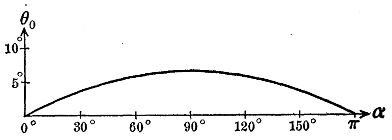


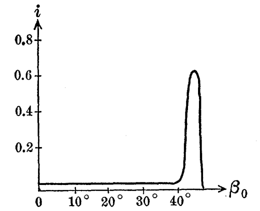
Fig. 25.

$$i = \frac{1}{\pi} \left\{ \int_0^{\alpha_1} \cos (\alpha - \theta) d\alpha + \int_{\alpha_1-(\pi/2)}^{\alpha_2-(\pi/2)} \cos (\alpha - \theta) d\alpha + \int_{\alpha_2}^{\pi} \cos (\alpha - \theta) d\alpha \right\},$$

It may also occur that some complexes, whose direction of magnetic axis lies between α_1 and α_2, make the abrupt rotations twice or thrice; in such cases, we must take for the limits $\alpha_1 - \pi$ and $\alpha_2 - \pi$ or $\alpha_1 - (3/2)\pi$ and $\alpha_2 - (3\pi/2)$, etc. In this way, under a given field, the value of i corresponding to different values of β_0 can be calculated. If we multiply these values of i by the ratio

$$\frac{I_m}{I} = \frac{\sqrt{2}}{\sqrt{1 + k'}} \frac{K(k/(1 + k'))}{K(k)},$$

which represents the mean effect of rotatory vibrations, the resultant intensity of magnetization will be those as affected by temperature.



F 26. ig.

Fig. 26 shows the result of our calculation for $h = 0.5$; the ordinate represents the magnetization in question and the abscissa the angle β_0. The temperature is obviously some function of β_0 increasing with it. If we consider $\beta = \pi/4$ to correspond to the critical point, the course of the curve is quite similar to that obtained by J. Hopkinson for a very weak field.

If h gets greater, the increased number of complexes turns abruptly towards the field, even if there is no thermal motion; and consequently the increase of magnetization due to the thermal vibration becomes always less. In a sufficiently strong field, where all the complexes have finished their possible abrupt turning, the effect of temperature in increasing magnetization must vanish, and there exists only the effect of diminishing magnetization due to rotational vibrations. Thus the effect of temperature on magnetization is explained by our theory, at least qualitatively.

In the above calculation, the A_2 transformation was not taken into account. This transformation obviously affects in reducing the magnetization at high temperatures.

The theory so far explained strongly confirms the general view that *the magnetic phenomena are really due to the rotation of the molecules about their own centers*. This fact has an important bearing to the molecular structure of ferromagnetic crystals, the discussion of which will be given in a next paper to be published shortly.

§ 8. MOLECULAR FIELD INTRODUCED BY PROF. P. WEISS.

Lastly the molecular field introduced by Prof. P. Weiss[1] will be considered in the light of the present investigations. According to him, it is a uniform field acting on each molecule of a ferromagnetic substance, its magnitude being assumed to be proportional to the intensity of magnetization and having an enormous value amounting to several ten millions of gauss. This molecular field was introduced by Weiss to extend Langevin's theory of paramagnetism to the ferromagnetic substances; one of the present writers[2] has however shown that the same extension can be made quite naturally by considering the molecules of the ferromagnetic substances to be nearly spherical in form. The introduction of the molecular field into the theory of magnetism meets with great difficulties; namely his theory cannot explain very fundamental and important facts in the theory of magnetism, such as the curve of magnetization and hysteresis phenomenon.

The evidence, which Weiss sets forth as proof of his theory, is:

(i) The explanation of the magnetic properties of magnetite and pyrrhotine by means of the demagnetizing field.

(ii) The existence of the corresponding magnetic states in ferromagnetic substances.

(iii) The applicability of the relation

$$x(T - \theta) = \text{const},$$

where x is the specific susceptibility at a temperature T higher than the critical temperature θ.

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[1] Conférrence à la Soc. francai. de Phys., April 4 (1907). Arch. des Sci., No. 5, 31 (1911), 401.

[2] K. Honda, Sci. Rep., 3 (1914), 171.

(iv) The change of specific heat in the critical range of iron, nickel and magnetite.

In explaining the magnetic properties of crystals, P. Weiss was led to assume a uniform demagnetizing field of considerable magnitude. In addition, with some improbable assumptions, he explained the complicated magnetic properties of crystals; but we have shown in a paper,[1] that these properties can be very simply explained without assuming any demagnetizing field.[1] Hence (1) can not be considered to support his theory.

Secondly he obtained from his theory the relations:

$$\frac{T}{\theta} = \frac{3}{a}\frac{I}{I_0} \quad \text{and} \quad \frac{I}{I_0} = \coth a - \frac{1}{a},$$

where T and θ have the same meaning as before and a is a quantity depending on the nature of the substance. If we eliminate a from these equations, we obtain a relation giving a dependence of I/I_0 on I/θ; this relation is independent of the nature of the substance and therefore called *the relations for the corresponding states*. This consequence affords a means of verifying his theory. Weiss showed that this relation holds good in high temperatures for magnetite and ferronickel, but in low temperature the deviation between the theory and the result of the experiment is considerably great. He also remarked that for iron and nickel the agreement is only qualitative.

We have also examined the above relation for iron, nickel and nickel steels of different compositions. For this purpose, it is necessary to find the saturation value of magnetization at the absolute zero from the observed values at low temperatures. We have here two methods: Firstly, if we assume the above relations to hold good at least from the observed lowest temperature to the absolute zero, we can find the value of I_0 from the known values of I, T and θ. Secondly, we may also find I_0 by extrapolation from I, T curve actually observed. These two methods do not give the same result. We found therefore two values of I_0 and calculated two sets of values of I/I_0 and T/θ for each specimen. In our calculation, we availed ourselves of the results of experiments made by Mr. S. Shimizu and one of the present writers[2] for Swedish iron, nickel and nickel steels of 30, 36, 48, 50 and 60 per cent. of nickel. The experiment was made at different temperatures ranging from liquid air temperature to those above their critical points, and under constant fields up to 700 gauss. For these specimens, the magnetization at liquid

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[1] Sci. Rep., 5 (1916), 153.

[2] Jour. Coll. Sci., 20, Art. 6 (1904).

air temperature nearly attained its saturation value in the highest field just referred to. The results are graphically shown in Figs. 27, 28, 29, 30. The broken curve in each figure represents the theoretical one, while other curves are the observed results. From these figures, we conclude that the relation for the corresponding state is here only qualitatively satisfied. Hence we can not regard the above relation as a confirmation of Weiss's theory.

Thirdly, Weiss obtained from his theory a relation

$$x(T - \theta) = \text{const.}$$

One[1] of the present writers made however a thorough investigation of this subject, and showed that the relation is approximately true for iron, nickel and cobalt and fails to be applicable in the case of magnetite. He also showed that this relation can be obtained as a special case from his theory, which does not take any account of the molecular field.

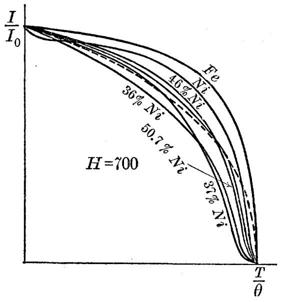

Fig. 27.

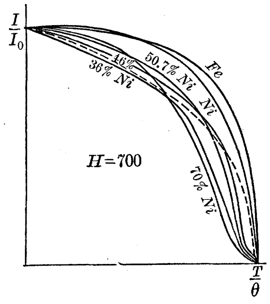

Fig. 28.

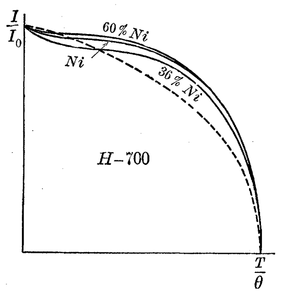

Fig. 29.

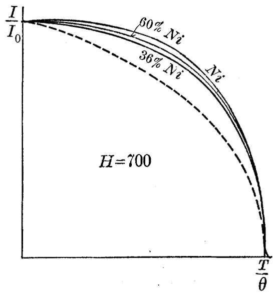

Fig. 30.

Hence as evidence for the existence of the molecular field, the above relation has a little importance.

Lastly the change of specific heat at critical range of ferromagnetic substances will be considered. It was shown by P. Weiss and P. N.

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[1] K. Honda, Sci. Rep., 3, l. c.; Sci. Rep., 4 (1915), 248.

Beck[1] that the specific heat of iron, nickel and magnetite considerably increases in the critical range. As however these metals evolve heat by cooling through the critical range, what they measured is not properly termed the change of specific heat by temperature, but the quantity of heat evolved during the transformation[2] as measured calorimetrically. This heat evolution was early measured by Pionchon,[3] Standfield,[4] and recently by Meuten.[5] Weiss explained the heat evolved or absorbed during the transformation as due to magnetic energy. Thus he calculated on one hand the change of magnetic energy per degree at different high temperatures, using Curie's result on the magnetic measurement at high temperatures, and on the other hand, in coöperation with P. N. Beck, he measured calorimetrically the heat evolution at high temperatures up to the critical point. In this way, the change of magnetic energy δc_m per degree and that of the specific heat δc were compared with each other for iron, nickel and magnetite; the results of his calculation are given in the following table:

Substance.	θ.	N.[6]	I.	δC.	δC_m.
Fe	753° C.	3,840	1,700	0.112	0.136
Ni	376°	12,700	500	0.027	0.025
Fe_3O_4	588°	33,200	430	0.050	0.048

The agreement between δc and δc_m is apparently as good as we can desire. But it should be remarked that the thermomagnetic properties of the ferromagnetic substances, and therefore the values of N, vary for different specimens of the same metal, as the following table shows:

Substance.	N (Curie).	N (Honda, Takagi).
Fe	3,840	5,910[7]
Ni	12,700	10,730[8]
Fe_3O_4	33,200	37,200–10,600

For magnetite, the quantity $x(T - \theta)$ is far from being constant,[9] so that N varies considerably with temperature. If we use the values of

[1] Jour. de Physique, 7 (1908), 249.

[2] K. Honda, Sci. Rep., 4 (1915), 169.

[3] Ann. Chim. Phys., 6th series, II. (1887), 33.

[4] Ferrum, 1 (1912), 1.

[5] Jour. Iron and Steel Inst., No. 2 (1899), 169.

[6] N = coefficient of molecular field. The change of magnetic energy per degree =

$$\frac{1}{2}\frac{d}{dt}(NI^2) = NI\frac{dI}{dt}.$$

[7] Sci. Rep., 4 (1915), 261.

[8] Sci. Rep., 1 (1912), 229.

[9] Curie, Oeuvres (1908), 322; H. Takagi, Sci. Rep., 2 (1913), 117; P. Weiss and G. Foex, Arch. des Sci., 31 (1911), 89.

N given in the above table for the calculation of δc_m, the deviation between the theory and the experiment becomes considerable. In the calculation of Weiss, the data for magnetic and calorimetric measurements belong to different specimens, and therefore the coincidence in his case may be accidental. It is, however, a remarkable fact that the two quantities δc_m and δc, which are obtained from the quantities of quite a different nature, coincide with each other at least in the order of magnitude, and therefore this instance may be regarded as the most favorable case put forward by Weiss. But the heat evolution or absorption in the critical range can also be explained by another theory, as was actually done by one of the present writers, and again, as shown in the present theory, the principal features of magnetic phenomena, that is, the magnetization curve, the hysteresis phenomena, and the temperature effect on magnetization are satisfactorily explained without assuming Weiss's molecular field; hence the necessity for assuming the molecular field will not only disappear, but the difficulties involved in assuming it remain undiminished as before. Hence, it seems to us that the existence of the molecular field put forward by Weiss is not consistent with the observed facts.

SENDAI, JAPAN,
1917.