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ON A GENERAL EXPANSION THEOREM FOR THE TRANSIENT OSCILLATIONS OF A CONNECTED SYSTEM.

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IN the usual solution of the problem of the transient oscillations of a connected mechanical or electrical system in response to a suddenly impressed set of forces, the determination of the characteristic modes of oscillation (periodicities and dampings) is comparatively easy, since it involves only the determination of the roots of a polynomial. As regards the amplitudes of the transient oscillations the case is different. The usual procedure is to designate the amplitude of each mode of oscillation of each coördinate of the system by an undetermined constant, substitute in the equations which describe the system, and then determine the unknown constants in accordance with the given initial configuration of the sysem. This method of determination, while perfectly straightforward, is extremely laborious, and the difficulty increases rapidly with the number of degrees of freedom of the system. When the initial configuration is arbitrary no other method than that outlined above is known to the writer; when, however, a set of forces is impressed on a system at rest or in equilibrium configuration the amplitudes of the transient oscillations admit of much simpler determination by the expansion theorem developed in this paper.

So far as the writer is aware no one, with the exception of Heaviside, has attacked the problem of a general formulation of the transient oscillation as regards their amplitudes as well as periodicities. Heaviside in his Expansion Theorem[1] gave a very valuable formulation of the transient oscillation of an electrical network when the oscillations are excited by the sudden application to the system of an electromotive force which is not a function of time; that is a steady uniform electromotive force.

[1] See Heaviside, Electromagnetic Theory, Vol. II., p. 127.

In the present paper the general solution of the problem considered by Heaviside will be developed and an Expansion Theorem derived which formulates the resultant (forced and transient oscillations) of a connected dynamical system in response to an arbitrary driving force applied to any coördinate of the system. The only limitation imposed on the form of the driving force is that it shall be capable of expansion in a Fourier's integral or else expressible as a complex exponential function of time, a limitation which constitutes no restriction of practical importance. The Expansion Theorem to be derived is thus of broader application than Heaviside's Theorem which constitutes a particular case of the general theorem;[1] furthermore its derivation may be of interest since Heaviside states his theorem without proof.[2]

The dynamical system to be considered may be either a mechanical or electrical system characterized by a symmetrical system of linear differential equations of the following form:[3]

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= F_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= F_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots \qquad\qquad + a_{nn}x_n &= F_n, \end{aligned} \tag{1}$$

In equations (1) $x_1, x_2, \cdots x_n$ are the displacements from equilibrium or zero configuration of the n coördinates $X_1, X_2, \cdots X_n$ specifying the system while $F_1, F_2, \cdots F_n$ are impressed forces. The general coefficient a_{jk} is of the form

$$a_{jk} = q_{jk}\frac{d^2}{dt^2} + r_{jk}\frac{d}{dt} + s_{jk}, \tag{2}$$

where q_{jk}, r_{jk} and s_{jk} are constants. The q coefficients will be termed the inertia factor, the r coefficient the resistance factor and the s coefficient the stiffness factor. From a mathematical standpoint no limit is placed on the order of a_{jk} in (d/dt); it is only necessary that it be capable of expression as a polynomial of the nth order in (d/dt). However in cases of practical importance the polynomial a_{jk} is of the second order in d/dt as indicated by equation (2).

To simplify the following work it will be assumed that only one driving

[1] In El. Th., p. 131–2, Vol. II., the case of simply periodic forces is treated by the operationa method.

[2] Since the above was written Mr. H. W. Nichols has called my attention to the fact that Heaviside derives his Expansion Theorem in his Electrical Papers, Vol. II., p. 373. Mr. H. W. Malcolm in a series of papers entitled "The Theory of the Submarine Telegraph" appearing in the Electrician during 1912 attempts to prove the Heaviside Theorem. The method of derivation is, however, quite defective.

[3] See Webster's Dynamics, 2d edition, pp. 173, 174.

force F_1 is impressed and consequently F_2, F_3, $\cdots F_n$ are put equal to zero. It may be readily shown that this simplification involves no loss of generality whatsoever, since the complete solution may be built up at once from the formulæ to be derived.

The driving force is assumed of the form[1]

$$F_1 = \tfrac{1}{2}\{E_1\epsilon^{pt} + \overline{E}_1\epsilon^{\bar{p}t}\} \tag{3}$$

$$= R\{E_1\epsilon^{pt}\} \tag{4}$$

where E and p are constants. In formula (3) the bar denotes the conjugate imaginary of the unbarred symbol, while in formula (4) R indicates that the real part of the expression alone is to be retained. For convenience the symbol R will be omitted and it will be understood that the real part of the final expression is the solution.

The forced oscillations of the system are gotten by the well-known method[2] of replacing d/dt by p in (1). If y_1, y_2, $\cdots y_n$ denote the forced components of x_1, x_2, $\cdots x_n$, then:

$$y_k = E_1\frac{M_{1k}(p)}{D(p)}\epsilon^{pt}. \tag{5}$$

In formula (5), $D(p)$ is the value of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & \cdots & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \tag{6}$$

when the operator d/dt is replaced by p. $M_{1k}(p)$ is the minor of the first row and Kth column of $D(p)$. The solution for the forced oscillations is of course well known.

The complementary solution of equation (1) gives the transient oscillations. If z_k denotes the transient component of x_k, then z_k is expressible as:

$$z_k = \sum_{m=1} A_k^{(m)} \cdot \epsilon^{p_m t} \tag{7}$$

where p_m is a root of the equation $D(p) = 0$ and $A_k^{(m)}$ is an integration constant to be determined by the connections of the system and the initial configuration at time $t = 0$. The summation is extended over the roots of $D(p) = 0$.

[1] When the driving force is the arbitrary time function $f(t)$ it can of course be expressed as a Fourier Integral or Series, each of whose components is of the form given by (4) when p is a pure imaginary. The explicit treatment of this case is reserved for a future paper.

[2] *Loc. cit.*

The solution is then:

$$x_k = y_k + z_k = E_1 \frac{M_{1k}(p)}{D(p)} \epsilon^{pt} + \sum_{m=1}^{m=n} A_k^{(m)} \cdot \epsilon^{p_m t}. \tag{8}$$

In general the conditions to be satisfied by the solution are as follows:

1. The initial displacement of every coördinate shall be zero; that is $x_k = 0$ at $t = 0$ for all values of k.

2. The initial velocity of every coördinate shall be zero; that is

$$\frac{dx_k}{dt} = \dot{x}_k = 0$$

at $t = 0$ for all values of k.

3. The ratio of z_j to z_k for the mth mode of oscillation shall be equal $M_{1j}(p_m)/M_{1k}(p_m)$. This last condition may be readily seen to be necessary by substitution of (7) in (1), and is perfectly general. The first two conditions follow from the fact that the initial configuration is one of equilibrium. Certain particular cases when these conditions do not hold are examined below.

We shall now proceed to a determination of the integration constants of (7). The initial value ($t = 0$) of y_k is by (5):

$$(y_k)_{t=0} = E_1 \frac{M_{1k}(p)}{D(p)}. \tag{9}$$

Now from equation (2) and formula (6) $D(p)$ is in general a polynomial of the $2n$th order in p while $M_{1k}(p)$ is a polynomial of the $(2n - 2)$ order in p. The right-hand side of (9) may be expanded by means of the following theorem:[1]

If $Q_{(x)}$ and $P_{(x)}$ are polynomials in x and if $P_{(x)}$ is of higher order than $Q_{(x)}$, then:

$$\frac{Q(x)}{P(x)} = \sum_{m=1}^{m=n} \frac{Q(x_m)}{(x - x_m)P'(x_m)} \tag{10}$$

where X_m is a root of $P_{(x)} = 0$, and

$$P'(x_m) = \left(\frac{dP(x)}{dx}\right)_{x=x_m}, \tag{11}$$

provided $P_{(x)}$ does not contain repeated roots. The special case of repeated roots will be briefly discussed later.

In general $M_{lk}(p)$ and $D(p)$ satisfy the conditions of expansion, whence

$$(y_k)_{t=0} = E_1 \frac{M_{1k}(p)}{D(p)} = E_1 \sum_{m=1}^{m=n} \frac{M_{1k}(p_m)}{(p - p_m)D'(p_m)}, \tag{12}$$

[1] See Williamson, Integral Calculus, pp. 42, 43.

where the summation is extended over the roots of $D(p)$, and

$$D'(p_m) = \left(\frac{d}{dp}D(p)\right)_{p=p_m}.$$

Clearly then, both conditions (1) and (3) are satisfied if we set

$$A_k^{(m)} = E_1 \frac{M_{1k}(p_m)}{(p_m - p)D'(p_m)} \tag{13}$$

since then $(y_k + z_k)_{t=0} = 0$ and[1]

$$\frac{A_k^{(m)}}{A_j^{(m)}} = \frac{M_{1k}(p_m)}{M_{1j}(p_m)}. \tag{14}$$

Hence the complete solution is

$$x_k = E_1 \left\{ \frac{M_{1k}(p)}{D(p)} \epsilon^{pt} - \sum_{m=1}^{m=n} \frac{M_{1k}(p_m)}{(p - p_m)D'(p_m)} \epsilon^{p_m t} \right\}, \tag{15}$$

provided this solution satisfies condition (2). That this is in general the case may be readily shown. Differentiating (15) we have

$$\frac{dx_k}{dt} = \dot{x}_k = E_1 \left\{ \frac{pM_{1k}(p)}{D(p)} \epsilon^{pt} - \sum_{m=1}^{m=n} \frac{p_m M_{1k}(p_m)}{(p - p_m)D'(p_m)} \epsilon^{p_m t} \right\} \tag{16}$$

and

$$(\dot{x}_k)_{t=0} = E_1 \left\{ \frac{pM_{1k}(p)}{D(p)} - \sum_{m=1}^{m=n} \frac{p_m M_{1k}(p_m)}{(p - p_m)D'(p_m)} \right\}.$$

Now $pM_{1k(p)}$ is in general a polynomial in p of lower order by one than $D(p)$ whence

$$\frac{pM_{1k}(p)}{D(p)} = \sum_{m=1}^{m=n} \frac{p_m M_{1k}(p_m)}{(p - p_m)D'(p_m)}, \tag{17}$$

so that condition (2) is satisfied.

It is now easy to extend formula (16) to the more general case when all the forces $F_1 \cdots F_n$ are finite. For let

$$F_1 = E_1 \epsilon^{pt}; \quad \cdots F_n = E_n \epsilon^{pt}.$$

Then the complete solution is

$$x_k = \sum_{j=1}^{j=n} E_j \frac{M_{jk}(p)}{D(p)} \epsilon^{pt} - \sum_{j=1}^{j=n}\sum_{m=1}^{m=n} E_j \frac{p_m M_{jk}(p_m)}{(p - p_m)D'(p_m)} \epsilon^{p_m t}.$$

Of course the different forces may be characterized by different exponential factors.

The conditions necessary that the partial fraction expansions given by (12) and (16) shall hold are satisfied in general; that is in the usual

[1] Equation (14) is equivalent to condition (3), and formulates the necessary relation among the constants of integration.

case when the inertia and stiffness factors q and s are all finite. No attempt will be here made to rigorously discuss the cases when the general expansion fails or when it must be specially interpreted. Two physically interesting cases will however be considered.

1. Assume that the inertia factors (q) are all zero. It will be clear then from physical considerations that condition (2) will not necessarily hold since finite velocities may be instantaneously established owing to the absence of inertia. The initial configuration of the coördinates must, however, be zero from physical considerations. We should, therefore, expect, from purely physical considerations, that the expansion given by (12) is still valid while the expansion given by (16) no longer holds. This is precisely the case since now $M_{1k(p)}$ is of order $(n - 1)$ and $D(p)$ of order n in p. Hence while the expansion of $M_{1k(p)}/D(p)$ is valid the expansion of $pM_{1k(p)}/D(p)$ is no longer valid since $pM_{1k(p)}$ is of the same instead of lower order than $D(p)$. Thus while the expansion formula (15) for the coördinates and consequently the expansion formula following for the velocity are correct, the initial velocities are no longer necessarily zero.

2. Assume that the stiffness factors (s) are all zero. Then physical considerations show that an equilibrium configuration of the coördinates is indeterminate but that the initial velocities are necessarily zero. We should therefore expect a priori that the expansion (12) is not necessarily true but that expansion (17) is still valid. This is precisely the case as results from the following consideration. If the stiffness factors (s) are all zero, zero is a repeated root of $D(p)$ of the nth order and a repeated root of $M_{1k(p)}$ of the $(n - 1)$st order. Then

$$\frac{pM_{1k}(p)}{D(p)} = \frac{p^n Q(p)}{p^n P(p)} = \frac{Q(p)}{P(p)},$$

where $Q(p)$ and $P(p)$ contain no zero roots. Then

$$\frac{pM_{1k}(p)}{D(p)} = \sum_{m=1}^{m=n} \frac{Q(p_m)}{(p - p_m)P'(p_m)},$$

when the summation is taken for all the roots of $D(p)$ exclusive of zero. It may then be readily shown that

$$\frac{Q(p_m)}{(p - p_m)P'(p_m)} = \frac{p_m M_{1k}(p_m)}{(p - p_m)D'(p_m)},$$

whence it follows that the expansion for the velocities is valid. The expansion for the coördinates is meaningless.

The two foregoing particular cases serve to illustrate the fact that while the expansion is generally valid it will not hold for dynamic systems in which the initial conditions are not necessarily satisfied. If, therefore, the expansions (12) and (16) are not valid we may be sure that the initial conditions are not complied with by the system under consideration. Further elaboration of this point is not believed necessary and particular cases can be readily worked out from the general theory.

As stated above, the partial fraction expansion of equation (10) does not hold when the denominator $P(x)$ contains repeated roots. Cases, however, in which the characteristics determinant of the system contains repeated roots can be readily handled by letting the roots approach equality as a limit. A brief example will suffice to indicate the appropriate treatment. Assume that the characteristic determinant is

$$D(p) = p^2 + 2\alpha p + \alpha^2 = (p + \alpha)^2$$

and let

$$y = \frac{1}{D(p)} \epsilon^{pt}.$$

The roots of $D(p)$ are then equal so that $p_1 = p_2 = -\alpha$.

To handle this problem consider the general case where

$$D(p) = (p - p_1)(p - p_2).$$

Then

$$x = \frac{\epsilon^{pt}}{D(p)} - \frac{\epsilon^{p_1 t}}{(p - p_1)(p_1 - p_2)} - \frac{\epsilon^{p_2 t}}{(p - p_2)(p_2 - p_1)}.$$

Now let $p_2 = -\alpha + e$, $p_1 = -\alpha$ and let e approach zero as a limit. The final expression for x is, in the limit

$$x = \frac{\epsilon^{pt}}{(p + \alpha)^2} - \epsilon^{-\alpha t}\left(\frac{t}{p + \alpha} + \frac{1}{(p + \alpha)^2}\right).$$

For the sake of generality the foregoing formulæ have been derived in terms of a general dynamic system; since, however the most important application of the expansion theorem is concerned with oscillations of electrical networks, the formulæ will therefore be translated into the terms of such a system. In formula (16) replace $\dot{x}_k$ by I_k, where I_k is the current in the Kth branch or mesh of the network; let q, r and $1/s$ be inductance, resistance and capacity and let $Z_{1k}(p)$ be the ratio of the E.M.F. of frequency p impressed on branch or mesh 1 to the forced current flowing in branch or mesh K. Clearly $Z_{1k}(p)$ may be termed the impedance of the Kth with respect to the first branch and is given by

$$Z_{1k}(p) = \frac{D(p)}{pM_{1k}(p)}.$$

Also

$$\frac{d}{dp} Z_{1k}(p) = \frac{D'(p)}{M_{1k}(p)} - D(p) \frac{pM_{1k}'(p) + M_{1k}(p)}{p^2(M_{1k}(p))^2},$$

$$Z_{1k}'(p_m) = \frac{D'(p_m)}{p_m M_{1k}(p_m)},$$

since $D(p_m) = 0$.

Formula (16) may then be replaced by

$$I_k = E_1 \left\{ \frac{\epsilon^{pt}}{Z_{1k}(p)} - \sum_{m=1}^{m=n} \frac{\epsilon^{p_m t}}{(p - p_m) Z_{1k}'(p_n)} \right\}, \tag{18}$$

where p_m is a root of $Z_{1k}(p)$ since the roots of $Z_{1k}(p)$ are likewise the roots of $D(p)$. Formula (18) is the generalized form of Heaviside's Theorem, into which it degenerates when p is put equal to zero.

The expansion formula gives explicitly the resultant oscillations when a driving force is suddenly impressed on the system. It may be also used to formulate the subsidence to equilibrium of a system having any initial configuration, provided such configuration is producible without changing the connections or constraints of the system. This limitation is equivalent to the statement that the initial configuration may be formulated by sums of expressions of the form:

$$(I_k)_{t=0} = E_1 \left\{ \frac{\epsilon^{p\tau}}{Z_{1k}(p)} - \sum \frac{\epsilon^{p_m \tau}}{(p - p_m) Z_{1k}'(p_m)} \right\},$$

when τ is to be regarded as a constant. The free oscillations back to equilibrium are then given by

$$I_k = E_1 \sum \frac{\epsilon^{p\tau} - \epsilon^{p_m \tau}}{(p - p_m) Z_{1k}'(p_m)} \epsilon^{p_m t}. \tag{19}$$

The expansion theorem formulated by (18) is derived in terms of system which is specified by a finite number of coördinates. That it holds for a system characterized by an infinite number of coördinates is a fair inference, since it seems permissible to let the number of coördinates approach infinity as a limit, though doubtless a rigorous proof of this is necessary. However the Expansion Theorem does hold for a number of problems involving an infinite number of coördinates which have been examined by the writer; in particular the Expansion Theorem may be applied to the oscillations of a transmission line having distributed constants as well as to an artificial line having a finite number of lumped or localized elements.

To illustrate the application of the Expansion Theorem to the oscillations of a transmission line, assume an electromotive force expressible as

$R\{E\epsilon^{pt}\}$ to be impressed at time $t = 0$ on a transmission line of inductance L, capacity C, resistance R and leakage G per unit length. Let the length of the transmission line be l and let the e.m.f. be impressed through an impedance Z_1 at $s = 0$ which the line is closed by an impedance Z_2 at $s = l$. The " forced " component current at point s on the line, corresponding to the impressed e.m.f. is then expressible as

$$J_s = \frac{E\epsilon^{pt}}{\varphi_s(p)},$$

where

$$\varphi_s(p) = \frac{K(Z_1 + Z_2) + (K^2 + Z_1 Z_2)\tanh(\gamma l)}{\cosh(\gamma s)[K(Z_1 + Z_2) + (K^2 - Z_1 Z_2)\tanh(\gamma l)] - \sinh \gamma s[Z_2 + K\tanh(\gamma)l]}. \quad (20)$$

In the foregoing formula:

$$\gamma = \sqrt{\{R + Lp\}\{G + Cp\}}, \quad (21)$$

$$K = \sqrt{\left\{\frac{R + Lp}{G + Cp}\right\}}. \quad (22)$$

Z_1 and Z_2 are, of course, preassigned explicit functions of p.

In accordance then with equation (18) the expression for the current at any point s along the line, valid for positive values of t, is

$$I_s = E\left\{\frac{\epsilon^{pt}}{\varphi_s(p)} - \sum_{m-1}^{m=\infty} \frac{\epsilon^{p_m t}}{(p - p_m)\varphi_s'(p_m)}\right\}, \quad (23)$$

where $\varphi_s(p)$ is given by (20); p_m is the mth root and $\varphi_s'(p)$ is the derivative of $\varphi_s(p)$ with respect to p, and the summation is extended over all the roots. There are of course an infinite number of roots of the transcendental function $\varphi_s(p)$ so that in general the solution is practically unmanageable. It is however, a formal compact solution of the problem. Moreover for particular terminal arrangements, such as $Z_1 = Z_2 = 0$, the roots admit of rather easy determination.

The chief utility of the Expansion Theorem will be seen to reside in the fact that by its use the solution for the transient oscillations of the system is reduced to formulæ which are functionally the same as those for steady state oscillations, so that the problem is always completely solvable provided the roots of the characteristic $D(p)$ admit of determination.

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