Symmetry-Enforced Many-Body Separability Transitions

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(Received 9 December 2023; accepted 28 May 2024; published 17 July 2024)

We study quantum many-body mixed states with a symmetry from the perspective of *separability*, i.e., whether a mixed state can be expressed as an ensemble of short-range-entangled symmetric pure states. We provide evidence for "symmetry-enforced separability transitions" in a variety of states, where in one regime the mixed state is expressible as a convex sum of symmetric short-range-entangled pure states, while in the other regime, such a representation is not feasible. We first discuss the Gibbs state of Hamiltonians that exhibit spontaneous breaking of a discrete symmetry, and argue that the associated thermal phase transition can be thought of as a symmetry-enforced separability transition. Next we study cluster states in various dimensions subjected to local decoherence, and identify several distinct mixed-state phases and associated separability phase transitions, which also provides an alternative perspective on recently discussed "average symmetry-protected topological order." We also study decohered $p + ip$ superconductors, and find that if the decoherence breaks the fermion parity explicitly, then the resulting mixed state can be expressed as a convex sum of nonchiral states, while a fermion parity–preserving decoherence results in a phase transition at a nonzero threshold that corresponds to spontaneous breaking of fermion parity. Finally, we briefly discuss systems that satisfy the no low-energy trivial state property, such as the recently discovered good low-density parity-check codes, and argue that the Gibbs state of such systems exhibits a temperature-tuned separability transition.

DOI: 10.1103/PRXQuantum.5.030310

I. INTRODUCTION

Suppose one has the ability to apply unitary gates that act in a geometrically local fashion on a many-body system. Starting from a product state, a specific circuit composed of such gates results in a specific pure state, and an ensemble of such circuits can therefore be associated with the mixed state $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, where the pure state $|\psi_i\rangle$ is prepared with probability p_i. If one is limited to only constant-depth unitary circuits, then the corresponding mixed state can be regarded as "short-range entangled" (SRE) or "trivial" [1,2], which generalizes the notion of a short-range-entangled pure state [3–8]. In parallel with the notion of symmetry-protected topological (SPT) phases for pure states [9–12], it is then natural to define a trivial or SRE *symmetric* mixed state [a symmetric and SRE (sym-SRE) state] as one that can be obtained from an ensemble of pure states, where each element of the ensemble is prepared with only a constant-depth circuit consisting of local, symmetric gates under some given symmetry. Motivated by experimental progress in controllable quantum devices where both unitary quantum dynamics and decoherence play an important role [13–16], we explore in this paper mixed-state phase diagrams where in one regime a mixed state is sym-SRE, and in the other regime, it is not. We call such phase transitions "symmetry-enforced separability transitions," since a sym-SRE state is essentially separable [1] (i.e., a convex sum of unentangled states) up to short-distance correlations generated by constant-depth unitaries. In the absence of any symmetry constraint, analogues of such transitions were recently studied in Ref. [17] in the context of decohered topologically ordered mixed states [18–22]. To make progress, we try to leverage our understanding of the complexity of preparing pure many-body states using unitaries. Some of the questions that will motivate our discussion are as follows: Do there exist separability phase transitions when pure-state SPT phases are subjected to decoherence, and if the answers is "yes," what is the universality class of such transition? When a 2D chiral pure state (e.g., the ground state of an integer quantum Hall phase) is subjected to local decoherence, can the resulting density matrix be expressed as a convex sum of nonchiral states? Can the conventional, finite-temperature phase transitions corresponding to the spontaneous breaking of a global symmetry be also thought of as separability transitions?

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As an example, consider the transverse-field Ising model on a square lattice. We provide an argument (Sec. III) that the Gibbs state for this model can be prepared with use of an ensemble of finite-depth local unitary circuits at all temperatures, including at $T \leq T_c$, where T_c is the critical temperature for spontaneous symmetry breaking. It is crucial here that one is not imposing any symmetry constraint on the unitaries. This is consistent with previous work [23–25] where evidence was provided that the mixed-state entanglement corresponding to a Gibbs state that exhibits spontaneous symmetry breaking remains short-ranged at all nonzero temperatures, including at the finite-temperature critical point (assuming the absence of any coexisting finite-temperature topological order). However, if one allows access to an ensemble of short-depth unitary circuits composed of only *Ising symmetric* local gates, then using results from Ref. [21], we provide a rigorous argument that the Gibbs state cannot be prepared for any $T \leq T_c$. We expect similar results to hold for other symmetry-broken Gibbs states as well. Therefore, the conventional, finite-temperature symmetry-breaking phase transition in a transverse-field Ising model can be thought of as a symmetry-enforced separability transition. This statement is true even when the transverse field is zero (i.e., for a classical Ising model)—the quantum mechanics still plays a role since the imposition of symmetry implies that one is forced to work with "cat" (GHZ) states, which are long-range entangled (LRE).

In the context of pure states, a well-known example of symmetry-enforced complexity is an SPT phase whose ground state cannot be prepared with use of a finite-depth circuit composed of symmetric local gates [9–12]. Recent studies have provided a detailed classification of SPT phases protected by zero-form symmetries that are being subjected to decoherence with use of spectral sequences and obstruction to an SRE purification [26,27]. Progress has also been made in understanding nontrivial decohered SPT orders with use of string operators [28] and "strange correlators" [29,30], concepts that were originally introduced to characterize pure SPT states [10,31,32]. Here we are interested in understanding decohered SPT states from the viewpoint of separability, which, as we discuss in Sec. II, is a notion of entanglement of mixed states different from that based on SRE purification considered in Refs. [26,27]. As hinted above, we define a symmetric, LRE (sym-LRE) state as one that does not admit a decomposition as a convex sum of pure states that can all be prepared via a finite-depth circuit made of symmetric local gates. If this is the case, it is interesting to ask if there exist separability transitions between sym-LRE and sym-SRE states as a function of the decoherence rate, analogous to the phase transitions in mixed states with intrinsic topological order [17]. We will not consider a general SPT state, and will focus primarily on cluster states in various dimensions to illustrate the broad idea. A key step in our analysis is the following result, which was also briefly mentioned in Ref. [17] and which we discuss in detail in Sec. IV: for a large class of SPT orders, including the cluster states in various dimensions, a Kitaev chain in on dimension, and several 2D topological phases protected by zero-form Z_2 symmetry, one can find local, finite-depth channels that map the pure state to a Gibbs state. We discuss decoherence-induced separability transitions due to such channels in Sec. IV.

When trying to understand the complexity of mixed SPT states, we will often find the following line of inquiry helpful. One first asks whether our assuming that a mixed state is trivial (i.e., decomposable as a convex sum of SRE pure states) leads to an obvious contradiction. If the answer is that it does, then we already know that the mixed state is necessarily nontrivial. In this case, there may still exist interesting transitions between two different kinds of nontrivial mixed state, and we will consider a couple of such examples as well. On the other hand, if the answer is that it does not, we will attempt to find an explicit decomposition of the mixed state as a convex sum of SRE states. The aforementioned relation between local and thermal decoherence will again be instrumental in making analytical progress.

As an example, consider the ground state of the 2D cluster-state Hamiltonian H subjected to a local channel that locally anticommutes with the terms in the Hamiltonian. One can show that the resulting decohered state ρ_d takes the Gibbs form: $\rho \propto e^{-\beta H}$, where $\tanh \beta = 1 - 2p$ and p is the decoherence rate. In this example, H has both a zero-form and a one-form Ising symmetry. We will provide arguments that this system undergoes a separability transition as a function of p: for $0 < p < p_c$, ρ cannot be decomposed as a sum of pure states that respect the aforementioned two symmetries, while for $p > p_c$, such a decomposition is feasible. Moreover, for $p > p_c$ we will express ρ_d explicitly as $\sum_m p_m |\psi_m\rangle\langle\psi_m|$, where $|\psi_m\rangle$ are pure, symmetric states that are statistically SRE. More precisely, one can define an ensemble-averaged string correlation, $[\langle S_C \rangle^2] \equiv \sum_m p_m \langle S_C \rangle_m^2$, where $\langle S_C \rangle_m = \langle\psi_m|S_C|\psi_m\rangle / \langle\psi_m|\psi_m\rangle$ and S_C is a string operator whose nonzero expectation value implies long-range entanglement. We will show that $[\langle S_C \rangle^2]$ precisely corresponds to a disorder-averaged correlation function in the 2D random-bond Ising model (RBIM) along the Nishimori line [33]. Therefore, in this example, the separability transition maps to the ferromagnetic transition in the random-bond Ising model. For the 3D cluster state, we will find an analogous relation between separability and 3D random-plaquette Ising gauge theory. We note that similar order parameters and connections to statistical mechanics models also appear in the setting of measurement protocols to prepare long-range-entangled SPT states [34,35]. We briefly discuss the connection to these studies.

As another by-product of the relation between local decoherence and Gibbs states, we also study a recently introduced nontrivial class of mixed states that are protected by a tensor product of "exact" and "average" symmetries [26–29]. One says that a density matrix ρ has an "exact symmetry" if $U_E\rho \propto \rho$ for some unitary U_E, while it has an "average symmetry" if $U_A^\dagger \rho U_A = \rho$ for some unitary U_A. References [26–29] provide several nontrivial examples of such mixed-state SPT orders by showing that they possess nontrivial correlation functions and/or cannot be purified to an SRE pure state. Here we focus on examples of such states that are based on cluster states in various dimensions, and using locality/the Lieb-Robinson bound [36–38], we show that the corresponding mixed states cannot be written as a convex sum of symmetric, pure states. For a 1D cluster state, we also provide an alternative proof of nonseparability by using the result from Ref. [39] that in one dimension if a state has an average Z_2 symmetry, and its connected correlation functions are short-ranged, then the corresponding "order parameter" and the "disorder parameter" cannot both be zero or nonzero at the same time.

In Sec. V we consider fermionic chiral states subjected to local decoherence. We primarily focus on the ground state of a 2D $p_x + ip_y$ superconductor ($p + ip$ SC) as our initial state (we expect integer quantum Hall states to have qualitatively similar behavior). We first consider subjecting this pure state to a finite-depth channel with Kraus operators that are linear in fermion creation/annihilation operators, so that the decoherence breaks the fermion-parity symmetry. In the pure-state classification of topological superconductors, fermion parity is precisely the symmetry responsible for the nontrivial topological character of the $p + ip$ SC [40,41]. Therefore, it is natural to wonder about the fate of the mixed state obtained by breaking this symmetry from exact down to average. One potential path to make progress on this problem is to map the mixed state to a pure state in the doubled Hilbert space by means of the Choi-Jamiołkowski (CJ) map [42,43] (we call such a state the "double state," similar to the nomenclature in Ref. [20]). There are interesting subtleties in applying the CJ map to fermionic Kraus operators that we clarify. Following the ideas in Refs. [18,20,22,29], one may then map the double state to a $(1 + 1)$ D theory of counterpropagating free conformal field theories (CFTs) coupled via a fermion bilinear term, which is clearly relevant and gaps out the edge states in this doubled picture. However, a short-depth channel cannot qualitatively change the expectation value of state-independent operators [i.e., $\mathrm{tr}(\rho O)$, where O is independent of ρ] [18,19], and it is not obvious what the gapping of edge modes implies for the actual mixed state. We conjecture that the physical implication of the gapping of the edge states in the doubled formulation is that the actual mixed state can now be expressed as a convex sum of SRE states with zero Chern number, which is equivalent to the statement that they can be obtained as a Slater determinant of Wannier states, unlike the pure $p + ip$ state, where such a representation is not possible [44–46]. Therefore, the transition from the pure state to the mixed state can be thought of as a "Wannierizability transition." We consider an explicit ansatz of such a decomposition, and provide numerical support for our conjecture by calculating the entanglement spectrum and modular commutator of the pure states whose convex sum corresponds to the decohered density matrix.

A more interesting channel that acts on the 2D $p + ip$ SC corresponds to Kraus operators that are *bilinear* in fermion creation/annihilation operators. To make progress on this problem, we use the CJ map to obtain a field-theoretic description for this problem in terms of two counterpropagating chiral Majorana CFTs interacting via a four-fermion interaction, where the strength of the interaction is related to the strength of the interacting decohering channel. This theory admits a phase transition at a critical interaction strength in the supersymmeteric tricritical Ising universality class, which can be thought of as corresponding to spontaneous breaking of the fermion parity. Although we do not have an understanding of this transition directly in terms of the mixed state in the nondoubled (i.e., original) Hilbert space, it seems reasonable to conjecture that at weak decoherence, the density matrix cannot be expressed as a convex sum of area law–entangled nonchiral states, while at strong decoherence, it is most naturally expressible as a convex sum of states with GHZ-like character that originates from the aforementioned spontaneous breaking of the fermion parity.

Incidentally, the kind of arguments we consider to rule out sym-SRE mixed states in the context of symmetry-broken phases or SPT phases also finds an application in an exotic separability transition where symmetry plays no role. In particular, we consider separability aspects of the Gibbs state of Hamiltonians that satisfy the "no low-energy trivial state" (NLTS) condition introduced by Freedman and Hastings [47]. Colloquially, if a Hamiltonian satisfies the NLTS condition, then any pure state with energy density less than a critical nonzero threshold cannot be prepared by a constant-depth circuit. Recently, Anshu *et al.* [48] showed that the "good low-density parity-check (LDPC) code" constructed in Ref. [49] satisfies the NLTS condition (we note that "good LDPC codes" [49–51] have the remarkable property that both the code distance and the number of logical qubits scale linearly with the number of physical qubits). Anshu *et al.* [48] showed that the NLTS condition holds also for mixed states, if one defines the circuit depth of a mixed state as the minimum depth of the unitary needed to prepare it by acting on system ⊗ ancillae, both initially in a product state, where the ancillae are traced out afterwards [52]. Under such a definition of a nontrivial mixed state (namely, a mixed state that cannot be prepared by a constant-depth circuit under the aforementioned protocol), even mixed states with long-range *classical correlations* (e.g., the Gibbs state of 3D classical Ising model) would be considered nontrivial. In contrast, under our definition of a nontrivial mixed state, such classical states will be trivial since they can be written as a convex sum of SRE states. Therefore, we ask the following question: assuming that one defines a trivial (nontrivial) mixed state as one that can (cannot) be expressed as a convex sum of SRE states, is the Gibbs state of a Hamiltonian that satisfies the NLTS property nontrivial at a low but nonzero temperature? Under reasonable assumptions, in Sec. VI we provide a short argument that this is indeed the case. This implies that one should expect a nonzero temperature separability transition in such Gibbs states.

In Sec. VII we briefly discuss connections between separability criteria and other measures of the complexity of a mixed state such as the ability to purify a mixed state to an SRE pure state, entanglement of the doubled state using a CJ map, and strange correlators.

Finally, in Sec. VIII we summarize our results and discuss a few open questions.

II. SEPARABILITY CRITERIA WITH AND WITHOUT SYMMETRY

Motivated by Werner and Hastings [1,2], we call a mixed state ρ "short-range entangled" if and only if it can be decomposed as a convex sum of pure states,

$$\rho = \sum_m p_m |\psi_m\rangle\langle\psi_m|, \qquad (1)$$

where each $|\psi_m\rangle$ is SRE, i.e., it can be prepared by one applying a constant-depth local unitary circuit to some product state. The physical motivation for this definition is rather transparent: if a mixed state can be expressed as Eq. (1), only then it can be prepared with use of an ensemble of unitary circuits (acting on the Hilbert space of ρ) whose depth does not scale with the system size. We note that this definition of an SRE mixed state has been used to understand phase transitions in systems with intrinsic topological order subjected to thermal or local decoherence [17,53].

One can generalize the notion of an SRE mixed state in the presence of a symmetry. Specifically, we say that a mixed state ρ satisfying $U(g)\rho U^\dagger(g) = \rho$ for all $g \in G$ is a sym-SRE state if and only if one can decompose it as a convex sum of pure states, where each of these pure states can be prepared by one applying a finite-depth quantum circuit made of local gates that all commute with U to a symmetric product state.

Several comments follow:

(1) The "only if" clause in our definition for a sym-SRE state or SRE state is a bit subtle. For example, consider a density matrix where there exists no decomposition that satisfies Eq. (1) but there exists a decomposition $\rho = \sum_{m,|\psi_m\rangle\in\text{SRE}} p_m|\psi_m\rangle\langle\psi_m| + \sum_{m,|\phi_m\rangle\notin\text{SRE}} q_m|\phi_m\rangle\langle\phi_m|$ such that the relative weight of the non-SRE states is zero in the thermodynamic limit [i.e., $\sum_m q_m/\sum_m (p_m + q_m) \to 0$ in the thermodynamic limit]. In this case, it might seem reasonable to regard ρ as SRE. One may also define an average circuit complexity of a density matrix as $\langle\mathcal{C}\rangle = \inf\{\sum_m p_m \mathcal{C}(\psi_m)\}$, where $\mathcal{C}(|\psi_m\rangle)$ is the minimum depth of a circuit composed of local gates to prepare the state $|\psi_m\rangle$ and the infimum is taken over all possible decompositions of the mixed state ρ. One may then consider calling a mixed state ρ SRE if and only if $\langle\mathcal{C}\rangle$ does not scale with the system size. But even then, there may be special cases where the average behavior is not representative of a typical behavior. We will not dwell on this subtlety further at this point, and will use physical intuition to quantify the separability of a density matrix should we encounter such a situation.

(2) Reference [2] also introduced a seemingly different definition of an SRE mixed state: Consider a "classical" state $\rho_{\text{cl}} \propto e^{-H_{\text{cl}}}$, where H_{cl} is a Hamiltonian composed of terms that are all diagonal in a product basis, and that acts on an enlarged Hilbert space $a \otimes s$, where s denotes the system of interest and a denotes ancillae. Then a mixed state ρ may be regarded as SRE if it can be obtained from ρ_{cl} by one applying a finite-depth unitary on $s \otimes a$, followed by one tracing out a. That is, one may consider ρ as SRE if

$$\rho = \text{tr}_a\left(U^\dagger e^{-H_{\text{cl}}}, U/Z\right) \qquad (2)$$

where U is a finite-depth circuit and $Z = \text{tr}\left(e^{-H_{\text{cl}}}\right)$. We are unable to show that the definition in Eq. (1) is equivalent to Eq. (2). Although we will primarily use the former definition [Eq. (1)], in Sec. VII we briefly discuss potential connections between the two definitions, and also the relation with other diagnostics of mixed-state entanglement.

(3) The symmetry $[(U(g)\rho U^\dagger(g) = \rho$ for all $g \in G)]$ we consider is called "weak symmetry" in Ref. [28] and "average symmetry" in Ref. [26], which highlights its difference from the stronger symmetry $U(g)\rho = \rho U(g) = e^{i\theta(g)}\rho$ for all $g \in G$ termed "strong symmetry" in Ref. [28] and "exact symmetry" in Ref. [26]. Physically, exact symmetry enforces the constraint that the density matrix *must* be written as an incoherent sum of pure states, where each of them is an eigenstate of $U(g)$ with the same eigenvalue $e^{i\theta(g)}$. On the other hand, while the mixed state ρ with only average symmetry can be written as a convex sum of symmetric pure states

having different charge under G, one may as well express ρ as a convex sum of nonsymmetric pure states. Therefore, our requirement that each of the pure states respects the symmetry puts a further constraint on a mixed state with only average symmetry.On that note, Ref. [54] defined a sym-SRE state for a symmetry U as one that satisfies Eq. (2) where $e^{-H_{\mathrm{cl}}}$ is replaced by $P_{\theta(g)}e^{-H_{\mathrm{cl}}}$, where $P_{\theta(g)}$ is a projector onto a given symmetry charge $\theta(g)$. Therefore, in this definition one is always working with a density matrix that has an *exact* symmetry. As already mentioned, we will instead impose the average symmetry only in our definition of a sym-SRE state (of course, there may be special quantum channels that happen to preserve an exact symmetry).

(4) An alternative definition of an SRE mixed state was considered in Refs. [26,27,52], whereby a mixed density matrix is considered SRE if it can be obtained from a *pure* product state in a system $\otimes$ ancillae Hilbert space via a finite-depth unitary followed by one tracing out ancillae. In contrast, as already mentioned in comment (2), Ref. [2] defines a mixed density matrix as SRE if it can be obtained from the "classical mixed state" $\rho_{\mathrm{cl}} \propto e^{-H_{\mathrm{cl}}}$ of system $\otimes$ ancillae via a finite-depth local quantum channel. Therefore, a mixed state can be trivial/SRE if one uses the definition in Ref. [2] while remaining nontrivial/LRE if one uses the definition in Refs. [26,27,52]. The physical distinction between these two definitions is most apparent when one considers a mixed state for qubits of the form $\rho = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)$, where $|\uparrow\rangle = \prod_i |\uparrow\rangle_i$ and $|\downarrow\rangle = \prod_i |\downarrow\rangle_i$. This state is clearly separable (unentangled). However, any short-depth purification of this state must be long-range entangled. This is because $\mathrm{tr}(\rho Z_i Z_j) - \mathrm{tr}(\rho Z_i)\mathrm{tr}(\rho Z_j)$ is nonzero and the purified state cannot change this correlation function due to the Lieb-Robinson bound [36,37] (this is also related to the fact the entanglement of purification [55] is sensitive to both quantum and classical correlations, and therefore is not a good mixed-state entanglement measure). Thus, the aforementioned ρ will be SRE if one uses the definition in Ref. [2] and will be LRE if one uses the definition in Refs. [26,27,52]. Of course, it will also be SRE via Eq. (1), which is the definition we will use throughout this paper.

III. ILLUSTRATIVE EXAMPLE: SEPARABILITY TRANSITION IN THE GIBBS STATE OF THE 2D QUANTUM ISING MODEL

Let us consider an example to illustrate the difference between an SRE mixed state and a sym-SRE mixed state, which will also provide one of the simplest examples of a separability transition. Consider the density matrix ρ for qubits [i.e., objects transforming in the spin-1/2 representation of SU(2)] given by $\rho(\beta) = e^{-\beta H}/Z$, where H is a local Hamiltonian that satisfies $U^\dagger H U = H$, with $U = \prod_i X_i$ being the generator of the Ising symmetry, and $Z = \mathrm{tr}e^{-\beta H}$ is the partition function. Let us further assume that $\rho(\beta)$ exhibits spontaneous symmetry breaking for $\beta > \beta_c$, where $0 < \beta_c < \infty$ (for a range of other parameters that specify the Hamiltonian). For concreteness, one may choose H as the nearest-neighbor transverse-field Ising model on the square lattice, i.e., $H = -\sum_{\langle i,j\rangle} Z_i Z_j - h\sum_i X_i$ although the only aspect that will matter in the following discussion is that H is local with a zero-form Ising symmetry, and the order parameter in the symmetry-breaking phase is a real scalar (e.g., one may also consider a transverse-field Ising model on a cubic lattice). Therefore, for a range of the transverse-field h and $\beta > \beta_c$ (where β_c depends on h), the two-point correlation function $\mathrm{tr}\left(\rho Z_i Z_j\right)$ is nonzero for $|i-j| \to \infty$. We will argue that ρ is SRE for all nonzero temperatures, while it is sym-SRE only for $\beta < \beta_c$. Partial support for ρ being an SRE at all nonzero temperatures was provided in Refs. [23–25], and we will argue for an explicit decomposition of ρ in terms of SRE states.

The statement that ρ is not sym-SRE for $\beta \geq \beta_c$ was also hinted at in Ref. [54], and intuitively follows from the fact that for $\beta > \beta_c$, spontaneous symmetry breaking implies that if one decomposes ρ as a convex sum of symmetric, pure states, those pure states must have GHZ-like entanglement. Let us first consider a rigorous argument for this statement that, up to small modifications, essentially follows the argument in Ref. [21] for a closely related problem of nontriviality of a density matrix with an exact symmetry and long-range order.

To show that for $\beta > \beta_c$, ρ cannot be a sym-SRE state, we first decompose ρ as $\rho = \rho_+ + \rho_-$, where $\rho_+ = (1 + U/2)\rho$ and $\rho_- = (1 - U/2)\rho$ are the projections of ρ onto even and odd charge of the Ising symmetry. ρ_+ and ρ_- are valid density matrices with an exact Ising symmetry; that is, they satisfy $U\rho_\pm = \pm\rho_\pm$. Now let us make the assumption that for $\beta > \beta_c$, ρ is a sym-SRE state. We will show that this assumption leads to a contradiction. Therefore, we write $\rho_\pm = \sum_\alpha p_{\alpha,\pm}|\psi_{\alpha,\pm}\rangle\langle\psi_{\alpha,\pm}|$, where $p_{\alpha,\pm}$ are positive numbers, and $|\psi_{\alpha,\pm}\rangle$ are SRE states for all values of α that satisfy $U|\psi_{\alpha,\pm}\rangle = \pm|\psi_{\alpha,\pm}\rangle$. Since U anticommutes with Z_i, $\langle\psi_{\alpha,\pm}|Z_i|\psi_{\alpha,\pm}\rangle = 0$. Further, since $|\psi_{\alpha,\pm}\rangle$ are all SRE states, correlation functions of all local operators decay exponentially (notably, we assume that the associated correlation length is bounded by a *system-size-independent* constant for all $|\psi_{\alpha,\pm}\rangle$), and therefore $\langle\psi_{\alpha,\pm}|Z_j Z_k|\psi_{\alpha,\pm}\rangle - \langle\psi_{\alpha,\pm}|Z_j|\psi_{\alpha,\pm}\rangle\langle\psi_{\alpha,\pm}|Z_k|\psi_{\alpha,\pm}\rangle = \langle\psi_{\alpha,\pm}|Z_j Z_k|\psi_{\alpha,\pm}\rangle$ vanishes as $|j-k| \to \infty$. However, this leads to a contradiction, because this implies that $\mathrm{tr}\left(\rho Z_j Z_k\right) = \sum_\pm \sum_\alpha p_{\alpha,\pm}\langle\psi_{\alpha,\pm}|Z_j Z_k|\psi_{\alpha,\pm}\rangle$ itself vanishes, which we know cannot be true since as mentioned above, for $\beta > \beta_c$, the system is in a spontaneous-symmetry-breaking phase with long-range order. Therefore, our assumption that ρ is a sym-SRE state for $\beta > \beta_c$ must be incorrect. The same conclusion also holds for $\beta = \beta_c$ since the correlations at the critical point decay as a power law.

As mentioned in Sec. I, our general approach would be to first look for general constraints that lead to a mixed state being necessarily nontrivial. If we are unable to find such a constraint, we will attempt to find an explicit decomposition of the density matrix as a convex sum of SRE states. For example, above we noted that ρ cannot be a sym-SRE state for $\beta \geq \beta_c$, and we also claimed that ρ is an SRE state for all nonzero temperatures. Let us therefore try to find an explicit decomposition of ρ as a convex sum of SRE pure states for any nonzero temperature, and as a convex sum of symmetric, pure SRE states for $\beta < \beta_c$. The key player in our argument will be a particular convex decomposition ansatz (CDA) that is motivated by the construction of "minimally entangled typical thermal states" (METTSs) introduced in Ref. [56], and which was used in Ref. [53] to show that the Gibbs state of 2D and 3D toric code is SRE for all nonzero temperatures. Note that despite the nomenclature, construction of METTSs as introduced in Ref. [56] does not involve minimization of entanglement over all possible decompositions, and is simply an ansatz that is physically motivated (which is why we prefer the nomenclature "CDA over METTSs" for our discussion).

First, let us specialize to zero transverse field. In this case, ρ is clearly an SRE state at any temperature since $\rho \propto \sum_m e^{-\beta E_m}|z_m\rangle\langle z_m|$, where $|z_m\rangle$ denotes a product state in the Z basis and $E_m = \langle z_m|H|z_m\rangle$. To obtain a symmetric convex decomposition, we write

$$\rho = \frac{\sum_m e^{-\beta H/2}|x_m\rangle\langle x_m|e^{-\beta H/2}}{Z} = \sum_m |\psi_m\rangle\langle\psi_m|, \quad (3)$$

where the set $\{|x_m\rangle\}$ corresponds to the complete set of states in the X basis and $|\psi_m\rangle = e^{-\beta H/2}|x_m\rangle/\sqrt{Z}$ is the unnormalized wave function. The states $|\psi_m\rangle$ are clearly symmetric under the Ising symmetry, and their symmetry charge ($= \pm 1$) is determined by the parity of the number of sites in the product state $|x_m\rangle$ where spins point along the negative-x direction. We will now argue that the states $|\psi_m\rangle$ are SRE for $\beta < \beta_c$ and LRE for $\beta \geq \beta_c$. To see this, we first consider the "partition function with respect to $|\psi_m\rangle$" defined as $\mathcal{Z}_m = \langle\psi_m|\psi_m\rangle$ and study its analyticity as a function of β. In this specific example, since the transverse field is set to zero, one finds that for all m, $\mathcal{Z}_m$ is simply proportional to the partition function of the 2D classical Ising model at inverse temperature β, and therefore is nonanalytic across the phase transition. Similarly, the two-point correlation function $\langle\psi_m|Z_iZ_j|\psi_m\rangle/\langle\psi_m|\psi_m\rangle$ is just the two-point spin-spin correlation function in the 2D classical Ising model, which is long-ranged for $\beta \geq \beta_c$ and exponentially decaying for $\beta < \beta_c$. These observations strongly indicate that $|\psi_m\rangle$ is SRE (and correspondingly, ρ is sym-SRE) if and only if $\beta < \beta_c$. Note that the states $|\psi_m\rangle$ are expected to be area-law entangled for all β. This is because one may represent the imaginary time evolution $e^{-\beta H}|m\rangle$ as a tensor network of depth β acting on $|m\rangle$ (which is a product state), which can generate only an area-law worth of entanglement. Further, even the state at $\beta = \infty$ is area-law entangled (which is the ground state of H). Therefore, short-range correlations are strongly suggestive of short-range entanglement.

Now let us consider a nonzero transverse field. To argue that ρ is SRE for any nonzero temperature, we again decompose it as $\rho = \sum_m |\psi_m\rangle\langle\psi_m|$, where $|\psi_m\rangle = e^{-\beta H/2}|z_m\rangle/\sqrt{Z}$. The corresponding $\mathcal{Z}_m = \langle\psi_m|\psi_m\rangle$ can now be expressed in the continuum limit as an imaginary-time path integral $\mathcal{Z}_m \sim \int_{\phi(\tau=0)=\phi(\tau=\beta)=\phi_0} D\phi\, e^{-S}$, where $S = \sum_n \int_{k_x,k_y} |\phi(k_x,k_y,n)|^2(k_x^2+k_y^2+\omega_n^2) + \int_{\tau=0}^{\beta}\int_{x,y}\left(r|\phi|^2+u|\phi|^4\right)$, $\omega_n = 2\pi n/\beta$ are the Matsubara frequencies, and, crucially, the Dirichlet boundary conditions $\phi(x,y,\tau=0) = \phi(x,y,\tau=\beta) = \phi_0(x,y)$ are imposed by the "initial" state $z_m \sim \phi_0(x,y)$. Since $\beta \neq \infty$, the discrete sum over the Matsubara frequencies will be dominated by $\omega_n = 0$, which corresponds to space-time configurations that are translationally invariant along the imaginary-time direction. Furthermore, the Dirichlet boundary conditions imply that there is just one such configuration, namely, $\phi(x,y,\tau) = \phi_0(x,y)$, such that $\mathcal{Z}_m \sim e^{S[\phi_0(x,y)]}$, and thus the fluctuations of ϕ will be completely suppressed at all nonzero temperatures (including at the finite-temperature critical point that corresponds to renormalized $r = 0$). Therefore, we expect that $\mathcal{Z}_m$ will not exhibit singularity across the finite-temperature critical point, which indicates that the states $|\phi_m\rangle$ are SRE.

To argue that ρ is sym-SRE for $\beta < \beta_c$, we now decompose ρ as $\rho = \sum_m |\psi_m\rangle\langle\psi_m|$, where $|\psi_m\rangle = e^{-\beta H/2}|x_m\rangle/\sqrt{Z}$. The corresponding $\mathcal{Z}_m = \langle\psi_m|\psi_m\rangle$ can again be expressed in the continuum limit as an imaginary-time path integral $\mathcal{Z}_m \sim \int D\phi\, e^{-S}$, where $S = \sum_n \int_{k_x,k_y} |\phi(k_x,k_y,n)|^2(k_x^2+k_y^2+\omega_n^2) + \int_{\tau=0}^{\beta}\int_{x,y}\left(r|\phi|^2+u|\phi|^4\right)$. Crucially, since the initial state is now a product state in the X basis, the fields at the two boundaries $\tau = 0, \beta$ are integrated over all possible configurations Again, the path integral will be dominated by $\omega_n = 0$, which implies that the dominant contribution comes only from configurations $\phi(\tau,x,y) = \phi(x,y)$. Therefore, unlike the aforementioned case when the CDA states corresponded to $e^{-\beta H/2}|z_m\rangle/\sqrt{Z}$, here the dominant contribution to $\mathcal{Z}_m$ precisely corresponds to the partition function of the 2D classical Ising model, which is in the paramagnetic phase for $\beta < \beta_c$. The correspondence with the 2D classical Ising model makes physical sense since the universality class of the phase transition at any nonzero temperature is indeed that of the 2D classical Ising model. Therefore, we expect that the states $|\psi_m\rangle = e^{-\beta H/2}|x_m\rangle/\sqrt{Z}$ are SRE for $\beta < \beta_c$ and LRE for $\beta \geq \beta_c$. Correspondingly, we expect that the Gibbs state is sym-SRE for $\beta < \beta_c$ and sym-LRE for $\beta > \beta_c$.

To summarize, we have provided arguments that the Gibbs state of a transverse-field Ising model is an SRE state at any nonzero temperature and is a sym-SRE state only for $\beta < \beta_c$. Therefore, we expect that it undergoes a separability transition as a function of temperature if one is allowed to expand the density matrix only as a convex sum of symmetric states. We expect similar statements for other models that exhibit a finite-temperature zero-form symmetry-breaking phase transition. In the following sections, we use logic broadly similar to that in this example, with the primary focus on topological phases of matter subjected to local decoherence. Specifically, we write $\rho = \Gamma\Gamma^\dagger$ and use the following CDA:

$$\rho = \sum_m \Gamma|m\rangle\langle m|\Gamma^\dagger = \sum_m |\psi_m\rangle\langle\psi_m| = \sum_m p_m|\tilde{\psi}_m\rangle\langle\tilde{\psi}_m|, \tag{4}$$

where $|\psi_m\rangle = \Gamma|m\rangle$, $p_m = \langle m|\Gamma^\dagger\Gamma|m\rangle = \langle\psi_m|\psi_m\rangle$, and $|\tilde{\psi}_m\rangle = |\psi_m\rangle/\sqrt{\langle\psi_m|\psi_m\rangle}$ are normalized versions of $|\psi_m\rangle$. We note that here Γ is not unique (note that Γ is not restricted to being a square matrix; see, e.g., Ref. [17]), and the CDA in Eq. (3) corresponds to our choosing $\Gamma = \rho^{1/2}$ for the Gibbs state ρ. We will sometimes call states $\{|\psi_m\rangle\}$ that enter a particular CDA "CDA states." We further note that, in general, we do not know how to find the matrix Γ that is "optimal," i.e., a matrix Γ that guarantees that the states $\Gamma|m\rangle$ are SRE whenever ρ is SRE. However, as we will see in the rest of the paper, for a large class of problems, after a judicious choice of the basis $\{|m\rangle\}$, the decomposition in the form of Eq. (4) turns out to be optimal.

IV. SEPARABILITY TRANSITIONS IN SPT STATES

The fundamental property of a nontrivial SPT phase is that it cannot be prepared with use of a short-depth circuit consisting of local, symmetric, unitary gates [9–12]. Therefore, it is natural to ask whether if an SPT phase is subjected to local decoherence whether the resulting mixed state is sym-SRE, i.e., can it be expressed as a convex sum of symmetric, SRE pure states? This is clearly a very challenging question for many-body mixed states, since to our knowledge, there does not exist an easily calculable measure of mixed-state entanglement that is nonzero if and only if the mixed state is unentangled [57] (if such a measure did exist, then it would be useful to study its universal, long-distance component, which is similar to the topological part of negativity [19,53,58]). As hinted at in Sec. I, our general scheme will be to first seek sufficient conditions that make a given mixed state sym-LRE (i.e., not sym-SRE). We will do this by decomposing the decohered state into its distinct symmetry sectors as $\rho = \sum_Q \rho_Q$, with ρ_Q the projection of the density matrix onto symmetry charge Q, and then examining whether the assumption of each ρ_Q being SRE leads to a contradiction. If we are unable to find an obvious contradiction, we will attempt to use the decomposition outlined in Eq. (4) to express ρ as a convex sum of sym-SRE states. In either of these steps, we will exploit the connection between local and thermal decoherence for cluster states that was briefly mentioned in Ref. [17], and which is described in the next subsection in detail.

A. A relation between local and thermal decoherence

Systems with intrinsic topological order typically behave rather differently when they are coupled to a thermal bath compared with when they are subjected to decoherence induced by a short-depth quantum channel. For example, when 2D and 3D toric codes are embedded in a thermal bath, so that the mixed state is described by a Gibbs state, the topological order is lost at any nonzero temperature [2,53,59,60]. In contrast, when 2D or 3D toric codes are subjected to local decoherence, the error-threshold theorems [61–66] imply that the mixed-state topological order is stable up to a nonzero decoherence rate [17–20,59,67]. Given this, it is interesting to ask if there exist situations where a local short-depth channel maps a ground state to a Gibbs state. Here we show that this is indeed the case if the corresponding Hamiltonian satisfies the following properties:

(1) It can be written as a sum of local commuting terms where each of them squares to identity:

$$H = \sum_j h_j, \quad [h_j, h_k] = 0, \quad h_j^2 = I, \text{ for all } j, k. \tag{5}$$

(2) There exists a local unitary O_j that anticommutes (commutes) with h_k if $j = k$ $(j \neq k)$:

$$\begin{aligned} O_j h_j O_j^\dagger &= -h_j, \\ O_j h_k O_j^\dagger &= h_k \ (j \neq k). \end{aligned} \tag{6}$$

Specifically, if we denore the total system size as N, the channel $\mathcal{E} = \mathcal{E}_1 \circ \cdots \circ \mathcal{E}_N$ with

$$\mathcal{E}_j[\rho] = (1-p)\rho + pO_j\rho O_j^\dagger \tag{7}$$

maps the ground-state density matrix ρ_0 to a Gibbs state for H.

To verify the claim, we first note that Eq. (5) implies that ρ_0 can be written as the product of the projectors on all sites $\rho_0 = 1/2^N \prod_j (I - h_j)$. With use of Eq. (6), it is straightforward to show that $\mathcal{E}_j[\rho_0] = 1/2^N[I - (1 - 2p)h_j] \prod_{k\neq j}(I - h_k)$. It then follows that the composition of $\mathcal{E}_j$ on all sites gives

$$\mathcal{E}[\rho_0] = \frac{1}{2^N} \prod_j [I - (1 - 2p)h_j]. \tag{8}$$

Since $h_j^2 = I$, which implies $e^{-\beta h_j} = \cosh(\beta)I - \sinh(\beta) h_j$, one may now exponentiate Eq. (8) to obtain $\mathcal{E}[\rho_0] = (1/\mathcal{Z})e^{-\beta H}$, where $\tanh\beta = (1 - 2p)$ and $\mathcal{Z} = \mathrm{tr}(e^{-\beta H})$. In Sec. VIII, we also discuss a $\mathbb{Z}_N$ generalization of this construction. For the rest of the paper, the aforementioned $\mathbb{Z}_2$ version will suffice. In Secs. IV B–IV D, we exploit the connection between local and thermal decoherence to study decoherence-induced separability transitions for the cluster states in various dimensions. In Sec. IV E, we briefly discuss a couple of examples where the pure state is protected by a single zero-form symmetry.

B. One-dimensional cluster state

The Hamiltonian for the 1D cluster state is given by

$$\begin{aligned} H &= -\sum_{j=1}^{N} (Z_{b,j-1}X_{a,j}Z_{b,j} + Z_{a,j}X_{b,j}Z_{a,j+1}) \\ &= \sum_{j=1}^{N} h_{a,j} + h_{b,j}, \end{aligned} \tag{9}$$

where a and b denote the two sublattices of the 1D chain [see Fig. 1(a)]. H has a global $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry generated by

$$U_a = \prod_j X_{a,j}, \quad U_b = \prod_j X_{b,j}. \tag{10}$$

We assume periodic boundary conditions, so that there is a unique, symmetric, ground state of H that is separated from the rest of the spectrum with a finite gap. It is obvious that H satisfies Eq. (5). To satisfy Eq. (6), we choose Kraus operators $O_{a/b,j} = Z_{a/b,j}$. Therefore, under the composition of the channel $\mathcal{E}_{a/b,j}[\rho] = (1 - p_{a/b})\rho + p_{a/b}Z_{a/b,j}\rho Z_{a/b,j}$ on all sites, the pure-state density matrix becomes

$$\begin{aligned} \rho(p_a, p_b) &= \left(\frac{1}{\mathcal{Z}_a} e^{-\beta_a \sum_j h_{a,j}}\right)\left(\frac{1}{\mathcal{Z}_b} e^{-\beta_b \sum_j h_{b,j}}\right) \\ &= \rho_a(p_a)\rho_b(p_b), \end{aligned} \tag{11}$$

with $\tanh\beta_{a/b} = (1 - 2p_{a/b})$ and $\mathcal{Z}_{a/b} = \mathrm{tr}(e^{-\beta_{a/b}\sum_j h_{a/b,j}})$. In the following, we suppress the arguments p_a and p_b in $\rho_a(p_a)$ and $\rho_b(p_b)$ if there is no ambiguity. Note that ρ_a and ρ_b commute with each other. To decompose ρ as a convex sum of symmetric states, we write $\rho = \sum_{Q_a,Q_b} \rho_{Q_a,Q_b}$, where each ρ_{Q_a,Q_b} is an unnormalized density matrix that carries exact symmetry: $U_a\,\rho_{Q_a,Q_b} = (-1)^{Q_a}\rho_{Q_a,Q_b}$, $U_b\,\rho_{Q_a,Q_b} = (-1)^{Q_b}\rho_{Q_a,Q_b}$, with $Q_a = 0, 1$ and $Q_b = 0, 1$, so the sum over Q_a, Q_b contains four terms. The explicit expression for ρ_{Q_a,Q_b} is given as $\rho_{Q_a,Q_b} = \rho_{Q_a}\rho_{Q_b}$, where $\rho_{Q_a} = \rho_a P_{Q_a}$ and $\rho_{Q_b} = \rho_b P_{Q_b}$, and $P_{Q_{a/b}} = (I + (-1)^{Q_{a/b}}U_{a/b})/2$ are projectors. Note that the probability for a given sector (Q_a, Q_b) is given by $\mathrm{tr}\left(\rho_{Q_a,Q_b}\right)$, which can be used to obtain the normalized density matrix $\tilde{\rho}_{Q_a,Q_b}$ for a sector (Q_a, Q_b) as $\tilde{\rho}_{Q_a,Q_b} = \rho_{Q_a,Q_b}/\mathrm{tr}\left(\rho_{Q_a,Q_b}\right)$.

To discuss whether the decohered mixed state ρ is trivial on the basis of our definition of a sym-SRE mixed state, we start by considering the special case $p_a > 0$, $p_b = 0$, i.e., the mixed state obtained by application of the aforementioned quantum channel only on sublattice a. This case was studied in detail in Ref. [27] from a different perspective and is an example of an "average-SPT-order phase" [26,27,29,30]. In particular, it was shown in Ref. [27] that this mixed state cannot be purified to an SRE pure state with use of a finite-depth local quantum channel. As discussed in Sec. II, our definition of an SRE mixed state is a bit different (namely, whether a mixed state can be written as a convex sum of SRE pure states), and therefore it is worth examining whether this state continues to remain an LRE mixed state with our definition.

When $p_a > 0$, $p_b = 0$, only the sector corresponding to $Q_b = 0$ survives, and in this sector, $\rho_{Q_a,Q_b} \propto \prod_j (I - h_{b,j})e^{-\beta_a \sum_j h_{a,j}} P_{Q_a}$. We now provide two separate arguments that show that ρ_{Q_a,Q_b} is a sym-LRE (i.e., not a sym-SRE) mixed state when $p_a > 0$, $p_b = 0$.

1. First argument

We want to show that $\rho_{Q_a,Q_b} \propto \prod_j (I - h_{b,j})e^{-\beta_a \sum_j h_{a,j}} P_{Q_a}$ cannot be written as $\sum_m p_m|\psi_m\rangle\langle\psi_m|$, where $|\psi_m\rangle$ are SRE states that can be prepared via a short-depth circuit consisting of symmetric, local gates. We use the result in Ref. [39], which shows that for an area law–entangled state in one dimension (which we take to be $|\psi_m\rangle$) that is symmetric under an Ising symmetry (which we take here to be $U_a = \prod_j X_{a,j}$), the order and disorder parameters cannot both vanish simultaneously. Note that we are assuming that $|\psi_m\rangle$ has an area-law entanglement, as otherwise it is certainly not SRE and there is nothing more to prove.

Therefore, following the results in Ref. [39], $|\psi_m\rangle$ must either (1) have a nonzero order parameter corresponding to the symmetry U_a, i.e., $\langle\psi_m|\tilde{Z}_j\tilde{Z}_k|\psi_m\rangle \neq 0$, where $|j - k| \gg 1$ and $\tilde{Z}$ is an operator that is odd under U_a, e.g., $\tilde{Z}_i = Z_{a,i}$, or (2) have a nonzero "disorder parameter" corresponding to the symmetry U_a, i.e., $\langle\psi_m|O_L\left(\prod_{l=j}^{k} X_{a,l}\right)O_R|\psi_m\rangle \neq 0$, where $|j - k| \gg 1$, and

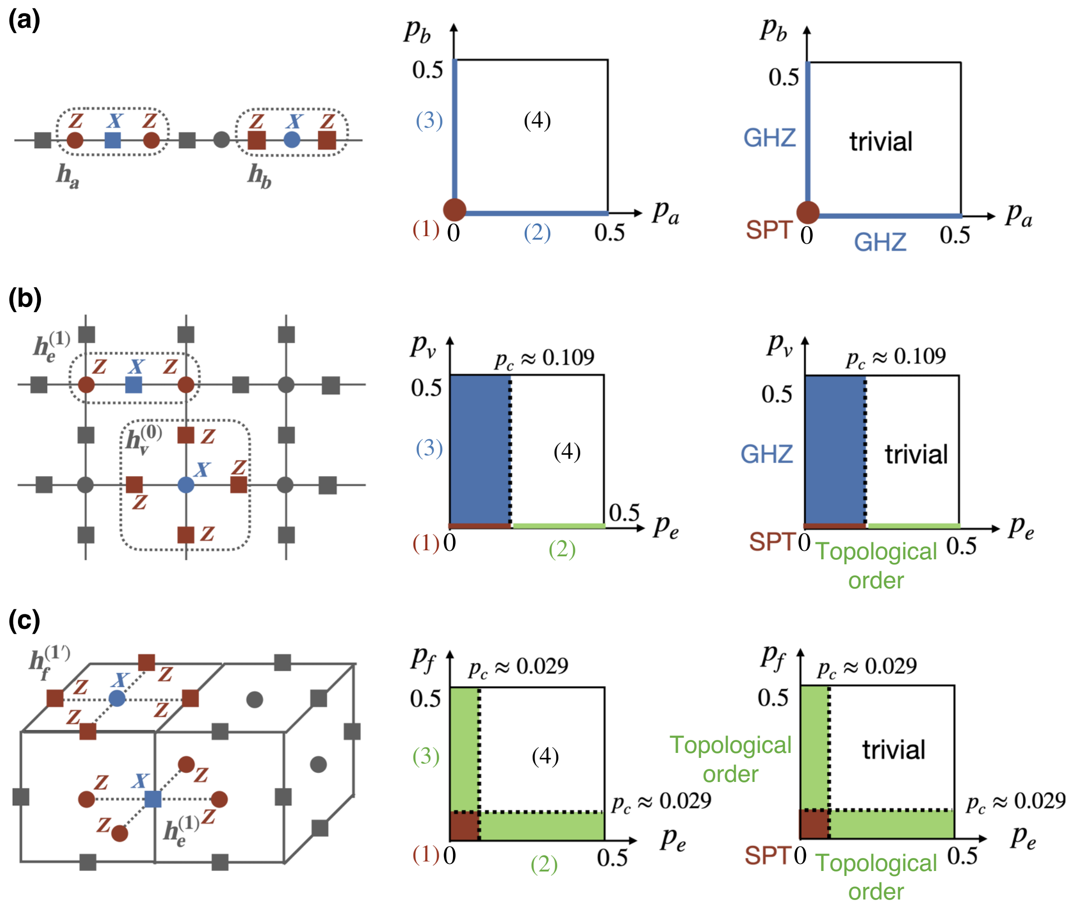

FIG. 1. Cluster states under decoherence in (a) one dimension, (b) two dimensions, and (c) three dimensions. In (a)–(c), the diagram on the left depicts the Hamiltonian of cluster states. The diagram in the middle in (a)–(c) divides the decohered mixed state as a function of error rates into several regimes that have qualitatively different behaviors. The white regions [region (4)] in the three phase diagrams denote phases where the mixed state is "sym-SRE" ("trivial"), i.e., it is expressible as a convex sum of symmetric, short-range-entangled pure states. In contrast, the colored regions or lines [regions (1), (2), and (3)] denote phases where such a decomposition is not possible ("sym-LRE"). There can be phase transitions from one kind of sym-LRE phase to a different kind of sym-LRE phase as depicted by different colors. The phase diagram is obtained by our calculating objects of the form $[\langle O\rangle^2] = \sum_Q P(Q)\left(\langle O\rangle_Q\right)^2$, where O corresponds to an appropriate observable that characterizes symmetry-enforced long-range entanglement and $P(Q)$ is the probability for obtaining the symmetry charge q. $p_c \approx 0.109$ in (b) corresponds to the ferromagnetic to paramagnetic phase transition in the 2D random-bond Ising model along the Nishimori line, while $p_c \approx 0.029$ in (c) corresponds to the critical point in the 3D random-plaquette gauge model along the Nishimori line. The diagram on the right in (a)–(c) shows the phase diagram obtained by our expressing ρ as a convex sum of symmetric states, where each symmetric state $|\psi_m\rangle = \rho^{1/2}|m\rangle$, with $|m\rangle$ the product state in the Pauli X basis. See the main text for more details.

O_L and O_R are operators localized close to site j and site k, respectively, that are *either* both even or both odd under U_a. In case (1), the system has a long-range GHZ-type order since the state $|\psi_m\rangle$ is symmetric under U_a. In case (2), we now argue that the system has an SPT order.

For $\langle\psi_m|O_L\left(\prod_{l=j}^k X_{a,l}\right)O_R|\psi_m\rangle$ to be nonzero, the operator $O_L \otimes O_R$ must carry no charge under the symmetry U_b as $|\psi_m\rangle$ is an eigenstate of U_b. Therefore, there are two disjoint possibilities for the operators O_L and O_R: they are either both charged under the symmetry U_b or neither of them is charged under U_b. If neither of them is charged under U_b, then $\langle\psi_m|O_L\left(\prod_{l=j}^k X_{a,l}\right)O_R|\psi_m\rangle$ must vanish. This is because $|\psi_m\rangle$ is an eigenstate of the string operator $S_b(l,r) = Z_{a,l}\left(\prod_{j=l}^r X_{b,j}\right)Z_{a,r+1}$ [this follows from the fact that $\rho_{Q_a,Q_b} \propto \prod_j (I - h_{b,j})$], which anticommutes with $O_L\left(\prod_{l=j}^k X_{a,l}\right)O_R$ for an appropriate choice of (l,r) whenever neither O_L nor O_R is charged under U_b. As a consequence, for $\langle\psi_m|O_L\left(\prod_{l=j}^k X_{a,l}\right)O_R|\psi_m\rangle$ to be nonzero, O_L and O_R must both be odd under U_b. If this is so, then the disorder parameter precisely corresponds to one of the two SPT string order parameters, namely, $S_a(j,k) = Z_{b,j-1}\left(\prod_{l=j}^k X_{a,l}\right)Z_{b,k+1}$ up to finite-depth symmetric unitary transformation. At the same time, the other SPT string order parameter $\langle\psi_m|S_b(j,k)|\psi_m\rangle$ is also nonzero [due to $\rho_{Q_a,Q_b} \propto \prod_j (I - h_{b,j})$], and therefore we arrive at the conclusion that in case (2), $|\psi_m\rangle$ must possess nontrivial SPT order since the string order parameters on both sublattices are nonzero. Therefore, in either case (1) or case (2), $|\psi_m\rangle$ cannot be prepared by a short-depth circuit composed of local gates that respect both U_a and U_b, starting with a symmetric product state.

2. Second argument

This argument is essentially the same as the argument introduced in Ref. [38] to show that the circuit depth of various states with a nontrivial string order parameter cannot be a system-size-independent constant due to locality/the Lieb-Robinson bound [36,37]. Again, recall that we want to show that $\rho_{Q_a,Q_b} \propto \prod_j (I - h_{b,j}) e^{-\beta_a \sum_j h_{a,j}} P_{Q_a}$ cannot be written as $\sum_m p_m|\psi_m\rangle\langle\psi_m|$, where $|\psi_m\rangle$ are SRE. Since ρ_{Q_a,Q_b} carries an exact symmetry charge of U_a, U_b, so do each of the pure states $|\psi_m\rangle$. As discussed above, the expectation value of the string order parameter $S_b(j,k) = \prod_{l=j}^{k}(-h_{b,l}) = Z_{a,j}\left(\prod_{l=j}^{k} X_{b,l}\right) Z_{a,k+1}$ is unity with respect to ρ_{Q_a,Q_b}, which implies that its expectation value is also unity with respect to each of the states $|\psi_m\rangle$. Let us assume that $|\psi_m\rangle$ can be obtained from a symmetric product state (i.e., an eigenstate of Pauli X on all sites) that we denote as $|x_{\mathbf{a},\mathbf{b}}\rangle = \otimes_j |x_{a,j}, x_{b,j}\rangle$, i.e., $|\psi_m\rangle = V|x_{\mathbf{a},\mathbf{b}}\rangle$ (here $x_{a/b,j} = \pm 1$ are chosen so as to satisfy the symmetry $U_{a/b}|\psi_m\rangle = (-1)^{Q_{a/b}}|\psi_m\rangle$). Note that $|x_{\mathbf{a},\mathbf{b}}\rangle$ satisfy not only the global symmetry $U_{a/b}$ but the "local" ones as well, i.e., $\prod_{j\in l} X_j |x_{\mathbf{a},\mathbf{b}}\rangle \propto |x_{\mathbf{a},\mathbf{b}}\rangle$ for any string l. Since each end point of S_b is charged under U_a (i.e., $U_a Z_{a,j/k+1} U_a^\dagger = -Z_{a,j/k+1}$), the local symmetry of $|x_{\mathbf{a},\mathbf{b}}\rangle$ implies $\langle x_{\mathbf{a},\mathbf{b}}|S_b(j,k)|x_{\mathbf{a},\mathbf{b}}\rangle = 0$. Moreover, since V is a finite-depth unitary, the operator $V^\dagger S_b(j,k) V$ is still a string operator with each "end-point operator" $V^\dagger Z_{a,j/k+1} V$ a sum of local operators (due to the locality of V) that are charged under U_a (due to V being a symmetric unitary, i.e., $[V, U_a] = [V, U_b] = 0$). Because of these properties, the expectation value $\langle x_{\mathbf{a},\mathbf{b}}|V^\dagger S_b(j,k) V|x_{\mathbf{a},\mathbf{b}}\rangle$ will be identically zero. However, $\langle x_{\mathbf{a},\mathbf{b}}|V^\dagger S_b(j,k) V|x_{\mathbf{a},\mathbf{b}}\rangle$ is nothing but $\langle\psi_m|S_b(j,k)|\psi_m\rangle$, which is unity, as discussed above. Therefore, we arrive at a contradiction. This implies that our assumption that $|\psi_m\rangle$ is a symmetric SRE state must be incorrect.

We now discuss the general case of both p_a and p_b being nonzero. On the basis of our discussion above, it is instructive to evaluate the string order parameter with respect to each ρ_{Q_a,Q_b}, i.e., $\mathrm{tr}(\rho_{Q_a,Q_b} S_{a/b})/\mathrm{tr}(\rho_{Q_a,Q_b})$. One finds (see Appendix A) that both string order parameters can be mapped to two-point correlation functions of spins in the 1D classical Ising model at nonzero temperature and hence they decay exponentially with the length of the strings. This result merely implies that the corresponding mixed state $\rho = \sum_{Q_a,Q_b} \rho_{Q_a,Q_b}$ does not satisfy the aforementioned sufficient condition for being a nontrivial sym-SRE state, and does not guarantee that ρ must be trivial. We now use the CDA in Eq. (4) to argue that ρ is indeed sym-SRE. In particular, we choose $\Gamma = \rho^{1/2}$ so that $\rho = \sum_m \Gamma|m\rangle\langle m|\Gamma^\dagger = \sum_m |\psi_m\rangle\langle\psi_m|$, with $|\psi_m\rangle \propto e^{-(\beta_a \sum_j h_{a,j} + \beta_b \sum_j h_{b,j})/2}|m\rangle$. To ensure that each $|\psi_m\rangle$ respects the global $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, we choose the set $\{|m\rangle\} = \{|x_{\mathbf{a}}, x_{\mathbf{b}}\rangle_m\}$. When $\beta_a = \beta_b = 0$, $|\psi_m\rangle = |x_{\mathbf{a}}, x_{\mathbf{b}}\rangle_m$ is a product state. To check whether $|\psi_m\rangle$ remains SRE for any noninfinite β_a and β_b, let us consider the "partition function with respect to $|\psi_m\rangle$"

$$\mathcal{Z}_m(\beta_a, \beta_b) = \langle\psi_m|\psi_m\rangle \tag{12}$$

as a function of β. As β_a and β_b are increased from zero, if the state $|\psi_m\rangle$ becomes long-range entangled, one expects that it will lead to a nonanalytic behavior of $\mathcal{Z}_m(\beta)$ as a function of β_a and β_b. The calculation for $\mathcal{Z}_m(\beta) = \langle x_{\mathbf{a}}, x_{\mathbf{b}}|\rho|x_{\mathbf{a}}, x_{\mathbf{b}}\rangle$ is quite similar to the one for $\mathrm{tr}(\rho_{Q_a,Q_b})$ detailed in Appendix A, and one finds that $\mathcal{Z}_m(\beta)$ is proportional to the product of two partition functions for the 1D classical Ising model at inverse temperatures β_a and β_b. Therefore, we expect that $|\psi_m(\beta)\rangle$ remains an SRE state as long as both $\beta_a < \infty$ and $\beta_b < \infty$, which confirms our expectation that ρ is sym-SRE for noninfinite β_a and β_b (i.e. $p_a, p_b > 0$).

One can also compute the string order parameters S_a (S_b) for $|\psi_m\rangle$ and show its equivalence to $\langle z_j z_k\rangle_{\text{1D Ising}}$ at inverse temperature β_a (β_b). Therefore, $|\psi_m\rangle$ does not develop string order as long as $\beta_{a/b} < \infty$. The triviality of $|\psi_m\rangle$ is also manifested by the *nonzero* expectation value of the disorder operator $U_{a/b}(k,j) = \prod_{l=j}^{k} X_{a/b,l}$. For example, consider the expectation value of the disorder operator on the a sublattice: $\langle U_a(k,j)\rangle_m = \langle\psi_m|U_a(k,j)|\psi_m\rangle/\langle\psi_m|\psi_m\rangle$. Using the fact that the only terms in $e^{-(\beta_a \sum_j h_{a,j} + \beta_b \sum_j h_{b,j})/2}$ that anticommute with $U_a(k,j)$ are $h_{b,j-1}$ and $h_{b,k}$, we find that $\langle U_a(k,j)\rangle_m = (\prod_{l=j}^{k} x_l)\sec h^2(\beta_a)$, which is nonvanishing except for $\beta_a = \infty$. This is, of course, expected on the basis of the result in Ref. [39], since $|\psi_m\rangle$ does not have any GHZ-type order. The result for $U_b(k,j)$ is similar.

It is also instructive to apply the aforementioned convex decomposition to the case $\beta_b = \infty, \beta_a \neq \infty$, i.e., the above-discussed case of "average SPT order." In this case we find that the corresponding state $|\psi_m\rangle$ develops GHZ-type long-range entanglement. To see this, one can rewrite $|\psi_m\rangle$ as $|\psi_m\rangle \sim e^{-\beta_a \sum_k h_{a,j}/2}|\chi_m\rangle$, where $|\chi_m\rangle \sim \prod_j (I - h_{b,j})|m\rangle = |x_{\mathbf{b}}\rangle \otimes \prod_j (I - x_{b,j} Z_{a,j} Z_{a,j+1})|x_{\mathbf{a}}\rangle$ exhibits GHZ-type long-range entanglement characterized by $|\langle\chi_m|Z_{a,j} Z_{a,k}|\chi_m\rangle| = 1$. Using the fact that the only terms in $e^{-\beta_a \sum_j h_{a,j}/2}$ that anticommute with $Z_{a,j} Z_{a,k}$ are $h_{a,j}$ and $h_{a,k}$, one finds that $|\langle\psi_m|Z_{a,j} Z_{a,k}|\psi_m\rangle| = \sec h^2(\beta_a)|\langle\chi_m|Z_{a,j} Z_{a,k}|\chi_m\rangle| = \sec h^2(\beta_a)$, which is nonvanishing except for $\beta_a = \infty$.

To summarize the results in this subsection, the decohered state ρ as a function of p_a and p_b can be divided into four regimes (see Fig. 1):

(1) $p_a = p_b = 0$: $\mathrm{tr}(\rho_{Q_a,Q_b} S_a(j,k)) = 1$ (in the $Q_a = 0$ sector) and $\mathrm{tr}(\rho_{Q_a,Q_b} S_b(j,k)) = 1$ (in the $Q_b = 0$ sector). This is just the pure-state SPT-order phase.

(2) $p_a > 0$ and $p_b = 0$: $\mathrm{tr}(\rho_{Q_a,Q_b} S_a(j,k))$ decays exponentially with $|j-k|$ and $\mathrm{tr}(\rho_{Q_a,Q_b} S_b(j,k)) = 1$ (in the $Q_b = 0$ sector). This regime is sym-LRE, i.e., a nontrivial mixed state, in agreement with the nontrivial "average-SPT-order phase" discussed in Ref. [27].

(3) $p_a = 0$ and $p_b > 0$: this is similar to case (2) with $a \leftrightarrow b$ and is again a sym-LRE state.

(4) $p_a, p_b > 0$: both $\mathrm{tr}(\rho_{Q_a,Q_b} S_a(j,k))$ and $\mathrm{tr}(\rho_{Q_a,Q_b} S_b(j,k))$ decay exponentially with $|j-k|$. This is a sym-SRE state.

On the basis of our discussion above, we also provide one possible "phase diagram" to express ρ as a convex sum of symmetric states using CDA states $|\psi_m\rangle = \rho^{1/2}|x_{\mathbf{a}}, x_{\mathbf{b}}\rangle$, as summarized on the right in Fig. 1(a). Note that the boundary of the phase diagram obtained with the CDA matches the boundary of regimes (1)–(4), and therefore the CDA is optimal in this sense. However, it is worth noting that the decomposition we chose is just one possible choice, and the label "GHZ" on the x and y axes in plot on the right in Fig. 1(a) is tied to this choice. One may also chose to expand ρ as a convex sum of SPT states. Therefore, the result that is independent of any specific choice of CDA is that regime (4) is sym-SRE, while regimes (1)–(3) are sym-LRE.

C. Two-dimensional cluster state

The 2D cluster state Hamiltonian $H_{\text{2D cluster}}$ is given by

$$\begin{aligned} H_{\text{2D cluster}} &= -\sum_v X_v \left(\prod_{e\ni v} Z_e\right) - \sum_e X_e (\prod_{v\in e} Z_v) \\ &= \sum_v h_v + \sum_e h_e. \end{aligned} \tag{13}$$

Here the Hilbert space consists of qubits residing on both the vertices v and the edges e of a 2D square lattice [see Fig. 1(b)]. The Hamiltonian has both a zero-form symmetry $Z_2^{(0)}$ and a one-form symmetry $Z_2^{(1)}$ with the corresponding generators

$$U^{(0)} = \prod_v X_v, \quad U_p^{(1)} = \prod_{e\in\partial p} X_e, \tag{14}$$

where p labels the plaquette on the lattice and ∂p is the boundary of p. We assume periodic boundary conditions, so that H has a unique, symmetric, gapped ground state. Using Eqs. (5) and (6), if one subjects the ground state of $H_{\text{2D cluster}}$ to Kraus operators $O_{v/e} = Z_{v/e}$ with respective probabilities $p_{v/e}$, the resulting decohered density matrix is given as $\rho = (1/Z)e^{-\left(\beta_v \sum_v h_v^{(0)} + \beta_e \sum_e h_e^{(1)}\right)}$, with $\tanh \beta_{e/v} = (1-2p_{e/v})$.

Let us decompose ρ as a convex sum of symmetric states by writing $\rho = \sum_{Q^{(0)},Q^{(1)}} \rho_{Q^{(0)},Q^{(1)}}$, where each $\rho_{Q^{(0)},Q^{(1)}}$ carries the exact symmetry: $U^{(0)}\rho_{Q^{(0)},Q^{(1)}} = (-1)^{Q^{(0)}}\rho_{Q^{(0)},Q^{(1)}}$, $U_p^{(1)}\rho_{Q^{(0)},Q^{(1)}} = (-1)^{Q_p^{(0)}}\rho_{Q^{(0)},Q^{(1)}}$. Here, the one-form symmetry charge is labeled by the set $Q^{(1)} = \{Q_p^{(1)}\}$, with $Q_p^{(1)} = 0, 1$ defined on each plaquette p. Crucially, the number of one-form symmetry sectors grows exponentially as a function of the system size, and this implies that the probability for a given sector $(Q^{(0)}, Q^{(1)})$, i.e., $\mathrm{tr}(\rho_{Q^{(0)},Q^{(1)}})$, is exponentially small in general. It follows that even if there exists some $\rho_{Q^{(0)},Q^{(1)}}$ that is not sym-SRE, the decohered state ρ may still be well approximated by a sym-SRE mixed state as long as the total probability corresponding to the nontrivial sectors is exponentially small. Therefore, the notion of ρ being sym-SRE must take into account the probability for each symmetry sector, and can be made precise only in a statistical sense (a similar situation arises for a certain nonoptimal decomposition for decohered toric code [17]. We will return to this point in detail below. For now, let us focus on the physical observables in each symmetry sector.

The observables that characterize the 2D cluster ground state are the expectation value of the membrane operator $M_S = \prod_{v\in S}(-h_v^{(0)})$, with S a surface (for simplicity, we assume that the boundary ∂S of this surface is contractible), and the string operator $S_C = \prod_{e\in C}(-h_e^{(1)})$, with C a curve (the expectation value of either of these operators equals unity in the 2D cluster ground state). To detect whether $\rho_{Q^{(0)},Q^{(1)}}$ is sym-SRE, i.e., it can be expanded as a convex sum of pure SRE states that each carries a definite symmetry charge $(Q^{(0)}, Q^{(1)})$, it is instructive to calculate the expectation value of these operators with respect to $\rho_{Q^{(0)},Q^{(1)}}$, i.e., $\mathrm{tr}(\rho_{Q^{(0)},Q^{(1)}} M_S)/\mathrm{tr}(\rho_{Q^{(0)},Q^{(1)}})$ and $\mathrm{tr}(\rho_{Q^{(0)},Q^{(1)}} S_C)/\mathrm{tr}(\rho_{Q^{(0)},Q^{(1)}})$. To proceed, we first compute the denominator in these expressions, i.e., $\mathrm{tr}(\rho_{Q^{(0)},Q^{(1)}})$. Similarly to the 1D cluster state, this can be easily done by insertion of the complete basis $\{|x_{\mathbf{e,v}}\rangle\}$ and $\{|z_{\mathbf{e,v}}\rangle\}$, where $|x_{\mathbf{e,v}}\rangle = \otimes_{e,v}|x_e, x_v\rangle$ and $|z_{\mathbf{e,v}}\rangle = \otimes_{e,v}|z_e, z_v\rangle$ denote the product state in the Pauli X basis and the Pauli Z basis, respectively. Following a calculation quite similar to that in the 1D cluster state, one finds that $\mathrm{tr}(\rho_{Q^{(0)},Q^{(1)}}) \propto \sum_{x_{\mathbf{v}}\in Q^{(0)}} \mathcal{Z}_{\text{2D gauge},x_{\mathbf{v}}} \sum_{x_{\mathbf{e}}\in Q^{(1)}} \mathcal{Z}_{\text{2D Ising},x_{\mathbf{e}}}$. Here $\mathcal{Z}_{\text{2D gauge},x_{\mathbf{v}}} = \sum_{z_{\mathbf{e}}} e^{\beta_v \sum_v x_v (\prod_{e\ni v} z_e)}$ is the partition function of the 2D Ising gauge theory with the sign of interaction on each vertex given by $x_{\mathbf{v}}$, while $\mathcal{Z}_{\text{2D Ising},x_{\mathbf{e}}} = \sum_{z_{\mathbf{v}}} e^{\beta_e \sum_e x_e (\prod_{v\in e} z_v)}$ is the partition function of the 2D Ising model with the sign of Ising interaction given by $x_{\mathbf{e}}$. In the summation, the notation $x_{\mathbf{v}} \in Q^{(0)}$ denotes all possible $x_{\mathbf{v}}$ that satisfy $\prod_v x_{\mathbf{v}} = (-1)^{Q^{(0)}}$, while $x_{\mathbf{e}} \in Q^{(1)}$ denotes all possible $x_{\mathbf{e}}$ that satisfy $\prod_{e\in\partial p} x_{\mathbf{e}} = (-1)^{Q_p^{(1)}}$ for all p. For a system with periodic boundary conditions, all possible $x_{\mathbf{v}} \in Q^{(0)}(x_{\mathbf{e}} \in Q^{(1)})$ can be reached by the transformation $x_v \to x_v \prod_{e\ni v} \sigma_e, \sigma_e = \pm 1$ ($x_e \to x_e \prod_{v\in e} s_v, s_v = \pm 1$). One may verify that $\mathcal{Z}_{\text{2D gauge},x_{\mathbf{v}}}$ ($\mathcal{Z}_{\text{2D Ising},x_{\mathbf{e}}}$) is invariant under the aforementioned transformation by changing the dummy variables $z_e \to \sigma_e z_e$ ($z_v \to s_v z_v$). It follows that $\mathcal{Z}_{\text{2D gauge},x_{\mathbf{v}}}$ ($\mathcal{Z}_{\text{2D Ising},x_{\mathbf{e}}}$) is a function of only the charge $Q^{(0)}(Q^{(1)})$, and therefore we label it as $\mathcal{Z}_{\text{2D gauge},Q^{(0)}}$ ($\mathcal{Z}_{\text{2D Ising},Q^{(1)}}$). Therefore, $\text{tr}(\rho_{Q^{(0)},Q^{(1)}}) \propto \mathcal{Z}_{\text{2D gauge},Q^{(0)}}\mathcal{Z}_{\text{2D Ising},Q^{(1)}}$ [68].

One may similarly compute $\text{tr}(\rho_{Q^{(0)},Q^{(1)}}M_S)$ and tr $(\rho_{Q^{(0)},Q^{(1)}}S_C)$, the numerators in the expectation value for the membrane and the string operators. Let us first consider the membrane order parameter in the sector (Q_0, Q_1), which we denote as $\langle M_S\rangle_{Q_0,Q_1}$. One finds

$$\begin{aligned}\langle M_S\rangle_{Q_0,Q_1} &= \frac{\text{tr}(\rho_{Q^{(0)},Q^{(1)}}M_S)}{\text{tr}(\rho_{Q^{(0)},Q^{(1)}})}\\ &= \frac{\sum_{z_{\mathbf{e}}}\left(\prod_{v\in S}x_v\prod_{e\in\partial S}z_e\right)e^{\beta_v\sum_v x_v\left(\prod_{e\ni v}z_e\right)}}{\mathcal{Z}_{\text{2D gauge},Q^{(0)}}}\Bigg|_{x_{\mathbf{v}}\in Q^{(0)}}\\ &= \left(\prod_{v\in S}x_v\right)\langle W_{\partial S}\rangle_{\text{2D gauge},x_{\mathbf{v}}}\Big|_{x_{\mathbf{v}}\in Q^{(0)}}\\ &\sim e^{-\kappa A(S)} \quad \text{for } \beta_v < \infty,\end{aligned} \tag{15}$$

where $\langle W_{\partial S}\rangle_{\text{2D gauge},x_{\mathbf{v}}}$ is the expectation value of the Wilson loop operator along the curve ∂S for the 2D Ising gauge theory with interaction $x_{\mathbf{v}}$ while $A(S)$ is the area enclosed by the surface S. The area law follows because the 2D Ising gauge theory is confining at any nonzero temperature. We conclude that $\rho_{Q^{(0)},Q^{(1)}}$ has no membrane order as long as $p_v > 0$.

On the other hand, the string order parameter $\langle S_C\rangle_{Q_0,Q_1}$ is given by

$$\begin{aligned}\langle S_C\rangle_{Q_0,Q_1} &= \frac{\text{tr}(\rho_{Q^{(0)},Q^{(1)}}S_C)}{\text{tr}(\rho_{Q^{(0)},Q^{(1)}})}\\ &= \frac{\sum_{z_{\mathbf{v}}}\left(\prod_{e\in C}x_e\right)z_{v_1}z_{v_2}e^{\beta_e\sum_e x_e\left(\prod_{v\in e}z_v\right)}}{\mathcal{Z}_{\text{2D Ising},Q^{(1)}}}\Bigg|_{x_{\mathbf{e}}\in Q^{(1)}}\\ &= \left(\prod_{e\in C}x_e\right)\langle z_{v_1}z_{v_2}\rangle_{\text{2D Ising},x_{\mathbf{e}}}\Big|_{x_{\mathbf{e}}\in Q^{(1)}},\end{aligned} \tag{16}$$

where v_1 and v_2 label the end points of the curve C and $\langle z_{v_1}z_{v_2}\rangle_{\text{2D Ising},x_{\mathbf{e}}}$ is the spin-spin correlation function of the 2D Ising model with the sign of the Ising interaction determined by $x_{\mathbf{e}}$. Clearly, $\langle S_C\rangle_{Q_0,Q_1}$ can show long-range order at low temperature, and following the same argument as for the 1D cluster state, long-range order for a given sector implies that the (unnormalized) density matrix $\rho_{Q^{(0)},Q^{(1)}}$ is sym-LRE. For example, in the sector corresponding to all $x_e = 1$, the long-range order sets in below the 2D Ising critical temperature. However, since the ordering temperature clearly depends on the sector $Q^{(1)}$, to understand whether the full density matrix $\rho = \sum_{Q^{(0)},Q^{(1)}}\rho_{Q^{(0)},Q^{(1)}}$ is sym-LRE, one needs to statistically quantify the string order as a function of the error rate. To do so, we introduce the following "average string order parameter":

$$[\langle S_C\rangle^2] = \sum_{Q^{(0)},Q^{(1)}}\text{tr}(\rho_{Q^{(0)},Q^{(1)}})\left(\langle S_C\rangle_{Q_0,Q_1}\right)^2. \tag{17}$$

Equation (17) is equivalent to the disorder-averaged spin-spin correlation function of the RBIM along the Nishimori line [33]. It follows that $[\langle S_C\rangle^2]$ decays exponentially as a function of $|C|$ when $p_e > p_c \approx 0.109$ [69].

On the basis of the above analysis, the decohered state ρ as a function of p_e and p_v can be divided into four regimes with use of the qualitative behavior of the expectation values of membrane and average string order operators [see Fig. 1(b)]:

(1) $p_v = 0$ and $p_c > p_e \geq 0$: $\langle M_S\rangle_{Q_0,Q_1} = 1$ (in the sector $Q^{(0)} = 0$) and $[\langle S_C\rangle^2]$ is a nonzero constant as $|C| \to \infty$. In this regime, ρ must be sym-LRE.
(2) $p_v = 0$ and $p_e > p_c$: $\langle M_S\rangle_{Q_0,Q_1} = 1$ (in the sector $Q^{(0)} = 0$) and $[\langle S_C\rangle^2]$ decays exponentially as a function of $|C|$. In this regime, ρ must again be sym-LRE.
(3) $p_v > 0$ and $p_c > p_e \geq 0$: $\langle M_S\rangle_{Q_0,Q_1} \sim e^{-A(S)}$ and $[\langle S_C\rangle^2]$ is a nonzero constant as $|C| \to \infty$. In this regime, ρ must also be (statistically) sym-LRE.
(4) $p_v > 0$ and $p_e > p_c$: $\langle M_S\rangle_{Q_0,Q_1} \sim e^{-A(S)}$ and $[\langle S_C\rangle^2] \sim e^{-|C|}$. This is suggestive that in this regime ρ is (statistically) sym-SRE, and we provide an argument in favor of this conclusion below using an explicit convex decomposition.

We now use the CDA in Eq. (4) with $\Gamma = \sqrt{\rho}$ to argue that regime (4), namely, $p_v > 0$ and $p_e > p_c$, is indeed sym-SRE. To ensure that each CDA state $|\psi_m\rangle$ satisfies the $Z_2^{(0)} \times Z_2^{(1)}$ symmetry, we choose $\{|m\rangle = |x_{\mathbf{v}}, x_{\mathbf{e}}\rangle\}$. Similarly to the 1D case, we consider the singularity of the "partition function" $\mathcal{Z}_m = \langle\psi_m|\psi_m\rangle$ as a diagnostic for transition from SRE to LRE as β is increased from zero. Since $\mathcal{Z}_m = \langle x_{\mathbf{e}}, x_{\mathbf{v}}|\rho|x_{\mathbf{e}}, x_{\mathbf{v}}\rangle$, a calculation similar to that for $\text{tr}(\rho_{Q^{(0)},Q^{(1)}})$ shows that $\mathcal{Z}_m$ is proportional to $\mathcal{Z}_{\text{2D Ising gauge},x_{\mathbf{v}}}\mathcal{Z}_{\text{2D Ising},x_{\mathbf{e}}}$. One can also compute the expectation values of membrane and average string order operators with respect to $|\psi_m\rangle$ and find that $\langle M_S\rangle_m$ is proportional to the expression in Eq. (15), while $[\langle S_C\rangle_m^2]$ is proportional to the expression in Eq. (17), and therefore both vanish when $p_v > 0$ and $p_e > p_c$.

Alternatively, one may define an "average free energy" $[\ln\mathcal{Z}] = \sum_m P_m\ln(\mathcal{Z}_m) \propto \sum_m\mathcal{Z}_m\ln(\mathcal{Z}_m)$ with respect to

$|\psi_m\rangle$ to detect whether the ensemble $\{\psi_m\rangle\}$ encounters a phase transition as a function of the error rate. When $\beta = 0$, $|\psi_m\rangle = |x_{\mathbf{a}}, x_{\mathbf{b}}\rangle_m$ is the trivial product state. On the other hand, $|\psi_m\rangle$ becomes the 2D cluster state when $\beta \to \infty$. One expects that the phase transition point can be located by the singular behavior of $[\ln \mathcal{Z}]$. Since $[\ln \mathcal{Z}]$ is proportional to the disorder-averaged free energy of the 2D RBIM along the Nishimori line, it is singular at $p_e \approx 0.109$. This leads to the same conclusion that $\{|\psi_m\rangle\}$ remains SRE in regime (4) above.

Interestingly, if one adopts the aforementioned CDA in regimes (2) and (3), then $|\psi_m\rangle$ hosts intrinsic topological order and GHZ order, respectively. This can be argued by one first considering the extreme case $(p_v, p_e) = (0, 0.5)$ in regime (2) and $(p_v, p_e) = (0.5, 0)$ in regime (3). When $(p_v, p_e) = (0, 0.5)$, $|\psi_m\rangle \propto \prod_v (I + h_v^{(0)})|m\rangle \propto (|x_{\mathbf{v}}\rangle \otimes \prod_v (I + x_v \prod_{e \ni v} Z_e)|x_{\mathbf{e}}\rangle)$, which is an eigenstate of toric code. On the other hand, when $(p_v, p_e) = (0.5, 0)$, $|\psi_m\rangle \propto \prod_e (I + h_e^{(1)})|m\rangle \propto (|x_{\mathbf{e}}\rangle \otimes \prod_e (I + x_e \prod_{v \in e} Z_v)|x_{\mathbf{v}}\rangle)$ is the 2D GHZ state. The argument based on the analyticity of the average free energy $[\ln \mathcal{Z}]$ then indicates that regimes (2) and (3) continue to host topological order and GHZ order, respectively. The phase diagram obtained with the current decomposition is summarized in Fig. 1(b).

Finally, we note that order parameters similar to $[\langle S_C \rangle^2]$ [Eq. (17)] and the connections between the decohered cluster states and the RBIM have also appeared in the context of preparing long-range-entangled states by means of measurement protocols in Refs. [34,35]. In particular, our phase diagram [Fig. 1(b)] along the line $p_v = 0.5$ is similar to the finite-time measurement-induced phase transitions in Ref. [34,35]. However, one crucial difference is that the mixed states in Refs. [34,35] do not respect the $Z_2^{(1)}$ symmetry and therefore the corresponding transitions cannot be interpreted as separability transitions protected by $Z_2^{(0)} \times Z_2^{(1)}$ symmetry between a sym-LRE phase and a sym-SRE phase. Instead, the role of different sectors corresponding to the $Z_2^{(1)}$ symmetry is played by the flux $f_p = \prod_{e \in p} s_e$ through a plaquette p, where s_e is the measurement outcome. One may then regard the transition in Refs. [34,35] as a separability transition where in the nontrivial phase it is impossible to decompose the density matrix as a convex sum of SRE states that carry both definite $\mathcal{Z}_2^{(0)}$ charge and flux f_p. Similar statements hold true for the case of a 3D cluster state, which we discuss next.

D. Three-dimensional cluster state

The 3D cluster state Hamiltonian $H_{\text{3D cluster}}$ is given by

$$H_{\text{3D cluster}} = -\sum_e X_e \prod_{f \ni e} Z_f - \sum_f X_f \prod_{e \in f} Z_e = \sum_e h_e + \sum_f h_f. \tag{18}$$

The Hilbert space consists of qubits residing at both the faces f and the edges e of a cubic lattice [see Fig. 1(c)] or, equivalently, at the edges of a cubic lattice and the edges of its dual lattice [recall that each edge (plaquette) of the original lattice is in one-to-one correspondence with a plaquatte (edge) of the dual lattice]. We assume periodic boundary conditions. This model has a $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1')}$ symmetry whose generators are given by

$$U_c^{(1')} = \prod_{f \in \partial c} X_f, \; U_{\tilde{c}}^{(1)} = \prod_{e \in \partial \tilde{c}} X_e, \tag{19}$$

where c $(\tilde{c})$ specifies the cube in the lattice (dual lattice) and ∂c $(\partial \tilde{c})$ denotes the faces on the boundary of c $(\tilde{c})$. Choosing Kraus operators $O_{e/f} = Z_{e/f}$ with respective probabilities $p_{e/f}$, using Eqs. (5) and (6), one obtains the decohered state $\rho = 1/Z e^{-\beta_e \sum_e h_e^{(1)} - \beta_f \sum_f h_f^{(1')}}$, with $\tanh \beta_{e/f} = 1 - 2p_{e/f}$.

We now decompose ρ as a convex sum of symmetric states by writing $\rho = \sum_{Q^{(1')}, Q^{(1)}} \rho_{Q^{(1')}, Q^{(1)}}$, where each $\rho_{Q^{(1')}, Q^{(1)}}$ carries exact symmetry: $U_c^{(1')} \rho_{Q^{(1')}, Q^{(1)}} = (-1)^{Q_c^{(1')}} \rho_{Q^{(1')}, Q^{(1)}}$, $U_{\tilde{c}}^{(1)} \rho_{Q^{(1')}, Q^{(1)}} = (-1)^{Q_{\tilde{c}}^{(1)}} \rho_{Q^{(1')}, Q^{(1)}}$. Here two one-form symmetry charges are labeled by $Q^{(1')} = \{Q_c^{(1')}\}$, with $Q_c^{(1')} = 0, 1$ defined on each cube c, and $Q^{(1)} = \{Q_{\tilde{c}}^{(1)}\}$, with $Q_{\tilde{c}}^{(1)} = 0, 1$ defined on each cube $\tilde{c}$ in the dual lattice. Let us focus on the physical observables that characterize each sector. These are the membrane operators $M_S = \prod_{f \in S} (-h_f^{(1')})$, with S a contractible surface on the original lattice (by "contractible surface" we mean an open membrane whose boundary ∂S is nonzero and is a closed loop) and $M_{\tilde{S}} = \prod_{e \in \tilde{S}} (-h_e^{(1')})$, with $\tilde{S}$ a noncontractible surface on the dual lattice. Thus, we want to compute $\text{tr}(\rho_{Q^{(1')}, Q^{(1)}} M_S)/\text{tr}(\rho_{Q^{(1')}, Q^{(1)}})$ and $\text{tr}(\rho_{Q^{(1')}, Q^{(1)}} M_{\tilde{S}})/\text{tr}(\rho_{Q^{(1')}, Q^{(1)}})$.

Similarly to the cases in previous sections, we first compute the denominator $\text{tr}(\rho_{Q^{(1')}, Q^{(1)}})$ in these expressions by inserting the complete basis $\{|x_{\mathbf{f}, \mathbf{e}}\rangle\}$ and $\{|z_{\mathbf{f}, \mathbf{e}}\rangle\}$, and obtain $\text{tr}(\rho_{Q^{(1')}, Q^{(1)}}) \sim \sum_{x_{\mathbf{f}} \in Q^{(1')}} \mathcal{Z}_{\text{3D gauge}, x_{\mathbf{f}}} \sum_{x_{\mathbf{e}} \in Q^{(1)}} \mathcal{Z}_{\text{3D gauge}, x_{\mathbf{e}}}$. Here $\mathcal{Z}_{\text{3D gauge}, x_{\mathbf{f}}} = \sum_{z_{\mathbf{e}}} e^{\beta_f \sum_f x_f (\prod_{e \in f} z_e)}$ is the partition function of the 3D Ising guage theory with the sign of the interaction on each face labeled by $x_{\mathbf{f}}$, and $x_{\mathbf{f}} \in Q^{(1')}$ denotes all possible $x_{\mathbf{f}}$ satisfying $\prod_{f \in \partial c} x_{\mathbf{f}} = (-1)^{Q_c^{(1')}}$. For a system with periodic boundary conditions, all possible $x_f \in Q^{(1')}$ can be reached by the transformation $x_f \to x_f \prod_{e \ni f} \sigma_e, \sigma_e = \pm 1$. Further, one may verify that $\mathcal{Z}_{\text{3D gauge}, x_{\mathbf{f}}}$ is invariant under the aforementioned transformation by changing the dummy variables $z_e \to \sigma_e z_e$. It follows that $\mathcal{Z}_{\text{3D gauge}, x_{\mathbf{f}}} = \mathcal{Z}_{\text{3D gauge}, Q^{(1')}}$ is a function of only charge $Q^{(1')}$. Analogous statements hold true for $\mathcal{Z}_{\text{3D gauge}, x_{\mathbf{e}}}$. Therefore, we write

$$\text{tr}(\rho_{Q^{(1')}, Q^{(1)}}) \propto \mathcal{Z}_{\text{3D gauge}, Q^{(1')}} \mathcal{Z}_{\text{3D gauge}, Q^{(1)}}. \tag{20}$$

One may similarly compute $\mathrm{tr}(\rho_{Q^{(1')},Q^{(1)}} M_S)$, and obtain the following expressions:

$$\langle M_S\rangle_{Q^{(1')},Q^{(1)}} = \frac{\mathrm{tr}(\rho_{Q^{(1')},Q^{(1)}} M_S)}{\mathrm{tr}(\rho_{Q^{(0)},Q^{(1)}})} = \frac{\sum_{z_{\mathbf{e}}} \left(\prod_{f\in S} x_f \prod_{e\in\partial S} z_e\right) e^{\beta_f \sum_f x_f (\prod_{e\in f} z_e)}}{\mathcal{Z}_{\text{3D gauge},Q^{(1')}}}\Bigg|_{x_{\mathbf{f}}\in Q^{(1')}}$$
$$= \left(\prod_{f\in S} x_f\right) \langle W_{\partial S}\rangle_{\text{3D gauge},x_{\mathbf{f}}}\Big|_{x_{\mathbf{f}}\in Q^{(1')}}, \tag{21}$$

where $\langle W_{\partial S}\rangle_{\text{3D gauge},x_{\mathbf{f}}}$ is the expectation value of the Wilson loop operator (corresponding to $\prod_e z_e$ along a closed curve) along the boundary of S for the 3D classical Ising gauge theory whose Hamiltonian is defined by the term that multiplies β_f in the exponential in the second line of Eq. (21). Since the plaquette interaction term in this Ising gauge theory depends on $x_{\mathbf{f}} \in Q^{(1')}$, similarly to the discussion for 2D cluster state, we introduce an average membrane order parameter

$$[\langle M_S\rangle^2] = \sum_{Q^{(1')},Q^{(1)}} \mathrm{tr}(\rho_{Q^{(1')},Q^{(1)}}) \left(\langle M_S\rangle_{Q_1',Q_1}\right)^2. \tag{22}$$

Equation (22) precisely corresponds to the disorder-averaged Wilson loop of the 3D random-plaquette gauge model along the Nishimori line [67]. It follows that $[\langle M_S\rangle^2] \sim e^{-\kappa|\partial S|}$ ("perimeter law") when $p_f < p_c \approx 0.029$, while $[\langle M_S\rangle^2] \sim e^{-\kappa|S|}$ ("area law") when $p_f > p_c$. One can also define the average membrane order parameter $[\langle M_{\tilde{S}}\rangle^2]$ for $M_{\tilde{S}}$, and the results are analogous with the same critical error rate p_c.

Therefore, using the qualitative behaviors of $[\langle M_S\rangle^2]$ and $[\langle M_{\tilde{S}}\rangle^2]$, one can divide the decohered state ρ as a function of p_f and p_e into four regimes [see Fig. 1(c)]:

(1) $p_f, p_e < p_c$: both $[\langle M_S\rangle^2]$ and $[\langle M_{\tilde{S}}\rangle^2]$ satisfy the perimeter law.
(2) $p_f < p_c, p_e > p_c$: $[\langle M_S\rangle^2]$ satisfies he perimeter law, while $[\langle M_{\tilde{S}}\rangle^2]$ satisfies the area law.
(3) $p_f > p_c, p_e < p_c$: $[\langle M_S\rangle^2]$ satisfies the area law, while $[\langle M_{\tilde{S}}\rangle^2]$ satisfies the perimeter law.
(4) $p_f, p_e > p_c$: both $[\langle M_S\rangle^2]$ and $[\langle M_{\tilde{S}}\rangle^2]$ satisfy the area law.

Using an argument similar to the argument in Ref. [21], and also similar to arguments used in previous subsections for 1D and 2D cluster states, one can show that in regimes (1)–(3), ρ cannot be a convex sum of symmetric pure states where membrane operators exhibit only an area law. This suggests that these three regimes are sym-LRE. In regime (4), ρ does not develop any average membrane orders, which strongly suggests that it is a sym-SRE state. We now use a CDA to support this expectation.

We again choose a CDA [Eq. (4)] with $\Gamma = \sqrt{\rho}$. To ensure that each $|\psi_m\rangle$ that enters the CDA satisfies $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1')}$ symmetry, we choose the basis $\{|m\rangle = |x_{\mathbf{e}}, x_{\mathbf{f}}\rangle\}$. Similarly to the previous cases, we consider the "partition function" $\mathcal{Z}_m = \langle\psi_m|\psi_m\rangle$, whose singularities are expected to indicate the presence of a phase transition. The evaluation of $\mathcal{Z}_m = \langle x_{\mathbf{f}}, x_{\mathbf{e}}|\rho|x_{\mathbf{f}}, x_{\mathbf{e}}\rangle$ is quite similar to that for $\mathrm{tr}(\rho_{Q^{(1')},Q^{(1)}})$, and one finds that $\mathcal{Z}_m \sim \mathcal{Z}_{\text{3D gauge},x_{\mathbf{f}}}\mathcal{Z}_{\text{3D gauge},x_{\mathbf{e}}}$. One may also compute the expectation values of the two membrane operators and find $\langle\psi_m|M_S|\psi_m\rangle = (\prod_{f\in S} x_f)\langle W_{\partial S}\rangle_{\text{3D gauge},x_{\mathbf{f}}}$ and $\langle\psi_m|M_{\tilde{S}}|\psi_m\rangle = (\prod_{e\in\tilde{S}} x_e)\langle W_{\partial\tilde{S}}\rangle_{\text{3D gauge},x_{\mathbf{e}}}$. Using these, one may then define average membrane order parameters $[\langle M_S\rangle^2] = \sum_m P_m \langle\psi_m|M_S|\psi_m\rangle^2$ and $[\langle M_{\tilde{S}}\rangle^2] = \sum_m P_m \langle\psi_m|M_{\tilde{S}}|\psi_m\rangle^2$. Using the same arguments as those following Eq. (22), one concludes that both these order parameters vanish in regime (4).

One may also conclude that the aforementioned decomposition in regimes (2) and (3) corresponds to topologically ordered phases. This can be argued by one first considering the extreme case $(p_f, p_e) = (0, 0.5)$ in regime (2) and $(p_f, p_e) = (0.5, 0)$ in regime (3). When $(p_f, p_e) = (0, 0.5)$, $|\psi_m\rangle \sim \prod_f (I + h_f^{(0)})|m\rangle \sim (|x_{\mathbf{f}}\rangle \otimes \prod_f (I + x_f \prod_{e\in f} Z_e)|x_{\mathbf{e}}\rangle)$, which is an eigenstate of the 3D toric code. The argument based on the singularity of the average free energy $[\ln\mathcal{Z}]$ then indicates that in regime (2) CDA states are topologically ordered. Similar arguments hold for regime (3). The phase diagram obtained with such a convex decomposition is summarized in the plot on the right in Fig. 1(c).

It is interesting to compare our results with the results in Ref. [70], where the Gibbs state of the 3D cluster Hamiltonian was studied. The main difference between the decohered state we study, which also takes the Gibbs form, and the state studied in Ref. [60] is that in Ref. [60], the Gibbs state is projected to a *single* charge sector of both one-form symmetries [and therefore possesses an exact symmetry, see comment (3) in Sec. II], which results in a phase transition as a function of temperature that is in the 3D Ising universality. In contrast, the decoherence we are considering leads only to an average (instead of an exact) symmetry, and therefore we obtain an *ensemble* of density matrices $\rho_{Q^{(1')},Q^{(1)}}$ labeled by the symmetry charges $Q^{(1')}, Q^{(1)}$. As discussed above, this implies that the universality class of the transition is related to the 3D random-plaquette gauge model (and not the 3D Ising transition).

E. One-dimensional and two-dimensional topological phases protected by a $Z_2^{(0)}$ symmetry

Aside from the cluster states in several dimensions, Eqs. (5) and (6) also hold for various stabilizer models realizing 1D and 2D SPT phases protected by a $Z_2^{(0)}$ symmetry, which we now discuss briefly. An example in one dimension is the nontrivial phase of the Kitaev chain [71]:

$$H = -i \sum_j \gamma_{2j-1}\gamma_{2j}, \tag{23}$$

where γ_j denotes the Majorana operator satisfying $\{\gamma_j, \gamma_k\} = 2\delta_{ij}$. It is straightforward to see that the Hamiltonian satisfies Eq. (5), and one can choose O_j as γ_{2j-1} or γ_{2j} such that Eq. (6) is satisfied. Therefore, under the composition of the channel $\mathcal{E}_j[\rho] = (1-p)\rho + p\gamma_{2j-1}\rho\gamma_{2j-1}$, the pure-state density matrix becomes the finite-temperature Gibbs state with $\tanh\beta = 1-2p$. A 2D example is the Levin-Gu state [72], where the Hamiltonian is defined on the triangular lattice and can be written as

$$H = -\sum_p B_p, \quad B_p = -X_p \prod_{\langle pqq'\rangle} i^{(1-Z_q Z_{q'})/2}, \tag{24}$$

where the product runs over the six triangles $\langle pqq'\rangle$ containing the site p. The ground state has nontrivial SPT order for the $Z_2^{(0)}$ symmetry generated by $U = \prod_p X_p$. One can verify that $[B_p, B_{p'}] = 0$ and $B_p^2 = 1$ by straightforward algebra, and thus Eq. (5) is satisfied. Besides, one can choose $O_j = Z_j$ such that Eq. (6) is satisfied. Therefore, under the composition of the channel $\mathcal{E}_j[\rho] = (1-p)\rho + pZ_j\rho Z_j$, the pure-state density matrix becomes the finite-temperature Gibbs state with $\tanh\beta = 1-2p$. Using the CDA in Eq. (4), one may then argue that both the decohered Kitaev chain and the Levin-Gu state are sym-SRE for any nonzero p (we assume periodic boundary conditions so that there are no boundary modes).

V. SEPARABILITY TRANSITIONS FOR 2D CHIRAL TOPOLOGICAL STATES

A. Setup and motivation

In this subsection, we consider subjecting chiral fermions in two dimensions to local decoherence. The starting pure state we consider is the ground state of a $p+ip$ SC, although we expect that the results will qualitatively carry over to other noninteracting chiral states.

Our motivation is as follows: It is generally believed that the 2D $p+ip$ SC cannot be prepared from a product state with use of a constant-depth unitary circuit (as suggested by the fact that the thermal Hall conductance of a $p+ip$ SC is nonzero, while that for a trivial, gapped paramagnet is zero). Indeed, one may think of a $p+ip$ SC as an SPT phase protected by the conservation of fermion parity [40]. Therefore, it is natural to ask what happens if one applies a quantum channel to this system where Kraus operators anticommute with the fermion parity. This is conceptually similar to our discussion in Sec. IV, where we subjected a nontrivial SPT ground state to Kraus operators odd under the symmetry responsible for the existence of a (pure) SPT ground state. An example of such a Kraus operator is the fermion creation/annihilation operator, and we study this case in detail. Alternatively, one may consider subjecting a $p+ip$ ground state to decoherence with Kraus operators *bilinear* in fermion creation/annihilation operators. In this latter case, the fermion parity remains an exact symmetry. From our discussion in Sec. IV, one may expect a qualitative difference in these two cases, namely, Kraus operators linear versus bilinear in fermion creation/annihilation operators. We briefly outline such a qualitative difference as suggested by field-theoretic considerations, whose details are presented in Sec. V D.

Let us first consider Kraus operators linear in fermion operators. This is equivalent to one bringing in auxiliary fermions and entangling them with the fermions of the $p+ip$ SC by a finite-depth unitary. Since this is a finite depth unitary operation on the enlarged Hilbert space (including both the system and auxiliary qubits), the expectation value of any observable, including nonlocal ones that detect chiral topological order [73,74], cannot become zero. At the same time, intuitively, the resulting mixed state for the electrons belonging to the original $p+ip$ SC must somehow "lose its chirality" at infinitesimal coupling to the ancillae. This is indicated by our treating the density matrix as a pure state in the doubled Hilbert space using CJ isomorphism, which we discuss below in detail, where we also clarify subtleties pertinent to the mapping of Kraus operators linear in fermion operators. Under the CJ map, the effect of the channel becomes a coupling bilinear in fermion operators between two chiral Ising CFTs with opposite chirality, and which, therefore, gaps out the counterpropagating chiral CFTs. The gapping out of the edge states in the double state is also manifested in the entanglement spectrum of the double state, which we also study. In particular, we show that infinitesimal decoherence leads to a gap in the entanglement spectrum.

Although working with the double state obtained via the CJ map is insightful, it does not directly tell us the nature of the decohered mixed state. One of our central aims is to understand the difference between the original pure (non-decohered) state and the decohered state not in terms of the double state obtained via the CJ map, or in terms of nonlinear functions of the density matrix, but directly in terms of the separability properties of the mixed state. Our main result is that the resulting mixed state can be expressed as a convex sum of nonchiral states, and in this sense, it is nonchiral (i.e., it can be prepared with use of an ensemble of finite-depth unitaries that commute with fermion parity).

We next consider Kraus operators bilinear in the fermion operators. We study this problem using only the double-state formalism (i.e., the aforementioned CJ map), and obtain an effective action consisting of two counterpropagating free, chiral Majorana CFTs coupled via a four-fermion interaction. Such a Hamiltonian has already been studied (see, e.g., Refs. [75,76]), and we simply borrow the previous results to conclude that unlike the case for Kraus operators linear in Majorana operators, this system is *stable* against infinitesimal decoherence. Furthermore, the field theory corresponding to the double state indicates that this system undergoes a spontaneous symmetry breaking where the gapless modes corresponding to the CFT are gapped out. The universality class for this transition lies in the (supersymmetric) $c = 7/10$ tricritical Ising model. We discuss this in detail in Sec. V D. We note that recently Su *et al.* [22] studied chiral topological phases subjected to decoherence using a generalization of a strange correlator [32] to mixed states [29,30]. Although they did not study the problem of our interest (namely, $p + ip$ SC subjected to Kraus operators bilinear in Majorana fermions), the overall structure of the field theories obtained in Ref. [22] using a strange correlator bears resemblance to the one we derive using an entanglement spectrum in Sec. V D.

B. Separability of a $p + ip$ SC subjected to fermionic Kraus operators

Our starting point is the ground state of the $p + ip$ superconductor [45] described by the following Hamiltonian on a square lattice:

$$\begin{aligned} H = \sum_{x,y} & -t(\mathbf{c}^\dagger_{x+1,y}\mathbf{c}_{x,y} + \mathbf{c}^\dagger_{x,y+1}\mathbf{c}_{x,y} + \text{H.c.}) \\ & + \Delta(\mathbf{c}^\dagger_{x+1,y}\mathbf{c}^\dagger_{x,y} + i\mathbf{c}^\dagger_{x,y+1}\mathbf{c}^\dagger_{x,y} + \text{H.c.}) \\ & - (\mu - 4t)\mathbf{c}^\dagger_{x,y}\mathbf{c}_{x,y}. \end{aligned} \tag{25}$$

When $t = \Delta = 1/2$ and the chemical potential $\mu = 1$, the system is in the topologically nontrivial phase. This can be diagnosed, for example, by one studying the entanglement spectrum, which will exhibit chiral propagating modes [77,78], or by one studying the modular commutator [79–82], which is proportional to the chiral central charge of the edge modes that appear if the system had boundaries. Relatedly, in the topological phase, the ground state cannot be written as a Slater determinant of exponentially localized Wannier single-particle states [44–46]. In our discussion, we assume periodic boundary conditions, so that there are no physical edge modes.

We are interested in subjecting the ground state of Eq. (25) to the composition of the following single-Majorana-fermion channel on all sites:

$$\mathcal{E}_j[\rho] = (1-p)\rho + p\gamma_j\rho\gamma_j. \tag{26}$$

Is the chiral nature of the ground state ρ_0 stable under the channel? More precisely, can we express the decohered density matrix as a convex sum of pure states, where each of these pure states now does not exhibit chiral states in its entanglement spectrum, and relatedly, has a vanishing modular commutator in the thermodynamic limit?

Under the aforementioned channel [Eq. (26)], the density matrix will continue to remain Gaussian, and is fully determined by the covariance matrix M defined as $M_{jk} = -i\text{tr}(\rho(\gamma_j\gamma_k - \delta_{jk}))$. As shown in Appendix B 1, under the channel in Eq. (26), M evolves as $\mathcal{E}(M) = (1-2p)^2M$. We write the decohered density matrix ρ as $\rho(p) = e^{-H_\rho(p)}$, where $H_\rho(p)$ can be determined explicitly in terms of $\mathcal{E}(M) = (1-2p)^2M$ as detailed in Appendix B 1.

To write the decohered mixed state ρ as a convex sum of pure states, we consider the decomposition in Eq. (4), and write

$$\begin{aligned} \rho(p) &= \sum_m e^{-H_\rho(p)/2}|m\rangle\langle m|e^{-H_\rho(p)/2} \\ &= \sum_m |\psi_m\rangle\langle\psi_m|, \end{aligned} \tag{27}$$

where $|m\rangle$ are product states in the occupation-number basis, $|m\rangle = |m_1, \ldots, m_N\rangle$, $m_j = 0, 1$, and $|\psi_m\rangle = \sqrt{\rho}|m\rangle = e^{-H_\rho(p)/2}|m\rangle$. To build intuition for the states $|\psi_m\rangle$, let us consider the particular state $|\psi_0\rangle = \sqrt{\rho}|0\rangle$, where $|0\rangle$ is a state with no fermions. One can analytically show at any nonzero decoherence that the real-space wave function for this state is a Slater determinant of localized Wannier orbitals, unlike the (undecohered) ground state of the $p + ip$ SC [44–46]. The argument is as follows. One may write $|\psi_0\rangle \propto e^{-\beta\sum_\mathbf{k}\alpha^\dagger_\mathbf{k}\alpha_k}|0\rangle$, where $\tanh\beta = (1-2p)^2$ and $\alpha^\dagger_\mathbf{k} = u_\mathbf{k}c^\dagger_\mathbf{k} + v^*_\mathbf{k}c_{-\mathbf{k}}$ are the same (complex) fermionic operators that diagonalize the original $p + ip$ BCS Hamiltonian (see Appendix B 1), with $|u_\mathbf{k}|^2 + |v_\mathbf{k}|^2 = 1$ due to unitarity. Since $c_\mathbf{k}|0\rangle = 0$, this implies that

$$|\psi_0\rangle \propto \prod_\mathbf{k}\left[1 + \left(e^{-\beta} - 1\right)\left(|v_\mathbf{k}|^2 + u_\mathbf{k}v_\mathbf{k}c^\dagger_\mathbf{k}c^\dagger_{-\mathbf{k}}\right)\right]|0\rangle. \tag{28}$$

This expression may then be exponentiated to obtain the standard BCS-like form for $|\psi_0\rangle \propto e^{\sum_\mathbf{k} h(\mathbf{k})c^\dagger_\mathbf{k}c^\dagger_{-\mathbf{k}}}|0\rangle$, where

$$h(\mathbf{k}) = \frac{u_\mathbf{k}v_\mathbf{k}\left(e^{-\beta} - 1\right)}{|u_\mathbf{k}|^2 + |v_\mathbf{k}|^2e^{-\beta}}. \tag{29}$$

As $p \to 0$, $\beta \to \infty$ [recall $\tanh\beta = (1-2p)^2$], and one recovers the $p+ip$ ground state where $h(\mathbf{k}) \sim v_{\mathbf{k}}/u_{\mathbf{k}}$ diverges as $1/(k_x+ik_y)$ and results in a power-law decay of Wannier orbitals [45]. In contrast, at any noninfinite β (i.e., nonzero decoherence rate p), $h(\mathbf{k})$ is noninfinite for any $\mathbf{k}$ (since $|u_{\mathbf{k}}|^2+|v_{\mathbf{k}}|^2=1$), and therefore the Wannier orbitals corresponding to the state $|\psi_0\rangle$ are exponentially localized. As an aside, this same argument also applies to the decohered 1D Kitaev chain (Sec. IV E), and more generally to other decohered noninteracting fermionic topological superconductors.

The above argument applies only to the translationally invariant state $|\psi_0\rangle$ that enters the convex decomposition in Eq. (27). To make progress for general $|\psi_m\rangle$, we found it more helpful to consider diagnostics that directly access the topological character (or lack thereof) of a wave function, and which are also more amenable to finite-size scaling. In particular, we use the "modular commutator" introduced in Refs. [79–82]. The modular commutator is a multipartite entanglement measure that quantifies the chiral central charge for a *pure state*, and can be completely determined by the many-body wave funtion [79–82]. Specifically, it is defined as $J_{ABC} := i\mathrm{tr}(\rho_{ABC}[\ln\rho_{AC}, \ln\rho_{BC}])$, with ρ_X the reduced density matrix in region X obtained from a pure state $|\psi\rangle$ (i.e., $\rho_X = \mathrm{tr}_{\overline{X}}|\psi\rangle\langle\psi|$).

In the absence of decoherence, the modular commutator of $|\psi_m\rangle$ for this setup is $J_{0,ABC} = \pi c/3 = \pi/6$, as the chiral central charge $c = 1/2$ for the $p+ip$ superconductor. Figure 2 shows the modular commutator $J_{ABC}/J_{0,ABC}$ on a $L\times L$ torus as a function of L. We choose the error rate $p = 0.04$ and several different initial states, including $|m\rangle = |0,\ldots,0\rangle$ (uniform), $|0,1,0,1,\ldots,0,1\rangle$ (staggered), and also a random bit string in the occupation-number basis. We find that in all cases, J_{ABC} vanishes in the thermodynamic limit. We also studied other values of p, and our results are again consistent with the claim that at any nonzero p, the modular commutator for the states $|\psi_m\rangle$ vanishes in the thermodynamic limit. This provides numerical evidence that at any nonzero error rate, the decohered mixed state can be expressed as a convex sum of states that do not have any chiral topological order, and hence must be representable as Slater determinants of single-particle localized Wannier states [44] (note that all states $|\psi_m\rangle$ are area-law entangled).

It is important to note that in contrast to the pure states $|\psi_m\rangle$, the modular commutator for the decohered mixed state ρ *does not* show any abrupt behavior change at $p = 0$ (dashed line in Fig. 2). This is consistent with the fact that the arguments relating the modular commutator to the chiral central charge rely on the overall state being pure [79–82], and therefore we do not expect that the modular commutator for the mixed state ρ captures the separability transition at $p = 0$. This again highlights the utility of the convex decomposition of ρ into pure states.

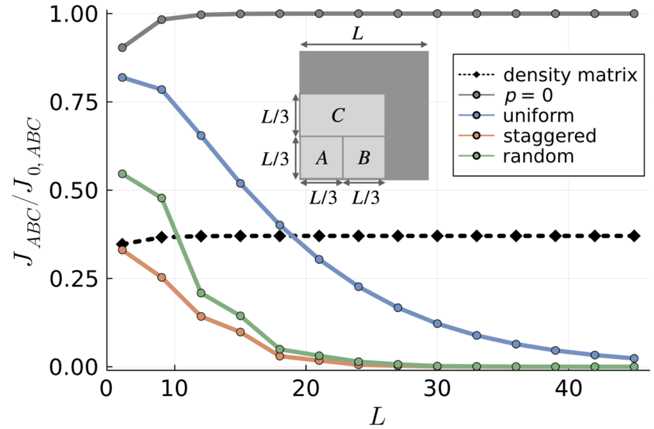

FIG. 2. Modular commutator $J_{ABC}/J_{0,ABC}$ on an $L\times L$ torus as a function of L corresponding to several different pure states $|\psi_m\rangle$ that enter the convex decomposition of the $p+ip$ SC subjected to decoherence with Kraus operators linear in Majorana fermions [Eq. (27)], as well as the modular commutator of the decohered mixed state itself. We choose error rate $p = 0.04$, and the following initial states $|m\rangle$ in Eq. (27): $|m\rangle = |0,\ldots,0\rangle$ (uniform), $|0,1,0,1,\ldots,0,1\rangle$ (staggered), and $|m\rangle$ = a random bit string in the occupational number basis. The inset shows the geometry of regions A, B, and C used to define the modular commutator. We use antiperiodic boundary conditions along both directions so that the ground state is unique.

In addition, we also numerically compute the entanglement spectrum of $|\psi_m\rangle$, with $|m\rangle$ the uniform product state (so that momentum along the entanglement bipartition is a good quantum number). For a chiral topological state, one expects that the edge spectrum of a physical edge will be imprinted on the entanglement spectrum of a subregion [77]. Since $|\psi_m\rangle$ is Gaussian, the entanglement spectrum is encoded in the spectrum of the matrix iM_A, where M_A is the restriction of the covariance matrix M to the region A in the inset in Fig. 3. Figure 3 shows the spectrum of iM_{ABC} (denoted as ν) as a function of the momentum k_y with error rate $p = 0$ and $p = 0.04$. The geometry is again chosen as a torus, with length $L_x = 60$, and height $L_y = 30$. In the absence of error ($p = 0$), all states $|\psi_m\rangle$ are projected to the $p+ip$ ground state, and thus the spectrum shows chirality, resembling the edge states of the $p+ip$ SC (note that we have two entanglement boundaries, resulting in counterpropagating chiral states in the entanglement spectrum). After the decoherence is introduced, one finds that the chiral mode in the entanglement spectrum is gapped out (see Fig. 3). We also confirmed that the gap between the two "bands" of the entanglement spectrum increases with the system size (not shown). Overall, both the modular commutator and the entanglement spectrum provide numerical evidence that the decohered density matrix can be written as a convex sum of free-fermion, pure states that have no chiral topological order.

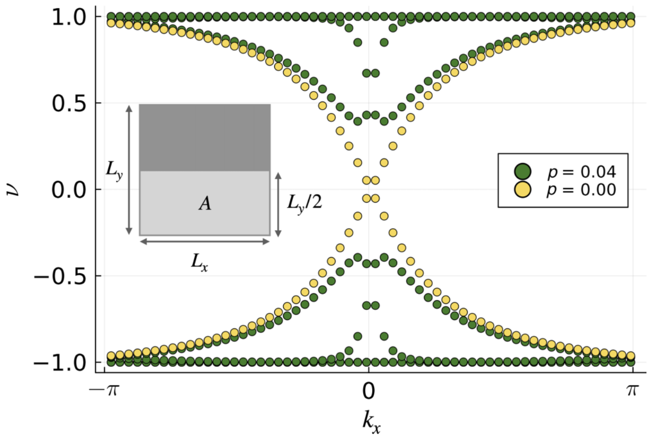

FIG. 3. Spectrum of iM_A (equals the restriction of the covariance matrix to region A in the inset) for a state $|\psi_m\rangle$ obtained from $|m\rangle = |0,\ldots,0\rangle$ [see Eq. (27)] as a function of the momentum k_y for error rates $p = 0$ (i.e., nondecohered) and $p = 0.04$ (i.e., decohered). Here we put the system on an $L_x \times L_y$ torus with $L_x = 60$ and $L_y = 30$.

C. Double-state formalism for fermions

The previous subsection focused on the single-Majorana channel that breaks the fermion-parity symmetry of the initial density matrix from exact ($U\rho = \rho U = \rho$) down to average ($U^\dagger \rho U = \rho$). As briefly mentioned above, if one instead uses a channel where Kraus operators are bilinear in Majorana operators (so that the fermion parity remains an exact symmetry), one might expect a more interesting behavior, in particular the possibility of a phase transition between different nontrivial mixed states. One way to make progress on this case is to study appropriate nonlinear functions of the density matrix [18–20,22,29,83]. Relatedly, one may use the double state obtained with the CJ map, which was used in Refs. [18,20] to study decoherence in bosonic problems. Specifically, given a density matrix $\rho_{\mathcal{H}}$ acting on the Hilbert space $\mathcal{H}$, one can define a state vector $|\rho\rangle_{\mathcal{H}\otimes\bar{\mathcal{H}}}$ in the doubled Hilbert space $\mathcal{H}\otimes\bar{\mathcal{H}}$ (with $\bar{\mathcal{H}}$ having the same dimension as $\mathcal{H}$) using the CJ map [42,43,84]:

$$|\rho\rangle_{\mathcal{H}\otimes\bar{\mathcal{H}}} = \rho_{\mathcal{H}} \otimes I_{\bar{\mathcal{H}}} |\Phi\rangle_{\mathcal{H}\otimes\bar{\mathcal{H}}}. \tag{30}$$

Here $I_{\bar{\mathcal{H}}}$ denotes the identity in $\bar{\mathcal{H}}$ and $|\Phi\rangle_{\mathcal{H}\otimes\bar{\mathcal{H}}}$ is the product of (unnormalized) maximally entangled pairs connecting $\mathcal{H}$ and $\bar{\mathcal{H}}$, i.e., $|\Phi\rangle_{\mathcal{H}\otimes\bar{\mathcal{H}}} = \otimes_j |\phi\rangle_{j,\mathcal{H}\otimes\bar{\mathcal{H}}}$, with $|\phi\rangle_{j,\mathcal{H}\otimes\bar{\mathcal{H}}} = \otimes_j (\sum_{p=1}^{d} |p_{\mathcal{H}}, p_{\bar{\mathcal{H}}}\rangle_j)$ and d the Hilbert space dimension on a single site. Henceforth, for notational simplicity, we omit the subscript labeling the Hilbert space if there is no confusion. For bosons, it is straightforward to see that under Eq. (30), the density matrix $\rho = \sum_{p,q} \rho_q^p |p\rangle\langle q|$ is mapped to $|\rho\rangle = \sum_{p,q} \rho_q^p |p,q\rangle$. On the other hand, the channel $\mathcal{E}[\cdot] = \sum_\alpha K_\alpha(\cdot)K_\alpha^\dagger$ is mapped to the operator

$$\mathcal{N}_{\mathcal{E}} = \sum_\alpha K_\alpha \otimes \bar{K}_\alpha. \tag{31}$$

This can be derived by one expressing $|\mathcal{E}[\rho]\rangle$ as an operator acting on $|\rho\rangle$, i.e., $|\mathcal{E}[\rho]\rangle = \mathcal{N}_{\mathcal{E}}|\rho\rangle$. See Appendix B 2 for details. However, a similar correspondence for fermions is a bit subtle. For example, naively applying Eq. (31) to the single-Majorana channel in Eq. (26), one obtains

$$\begin{aligned}\mathcal{E}_j|\rho\rangle &\overset{?}{=} [(1-p)I_j \otimes I_j + p\gamma_j \otimes \bar{\gamma}_j)]|\rho\rangle \\ &= [(1-p)I + p\gamma_j\eta_j)]|\rho\rangle \\ &\sim e^{-i\mu(i\gamma_j\eta_j)}|\rho\rangle, \quad \mu = \tan(p/(1-p)),\end{aligned} \tag{32}$$

where we denote $\eta = \bar{\gamma}$ as the Majorana operators in the Hilbert space $\bar{\mathcal{H}}$. Equation (32) suggests that the channel generates a *real* time evolution for the double state, which contradicts our intuition that the channel instead gives rise to an imaginary time evolution. Another hint that Eq. (32) is incorrect comes from our setting $p = 1/2$, where the relation $\mathcal{E}_j[\mathcal{E}_j[\rho]] = \mathcal{E}_j[\rho]$ holds. However, Eq. (32) gives $\mathcal{E}_j\mathcal{E}_j|\rho\rangle = \gamma_j\eta_j|\rho\rangle/2$, which is not equal to $\mathcal{E}_j|\rho\rangle$. Therefore, to find the correct correspondence between $\mathcal{E}[\cdot]$ and $\mathcal{N}_{\mathcal{E}}$ for fermions, one should begin with the more fundamental definition of the double state, i.e,, $|\rho\rangle = \rho \otimes I|\Phi\rangle$. Because of the linearity of Eq. (30), one can consider each $K_\alpha(\cdot)K_\alpha^\dagger$ individually. Using $|K_\alpha\rho K_\alpha^\dagger\rangle = (K_\alpha\rho K_\alpha^\dagger) \otimes I|\Phi\rangle = K_\alpha(\rho K_\alpha^\dagger \otimes I|\Phi\rangle) = K_\alpha|\rho K_\alpha^\dagger\rangle$, one finds K_α is unchanged under Eq. (30). On the other hand, since one can always write $K_\alpha^\dagger$ as a function of $\mathbf{c}$ and $\mathbf{c}^\dagger$, it suffices to consider how to express $|\rho\mathbf{c}\rangle$ and $|\rho\mathbf{c}^\dagger\rangle$ as an operator applying to $|\rho\rangle$. In Appendix B 2, we find that

$$|\rho\mathbf{c}\rangle = \mathbf{d}^\dagger|\rho\rangle, \quad |\rho\mathbf{c}^\dagger\rangle = -\mathbf{d}|\rho\rangle. \tag{33}$$

One can then use Eq. (33) to derive $\mathcal{N}_{\mathcal{E}}$ given $\mathcal{E}[\cdot]$. For example, for the Kraus operator given by $K = \gamma_1 \equiv (\mathbf{c} + \mathbf{c}^\dagger)$, one finds

$$\begin{aligned}|(\mathbf{c}+\mathbf{c}^\dagger)\rho(\mathbf{c}+\mathbf{c}^\dagger)\rangle &= (\mathbf{c}+\mathbf{c}^\dagger)|\rho(\mathbf{c}+\mathbf{c}^\dagger)\rangle \\ &= (\mathbf{c}+\mathbf{c}^\dagger)(\mathbf{d}^\dagger - \mathbf{d})|\rho\rangle.\end{aligned} \tag{34}$$

This implies the CJ transformed operator $\mathcal{N}_{\mathcal{E}} = (\mathbf{c} + \mathbf{c}^\dagger)(\mathbf{d}^\dagger - \mathbf{d}) = -i\gamma_1\eta_1$, where $\eta_1 = (\mathbf{d} - \mathbf{d}^\dagger)/i$ [85].

D. Phase transition induced by an interacting channel in a $p + ip$ SC

Being equipped with the correspondence between $\mathcal{E}[\cdot]$ and $\mathcal{N}_{\mathcal{E}}$, we now return to our discussion of decoherence-induced transitions in chiral topological states of fermions.

We first revisit the problem discussed in Sec. V B, and then consider a more interesting problem where the Kraus operators are bilinear in fermions so that the decohered density matrix is not Gaussian.

There are different ways to use the double state to probe the effect of decoherence. For example, one may consider nonlinear functions such as the normalization of the double state [18,20,29,83]. Here we will use the entanglement spectrum of a state obtained from the double state $|\rho\rangle$ (after space-time rotation) as a probe of the decoherence-induced phase transitions.

To begin with, consider the normalization of the double state

$$\langle\rho|\rho\rangle = \langle\rho_0|\mathcal{E}^\dagger\mathcal{E}|\rho_0\rangle. \tag{35}$$

If the bulk action describing $|\rho_0\rangle = |\Psi_0, \Psi_0^*\rangle$ is rotationally invariant, one can map $\langle\rho|\rho\rangle$ to the path integral of the $(1+1)$ D boundary fields following the procedure in Ref. [20]:

$$\langle\rho|\rho\rangle = \int \mathcal{D}(\psi_L, \psi_L^*, \psi_R, \psi_R^*) \times e^{-S_{0,L}(\psi_L,\psi_L^*)-S_{0,R}(\psi_R,\psi_R^*)-S_{\text{int}}(\psi_L,\psi_L^*,\psi_R,\psi_R^*)}. \tag{36}$$

Here ψ_L and ψ_L^* denote the low-energy field variables in $\mathcal{H}$ and $\bar{\mathcal{H}}$, respectively. $S_{0,L}$ is the partition function on the left side of the spatial interface $x = 0^-$ [the meanings of (ψ_R, ψ_R^*) and $S_{0,R}$ are similar]. S_{int} describes the effect of the channel $\mathcal{E}^\dagger\mathcal{E}$ and has two contributions:

$$S_{\text{int}} = S_1 + S_{\mathcal{E}}, \tag{37}$$

where S_1 denotes the action that exists even in the absence of decoherence. In particular, S_1 strongly couples the fields ψ_L (ψ_L^*) and ψ_R (ψ_R^*) such that $\psi_L = \psi_R$ $(\psi_L^* = \psi_R^*)$ in the absence of decoherence. On the other hand, $S_{\mathcal{E}}$ describes the action that merely comes from the decoherence and vanishes when the error rate $p = 0$. We note that a similar field theory was discussed in Ref. [86] in a different context in the evaluation of the system-environment entanglement in the $(1+1)$ D system. In general, the exact form of $S_{\mathcal{E}}$ involves four fields $(\psi_L, \psi_L^*, \psi_R, \psi_R^*)$ and may be schematically captured by the following Hamiltonian:

$$H = (H_{0,L} + H_{\mathcal{E},L}) + (H_{0,R} + H_{\mathcal{E},R}) + H_1, \tag{38}$$

where H_1 strongly couples the left and right fields. One may then consider the reduced density matrix for left fields that is obtained after one has traced out the right fields. One expects [87,88] that the corresponding entanglement Hamiltonian (i.e., logarithm of the reduced density matrix) will essentially correspond to $H_{0,L} + H_{\mathcal{E},L}$. Working with the entanglement Hamiltonian has the advantage that the number of fields one needs to keep track of is now halved. Similar simplification occurs if one considers the fidelity $\text{tr}(\rho_d\,\rho_0)$ between the decohered density matrix ρ_d and the nondecohered density matrix ρ_0; see Ref. [22]. Since we are now working only with the left fields, in the following we omit the subscript L for notational simplicity.

As an example, let us first revisit the case of a $p + ip$ superconductor perturbed by a channel that is linear in Majorana fermions (Sec. V B). Recall that here the Kraus map corresponds to the composition of the following map on all sites: $\mathcal{E}_{\mathbf{x}}[\rho] = (1-p)\rho + p\gamma_{\mathbf{x}}\rho\gamma_{\mathbf{x}}$. From our discussion above on the CJ map for fermions, this translates to a term of the form $H_{\mathcal{E}} = ig\int dy\,\gamma\,\eta$, where $p \sim g$ and where γ and η, respectively, denote the fields corresponding to $\mathcal{H}$ and $\bar{\mathcal{H}}$ of the left fields. In the absence of any decoherence, the spatial boundary of the $p + ip$ superconductor has a simple description in terms of a chiral Majorana fermion. The entanglement Hamiltonian in the doubled Hilbert space then corresponds to stacking the boundary of $p + ip$ and $p - ip$ superconductors, and is given by $H_0 = i\int dy(\gamma\,\partial_y\gamma - \eta\partial_y\eta)$. Therefore, one expects that the entanglement Hamiltonian for the left fields in the presence of decoherence will take the form

$$H_E = i\int dy(\gamma\,\partial_y\gamma - \eta\partial_y\eta) + ig\int dy\gamma\eta. \tag{39}$$

The counterpropagating edge modes are gapped out for any nonzero $g\,(\propto p)$, in line with our earlier discussion where we provided evidence that at any nonzero p the density matrix can be written as a convex sum of pure states that are SRE. The gapping out of the edge modes can also be seen by pne numerically evaluating the entanglement spectrum of the double state obtained via the CJ map. Figure 4 shows the spectrum of iM_L (denoted as ν) as a function of the momentum k_y with different error rates p. Here we put the system on a cylinder with circumference $L_x = 60$ and height $L_y = 16$. In the absence of error $(p = 0)$, there are two counterpropagating modes, resembling the edge states of the initial double state $|\rho_0\rangle$. After the decoherence is introduced, one can clearly see from Fig. 4 that these counterpropagating modes are gapped out for an arbitrarily small error rate. Note that we *did not* perform any space-time rotation to obtain Fig. 4. This suggests that the entanglement Hamiltonian of the double state $|\rho\rangle$ may already have the same qualitative behavior as the entanglement Hamiltonian obtained after space-time rotation. We leave further investigation of this point for future work.

Let us return to the problem of our main interest in this subsection—namely, that of Kraus operators that *commute*

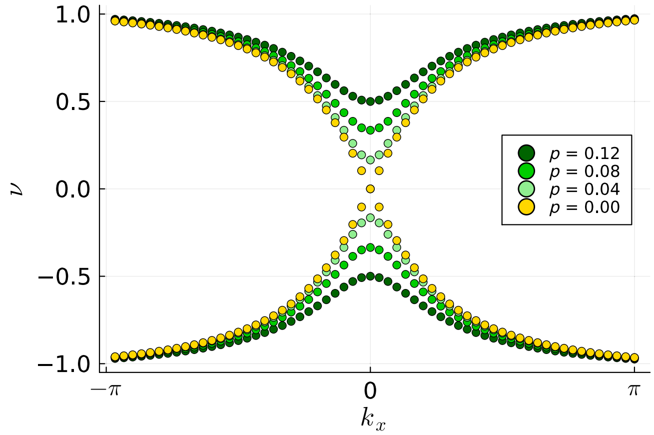

FIG. 4. Spectrum of iM_L for the double state $|\rho\rangle$, where M_L is the restriction of the covariance matrix M to the region L, as a function of the momentum k_y for different error rates p. Here we put the system on a cylinder with circumference $L_x = 60$ and height $L_y = 16$.

with the fermion-parity operator. The simplest possibility is the composition of the following Kraus map on all nearest-neighbor bonds $\langle \mathbf{x}, \mathbf{y}\rangle$ of the square lattice:

$$\mathcal{E}_{\langle \mathbf{x},\mathbf{y}\rangle}[\rho] = (1-p)\rho + p\gamma_{\mathbf{x}}\gamma_{\mathbf{y}}\rho\gamma_{\mathbf{y}}\gamma_{\mathbf{x}}. \tag{40}$$

The interaction term $H_{\mathcal{E}}$ induced by such a Kraus map in the double state should respect the following $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetries: $\gamma \to -\gamma$ and/or $\eta \to -\eta$. Since Majorana fermions square to identity, the simplest term that is bilinear in both γ and η and respects all the symmetries involves derivatives:

$$H_{\mathcal{E}} = g\int dy(\gamma\partial_y\gamma)(\eta\partial_y\eta), \tag{41}$$

where $g \propto p$. Therefore, the full entanglement Hamiltonian for the left fields in the presence of decoherence is given by

$$H_E = i\int dy(\gamma\partial_y\gamma - \eta\partial_y\eta) + g\int dy(\gamma\partial_y\gamma)(\eta\partial_y\eta). \tag{42}$$

This field theory was studied in Refs. [75,76]. At a particular $g = g_c$, the system undergoes a phase transition in the tricitical Ising universality class with central charge $c = 7/10$. For $g < g_c$, the interaction term is irrelevant, while above g_c, the system spontaneously breaks the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry down to the diagonal $\mathbb{Z}_2$ symmetry. Physically, this means that the exact fermion-parity symmetry (i.e., $U\rho = \rho$, where U is the generator of the fermion parity), has been spontaneously broken down to an average symmetry (i.e., $U\rho U^\dagger = \rho$). We note that a class of 2D chiral topological phases subjected to decoherence with fermion-bilinear Kraus operators was also studied in Ref. [22]. One notable difference between the examples considered in Ref. [22] and our problem is that in the examples considered in Ref. [22], the decoherence always reduces the effective central charge of the action corresponding to the double state. In contrast, in our problem, the effective central charge c increases from $1/2$ to $7/10$.

It is interesting to contemplate the implications of the phase transition described above in terms of the separability properties of the original mixed state ρ (instead of the double state $|\rho\rangle$). We conjecture that for $p \ll p_c$ there exists no decomposition of the density matrix as a convex sum of area law–entangled pure states without any chirality, while for $p > p_c$ the density matrix is expressible as a convex sum of area law–entangled pure states with GHZ-like entanglement (due to spontaneous breaking of fermion parity). Similarly to the case of intrinsic topological orders subjected to local decoherence [17,18,20,82], we anticipate that the universality class and the location of the critical point obtained from the double-state formalism will differ from those of the "intrinsic" mixed-state transition for the density matrix, e.g., when viewed from the perspective of separability. We do not know the universality for the latter transition, and we leave it as an open question.

VI. SEPARABILITY TRANSITION IN GIBBS STATES OF THE NLTS HAMILTONIAN

In this section we consider an exotic separability transition in a Gibbs state relevant to certain quantum codes. Although this transition does not require any symmetry, which has been a main ingredient in the rest of this work, the argument below to deduce the existence of a separability transition is broadly similar in spirit to that in Secs. III and IV.

Recently, "good LDPC codes," where the code distance and the number of logical qubits scale with the total number of qubits, have been discovered [49–51]. Moreover, Anshu *et al.* [48] showed that the construction of a good LDPC code in Ref. [49] satisfies the Freedman-Hastings NLTS conjecture [47], which, when satisfied by a Hamiltonian, means that any state $|\psi\rangle$ with energy density less than a nonzero value e_c cannot be prepared by a constant-depth unitary circuit [the energy density e of a state $|\psi\rangle$ is defined as $e = \lim_{N\to\infty} (\langle\psi|H|\psi\rangle - E_0)/N$, where E_0 is the groundspace energy of H]. Here we ask whether the Gibbs state of an NLTS Hamiltonian shows a separability transition at a nonzero temperature. That is, does there exist $T_c > 0$ so that for $T < T_c$ the Gibbs state cannot be written as a convex sum of SRE pure states?

Firstly, we note that Anshu *et al.* [48] have already proved that any mixed state whose energy density is less than a positive number e_c cannot be purified to a pure SRE state by a short-depth channel, i.e., it cannot be prepared by one first enlarging the Hilbert space to include ancillae, which are initially all in a product state, followed by a finite-depth unitary that entangles the "system" qubits (which are also initially in a product state) with the auxiliary qubits, and eventually integrating out the ancillae. However, as discussed in Sec. II, the inability to purify to an SRE state via a short-depth channel does not imply that a mixed state is SRE with our definition (i.e., expressibility of a mixed state as a convex sum of SRE pure states). We briefly reiterate the example discussed in Sec. II that illustrates these two different notions of mixed-state entanglement [see comment (4) in Sec. II] by stating that any Gibbs state that exhibits spontaneous symmetry breaking (which therefore has long-range correlations for the operator corresponding to the order parameter) cannot be purified to an SRE pure state via a short-depth channel. Therefore, such a mixed state will be SRE with our definition and LRE with the definition in Refs. [48,52]. Here we provide a simple argument that the Gibbs state of an NLTS-satisfying Hamiltonian shows a separability transition at a nonzero temperature.

Let us assume that the Gibbs state of an NLTS-satisfying Hamiltonian H can be expressed as a convex sum of SRE pure states for *any* temperature $T > 0$, i.e., $\rho(T) = e^{-H/T}/Z = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where each $|\psi_i\rangle$ can be prepared via a unitary whose depth is independent of the number of qubits N. For simplicity of notation, we set the ground-space energy E_0 to zero (this can always be achieved by one adding a constant Nc to the Hamiltonian, where c is a constant). We show that this assumption leads to a contradiction. Since all pure states $|\psi_i\rangle$ are SRE, by the NLTS condition, they must all satisfy $\langle\psi_i|H|\psi_i\rangle/N > e_c$ as $N \to \infty$. Therefore, $\mathrm{tr}(\rho(T)H)/N = \sum_i p_i \langle\psi_i|H|\psi_i\rangle/N > \sum_i p_i e_c = e_c$. This implies that if the Gibbs state can be expressed as a convex sum of SRE pure states, then its energy density is nonzero. However, nonzero energy density necessarily implies nonzero temperature. This is equivalent to showing that as $T \to 0$, $\mathrm{tr}(\rho(T)H)/N \to 0$. This is indeed the case because as $T \to 0$, $\mathrm{tr}(\rho(T)H)/N \approx E_1 e^{-E_1/T}/N$, which indeed vanishes as $T \to 0$ (E_1 denotes the energy of the first excited state, which is a constant independent of N since the LDPC code Hamiltonian under discussion is a sum of commuting projectors). Therefore, if we assume that the Gibbs state is separable for all nonzero temperatures, we arrive at a contradiction. Hence, the Gibbs state must be long-range entangled up to a nonzero temperature T. It seems reasonable to assume that at sufficiently high temperature, the Gibbs state is SRE. Therefore, one expects a separability transition at some temperature T_c that satisfies $0 < T_c < \infty$.

It is important to note that the above-argued separability transition does not necessarily imply that the Gibbs state has a *thermodynamic* phase transition, i.e., it need not be accompanied by a singularity of the partition function.

VII. SOME CONNECTIONS BETWEEN SEPARABILITY AND OTHER MEASURES OF MIXED-STATE COMPLEXITY

In this section, we comment on some connections among the separability criteria, purification, double states, and strange correlators.

A. Connections among separability, purification, and double states

In Sec. V C, we used the double-state formalism to probe decoherence-induced transitions. However, the connection between ρ being sym-SRE and the double state $|\rho\rangle$ being trivial remains unclear. In this subsection, we attempt to bridge the gap between them using purification of the mixed state.

We first recall the idea of purification: given a mixed state $\rho_{\mathcal{H}}$ in the Hilbert space $\mathcal{H}$, there exists a purification in the double Hilbert space $\mathcal{H}\otimes\bar{\mathcal{H}}$, with $\bar{\mathcal{H}}$ having the same dimension as $\mathcal{H}$:

$$|\rho^{1/2}\rangle = \rho_{\mathcal{H}}^{1/2} \otimes I_{\bar{\mathcal{H}}} |\Phi\rangle_{\mathcal{H}\otimes\bar{\mathcal{H}}}, \tag{43}$$

where $|\Phi\rangle_{\mathcal{H}\otimes\bar{\mathcal{H}}}$ is a maximally entangled state between $\mathcal{H}$ and $\bar{\mathcal{H}}$. It is straightforward to see that $\mathrm{tr}_{\bar{\mathcal{H}}}(|\rho^{1/2}\rangle\langle\rho^{1/2}|) = \rho_{\mathcal{H}}$. Besides, we note that Eq. (43) is somtimes called "standard purification" [55], and all possible purifications are equivalent up to an isometry applied merely in $\bar{\mathcal{H}}$. If one uses Eq. (2) as a definition of an SRE mixed state ρ, then $|\rho^{1/2}\rangle$ being SRE implies that ρ is SRE. However, it is not obvious to us how to show that this implies that ρ can be written as a convex sum of SRE states [Eq. (1)]. Instead, we are only able to show that if $|\rho^{1/2}\rangle$ is SRE, one can write the mixed state $\rho \otimes I/\dim(\bar{\mathcal{H}})$ (which lives in the Hilbert space system ⊗ ancillae) as a convex sum of SRE states. To see this, we first note that a complete basis for the Hilbert space $\mathcal{H}\otimes\bar{\mathcal{H}}$ can be obtained from a single maximally entangled state by application of local unitaries *merely* in $\bar{\mathcal{H}}$. Specifically, if we denote the complete basis of Bell pairs for a spin-1/2 system as $\{|\phi_{m,n}\rangle, m,n = 0,1\}$, all of them are related to $|\phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ through $|\phi_{m,n}\rangle = (Z_{\bar{\mathcal{H}}})^m (X_{\bar{\mathcal{H}}})^n |\phi\rangle$. It then follows that a complete basis for $\mathcal{H}\otimes\bar{\mathcal{H}}$ can be written as

$$|\Phi_{m,n}\rangle = \prod_j (Z_{j,\bar{\mathcal{H}}})^{m_j} (X_{j,\bar{\mathcal{H}}})^{n_j} |\Phi\rangle, \tag{44}$$

with $m = (m_1, m_2, \ldots)$ and $n = (n_1, n_2, \ldots)$. Since $|\Phi_{m,n}\rangle$ are obtained by application of the local unitary in $\mathcal{H}$ to a maximally entangled state, they are all also maximally entangled. We now use the same idea as we used to define CDA states [Eq. (4)] by writing $\rho \otimes I/\dim(H)$ as $1/\dim(\bar{\mathcal{H}}) \sum_{m,n} \left(\rho^{1/2} \otimes I\right) |\Phi_{m,n}\rangle\langle\Phi_{m,n}| \left(\rho^{1/2} \otimes I\right)$:

$$\begin{aligned}\rho\otimes\frac{I}{\dim(\bar{\mathcal{H}})}&=\frac{1}{\dim(\bar{\mathcal{H}})}\sum_{m,n}\rho^{1/2}\otimes I\Big[\prod_j(Z_{j,\bar{\mathcal{H}}})^{m_j}(X_{j,\bar{\mathcal{H}}})^{n_j}\Big]|\Phi\rangle\langle\Phi|\Big[\prod_k(Z_{k,\bar{\mathcal{H}}})^{m_k}(X_{k,\bar{\mathcal{H}}})^{n_k}\Big](\rho^{1/2}\otimes I)\\&=\frac{1}{\dim(\bar{\mathcal{H}})}\sum_{m,n}\Big[\prod_j(Z_{j,\bar{\mathcal{H}}})^{m_j}(X_{j,\bar{\mathcal{H}}})^{n_j}\Big](\rho^{1/2}\otimes I)|\Phi\rangle\langle\Phi|(\rho^{1/2}\otimes I)\Big[\prod_k(Z_{k,\bar{\mathcal{H}}})^{m_k}(X_{k,\bar{\mathcal{H}}})^{n_k}\Big]\\&=\frac{1}{\dim(\bar{\mathcal{H}})}\sum_{m,n}\Big[\prod_j(Z_{j,\bar{\mathcal{H}}})^{m_j}(X_{j,\bar{\mathcal{H}}})^{n_j}\Big]|\rho^{1/2}\rangle\langle\rho^{1/2}|\Big[\prod_k(Z_{k,\bar{\mathcal{H}}})^{m_k}(X_{k,\bar{\mathcal{H}}})^{n_k}\Big]\\&=\frac{1}{\dim(\bar{\mathcal{H}})}\sum_{m,n}|\rho^{1/2}_{m,n}\rangle\langle\rho^{1/2}_{m,n}|,\ \ |\rho^{1/2}_{m,n}\rangle=(\prod_j(Z_{j,\bar{\mathcal{H}}})^{m_j}(X_{j,\bar{\mathcal{H}}})^{n_j}|\rho^{1/2}\rangle.\end{aligned}\tag{45}$$

In the second line, we use the property that $\prod_j(Z_{j,\bar{\mathcal{H}}})^{m_j}(X_{j,\bar{\mathcal{H}}})^{n_j}$ and $\rho^{1/2}$ commute, as they act on different Hilbert spaces. Since $|\rho^{1/2}_{m,n}\rangle$ is related to $|\rho^{1/2}\rangle$ by a unitary acting solely in $\bar{\mathcal{H}}$, if $|\rho^{1/2}\rangle$ is SRE, then so is $|\rho^{1/2}_{m,n}\rangle$. Therefore, if there exists an SRE purification for ρ [Eq. (43)], then $\rho\otimes I/\dim(\bar{\mathcal{H}})$ can be written as a convex sum of SRE pure states [Eq. (45)].

However, we emphasize that the converse is not true: if $|\rho^{1/2}\rangle$ is not SRE, it does not rule out the possibility that the mixed state ρ is still SRE. This can be most easily seen by one considering the following counterexample that also appears in Sec. II. Let ρ be the convex sum of two product states $|0\rangle^N$ and $|1\rangle^N$, i.e.,

$$\rho=\frac{1}{2}[(|0\rangle\langle 0|)^{\otimes N}+(|1\rangle\langle 1|)^{\otimes N}].\tag{46}$$

It follows that the purified state is the GHZ state:

$$|\rho^{1/2}\rangle=\frac{1}{\sqrt{2}}[|00\rangle^{\otimes N}+|11\rangle^{\otimes N}],\tag{47}$$

which is clearly long-range entangled. This implies that $|\rho^{1/2}\rangle$ being trivial is a sufficient but not necessary condition for $\rho\otimes I/\dim(\bar{\mathcal{H}})$ being trivial.

The advantage of studying ρ using its purification is obvious: instead of finding the decomposition in Eq. (4), one needs to deal with only a single pure state $|\rho^{1/2}\rangle$. However, it is, in general, difficult to compute $|\rho^{1/2}\rangle$, as taking a square root of ρ is nontrivial if $H_\rho=-\ln(\rho)$ does not admit a simple compact form. An alternative is to consider the double state $|\rho\rangle=\rho\otimes I|\Phi\rangle$ in Eq. (30) (note that if the original density is pure, i.e., $\rho^2=\rho$, then the double state $|\rho\rangle$ is equivalent to the purified state $|\rho^{1/2}\rangle$). Heuristically, since the coefficient in front of H_ρ for $|\rho\rangle$ is greater than the coefficient for $|\rho^{1/2}\rangle$, we expect that if $|\rho\rangle$ is SRE, then $|\rho^{1/2}\rangle$ is SRE as well, but we do not know how to prove this. This is consistent with the result in Ref. [20], where the critical error rate for $|\rho\rangle$ being trivial is higher than the error rate at which the topological entanglement negativity drops to zero, and is also consistent with the results in Ref. [17] for the error threshold for separability of topologically ordered mixed states.

B. Connections between convex decomposition and strange correlators

In Sec. IV, we studied separability transitions for cluster-state SPT orders in various dimensions using the CDA [Eq. (4)] with the initial basis $\{|m\rangle\}$ as product states satisfying the corresponding symmetry of the cluster-state SPT (which was the Pauli X basis in all the cases we considered). Fortuitously, as we discussed, the threshold for the CDA states being sym-SRE exactly corresponded to the error rate beyond which ρ must be sym-LRE when general arguments are used, indicating that our choice of CDA is optimal.

Intriguingly, the symmetric product state basis to generate the CDA has an apparently close connection with the strange correlator [32], which was originally devised as a diagnosis for the SPT pure states and has recently been used to probe the nontrivial SPT mixed states [29,30]. To see the connection between them, we briefly review the original strange correlator for SPT pure states and two types of strange correlator introduced in Ref. [29]. If we choose $|m\rangle$ as the disordered product state respecting the symmetry group G, the strange correlator for a pure state $|\psi\rangle$ is defined as [32]

$$C_m(j-k)=\frac{\langle m|O_jO_k|\psi\rangle}{\langle m|\psi\rangle},\tag{48}$$

where O is some operator that transforms nontrivially under G. The basic idea of the strange correlator is that the temporal edge of an SPT pure state (when the many-body wave function is expressed as an imaginary-time path integral) mimics its spatial edge. Since at least 2D SPT orders possess nontrivial spatial edge states (in three dimensions, there also exists a possibility of boundary topological order), one may also use the temporal correlation defined in Eq. (48) to probe nontrivial SPT-order phase. To generalize the strange correlator from pure states to mixed states, two types of strange correlator were introduced in Ref. [29]. The type-I strange correlator is defined as

$$C_m^I(j-k) = \frac{\langle m|\rho O_j O_k|m\rangle}{\langle m|\rho|m\rangle}. \tag{49}$$

In the pure-state limit $\rho = |\psi\rangle\langle\psi|$, the type-I strange correlator reduces to Eq. (48). Therefore, in the case of one subjecting local decoherence to an SPT pure state, C_m^I can be intuitively regarded as asking whether the local decoherence destroys the temporal edge states. However, it was shown in Ref. [29] that the type-I strange correlator is unable to detect the average SPT order mentioned in Ref. [26]. Instead, it was argued that the nontriviality of such an SPT order should be detected by the type-II strange correlator, defined as

$$C_m^{II}(j-k) = \frac{\langle m|O_k^\dagger O_j^\dagger \rho O_j O_k|m\rangle}{\langle m|\rho|m\rangle}. \tag{50}$$

In the pure-state limit, it reduces to $|\langle m|O_j O_k|\psi\rangle|^2/\langle m|\psi\rangle$. Roughly speaking, the type-II strange correlator is devised to capture the case that ρ can be written as an incoherent sum of pure states $|\psi_p\rangle$, where $\langle m|O_j O_k|\psi_p\rangle$ is nontrivial but can be either positive or negative depending on $|\psi_p\rangle$.

On the other hand, the necessary condition for the mixed state to be nontrivial with use of separability criteria is the nontriviality of CDA states $|\psi_m\rangle$, which may be probed by several physical observables S as discussed in Sec. IV:

$$\frac{\langle\psi_m|S|\psi_m\rangle}{\langle\psi_m|\psi_m\rangle} = \frac{\langle m|\rho^{1/2} S \rho^{1/2}|m\rangle}{\langle m|\rho|m\rangle}. \tag{51}$$

Comparing Eqs. (49)–(51), one finds that the denominator is always the fidelity between a symmetric product state and the mixed state of interest:

$$\begin{aligned}\mathcal{Z}_m &= \mathrm{tr}(\rho|m\rangle\langle m|)\\ &= \langle m|\rho|m\rangle = \langle\psi_m|\psi_m\rangle.\end{aligned} \tag{52}$$

For the numerator, Eq. (51) involves insertion of an operator between $\langle m|\rho^{1/2}$ and $\rho^{1/2}|m\rangle$, while the strange correlator involves insertion of an operator between $\langle m|\rho$ and $|m\rangle$.

VIII. SUMMARY AND DISCUSSION

In this work we explored the interplay of complexity and symmetry for many-body mixed states. Specifically, we asked whether a given mixed state can be expressed as a convex sum of symmetric short-range-entangled pure states, which we took as a definition of an SRE mixed state subjected to a given symmetry (a "sym-SRE" mixed state, Sec. II). Our primary aim was to identify "many-body separability transitions" as a function of an appropriate tuning parameter (e.g., decoherence rate or temperature) across which the nature of the mixed state changes qualitatively—on one side of transition the mixed state is sym-SRE and on the other side it is sym-LRE (i.e., not sym-SRE). Analogous phase diagrams for intrinsic topological orders subjected to local decoherence [18–22] were recently studied in Ref. [17]. Our general approach was to first seek constraints that imply that a mixed state is necessarily long-range entangled, and absent such constraints, we developed tools to find the regime where a mixed state can be shown to be sym-SRE. One of the tools that allowed us to make progress was that local decoherence converts ground states of several SPT orders, e.g., cluster states in various dimensions, to a Gibbs state.

In the context of SPT orders subjected to local decoherence, we focused on cluster states in various dimensions and obtained their "separability phase diagram" as shown in Fig. 1. As evident from Fig. 1, the phase diagram gets progressively richer as one moves up in spatial dimensionality. The paths solely along the x and y axes in these phase diagrams correspond to the special case of "average SPT" mixed states where one of the symmetries is exact, while the other is average [26–30]. It is crucial to note that although the decohered mixed state takes a Gibbs form, the corresponding partition function is *not* singular at any nonzero temperature for any of these cluster states. This is because any local channel can be Stinespring dilated as a local unitary circuit in the enlarged system, and thus any physical observables $\mathrm{tr}(\rho O)$ with O acting on a large but finite region must be a smooth function of the error rate [18,19,86]. Therefore, the different phases in Fig. 1 arise *only* because we are requiring that the density matrix be expressible as a convex sum of pure, symmetric states. As a consequence, these transitions are conceptually distinct from thermal phase transitions, and are more akin to "complexity phase transitions" for the mixed state, when a symmetry is enforced. We briefly discussed relation with other approaches to classifying mixed-state SPT orders [26–30] in Sec. VII.

It is also interesting to contrast the symmetry-enforced separability transitions in decohered 2D and 3D cluster states with decoherence-induced separability transitions in 2D and 3D toric codes, studied in Ref. [17]. In both cases, one finds the appearance of the same statistical mechanics models (e.g., RBIM in two dimensions). This similarity can be traced to the fact that the ground state of toric codes can be obtained from the ground state of the cluster states by one performing appropriate projective measurements [89–92], along with the equivalence between local and thermal decoherence for cluster states (this statement holds true also for the fractonic X-cube model [93] and its parent cluster state [91]).

We also studied nonstabilizer topological states subjected to local decoherence. In particular, for free-fermion chiral states corresponding to a $p + ip$ superconductor, we argued that if the quantum channel responsible for decoherence breaks the fermion parity, the resulting Gibbs state can be expressed as a convex sum of nonchiral states, and is therefore SRE at any nonzero decoherence rate (Sec. V). We also studied a case where the channel respects the fermion parity and identified a mixed-state phase transition as a function of the decoherence rate using the double-state formalism. This transition can be thought of as corresponding to spontaneous breaking of the fermion parity, and as far as we know, does not have a pure-state counterpart. Intuitively, in a pure-state context, breaking fermion parity spontaneously essentially requires assigning a nonzero expectation value to fermionic operators, which is unphysical. In contrast, in the context of a mixed state, breaking fermion parity spontaneously means that the environment can exchange fermions with the system "spontaneously," which is not unphysical (in the double-state formalism, this corresponds to assigning a nonzero expectation value to the bosonic order parameter $\eta\gamma$, where η and γ, respectively, denote the fields corresponding to bra and ket Hilbert spaces).

We also analyzed separability transitions in the Gibbs state of the quantum Ising model and argued that the Gibbs state is SRE at any nonzero temperature is and sym-SRE only for $T > T_c$, where T_c is the critical temperature corresponding to the spontaneous symmetry breaking (Sec. III). We expect similar results to hold for other models whose Gibbs state shows a spontaneous breaking of zero-form discrete symmetry.

Finally, in Sec. VI, we provided a short argument that the Gibbs states of Hamiltonians that satisfy the NLTS condition [47] must exhibit a separability transition at a nonzero temperature.

In the rest of this section, we discuss various aspects of our results and discuss questions for further exploration.

A. SPT and chiral states

The technique we used to study phase diagrams of various cluster states relied on the fact the quantum channel resulted in Gibbs states [Eqs. (5) and (6)]. It is not obvious how to generalize it to other SPT states. On that note, the following $\mathbb{Z}_N$ generalization may be helpful to study $\mathbb{Z}_N$ cluster states and topological orders produced by partial measurement of such states. Let us consider a commuting projector Hamiltonian of the form $H = \sum P_i$, where P_i are projectors ($P_i^2 = 1$) written as $P_i = 1/N \sum_{n=0}^{N-1} h_i^n$, with $h_i^N = 1$. Let us now introduce the following set of Kraus operators on each site i: $K_1(i) = \sqrt{1-p}\mathbb{1}$, $K_2(i) = \sqrt{p/2}K(i)$, and $K_3(i) = \sqrt{p/2}K^\dagger(i)$, where $K^\dagger(i)K(i) = K(i)K^\dagger(i) = \mathbb{1}$, and $K(i)$ are clock operators that satisfy $K(i)h_iK^\dagger(i) = e^{2\pi i/N}h_i, K^\dagger(i)h_iK(i) = e^{-2\pi i/N}h_i$. One may verify that the application of this channel on all sites again results in a Gibbs state for H.

It might also be interesting to study "intrinsically mixed" SPT states introduced in Refs. [26,27] from the point of view of separability. These are SPT states that can exist only in the presence of decoherence. Conversely, it will be interesting to understand our results on nontrivial mixed SPT orders protected by higher-form symmetries, such as 2D and 3D cluster state, by one using the techniques in Refs. [26,27], which primarily focused on zero-form symmetry SPT orders.

In the context of chiral states, we studied a phase transition driven by a channel where the Kraus operators were Majorana bilinears (Sec. V C). We analyzed this problem using only the double-state formalism. As suggested by the problem of decoherence in toric code, the double state is likely to overestimate the threshold for the actual transition, and it will be interesting to find a description of the aforementioned transition in a $p + ip$ SC directly in terms of the separability properties of the mixed state.

One important subtlety we point out is that we assumed periodic boundary conditions in our discussion of the SPT and chiral states. If instead one considers open boundaries such that the boundaries do not break the symmetry responsible for nontrivial SPT/chiral topological order, then the pure (nondecohered) state is always LRE, e.g., due to propagating edge modes or topolgical order at the boundary. In the presence of decoherence, our naive expectation is that the resulting mixed state is not sym-SRE, even if the decoherence breaks the symmetry from exact down to average. It will be interesting to study this aspect in the future.

B. Symmetry-broken states

The first example we discussed, primarily to illustrate the distinction between SRE and sym-SRE states, was the Gibbs state of the transverse-field Ising model in any dimension (Sec. III). We discussed an explicit decomposition of this state at a nonzero temperature as a convex sum of pure states that we argued are SRE at any nonzero temperature. This conclusion is consistent with numerical results on Renyi negativity [24] and mean-field arguments [23,25]. On the other hand, for the case where one imposes the Ising symmetry on the pure states into which the Gibbs state is being decomposed, we used an argument from Ref. [21] to show that these pure states must be long-range entangled for $T < T_c$. This implies that the Gibbs state is sym-LRE for $T \leq T_c$. In contrast, for $T > T_c$, we provided an explicit sym-SRE decomposition of the Gibbs state. The basic idea of the argument is to write $e^{-\beta H}$ as $\sum_\phi e^{-\beta H/2}|\phi\rangle\langle\phi|e^{-\beta H/2}$, where $\{\phi\}$ are chosen as a complete set of states in the $z(x)$ basis if one wants to expand the Gibbs state as a sum of sym-SRE pure states or SRE pure states.

There are several open questions along this direction. Firstly, the argument we provided for the aforementioned pure states being sym-SRE or SRE is not mathematically rigorous. To explicitly show that a state is SRE, one needs to construct a finite-depth circuit that prepares it starting from a product state. We provided arguments only in the continuum limit that the pure states under consideration have short-range correlations. It will be worthwhile to study the entanglement structure of the pure states we claimed to be SRE with use of numerical methods (e.g., quantum Monte Carlo methods) or by a detailed field-theoretic analysis. Secondly, as we discussed, the transverse-field Ising model for $d \geq 2$ must exhibit a separability transition from a sym-SRE state to a sym-LRE state as a function of temperature. It will be interesting to study the symmetry-resolved negativity [94] to quantify the nature of long-range entanglement across this transition. Finally, our arguments apply only to Gibbs states that break a discrete symmetry spontaneously, and it will be interesting to consider generalization to systems with spontaneously broken continuous symmetries that host Goldstone modes at a nonzero temperature.

C. Experimental and numerical implications

It is interesting to contemplate experimental implications of a symmetry-enforced separability transition. One perspective is that symmetry-resolved versions of mixed-state entanglement measures such as entanglement negativity or entanglement of formation, which are specifically designed to quantify the lack of separability, would likely experience a singularity across such a phase transition. For example, for the Gibbs state ρ of the transverse-field Ising model (Sec. III), one can, in principle, prepare the states $\rho_\pm = P_\pm \rho$, where $P_\pm$ are the projectors onto the even and odd sectors of the Ising symmetry. This can be done, for example, by one entangling an auxiliary qubit with the system qubits sequentially using controlled NOT gates, and by one measuring the auxiliary qubit at the end. As discussed in Sec. III, the resulting mixed state (i.e., ρ_+ or ρ_-, depending on the outcome of the measurement on the auxiliary qubit) will show long-range mixed-state entanglement for $T < T_c$, in contrast to the original density matrix ρ, which will be short-range entangled for any $T > 0$. The long-range entanglement of $\rho_\pm$ can, in principle, be quantified experimentally with use of the Renyi negativity [95].

One may also imagine a very patient, gedanken experimentalist who has access to local unitary gates with a finite fidelity, so that such an experimentalist has the ability to prepare only an ensemble of SRE pure states (i.e., pure states that can be prepared with a constant-depth unitary). If this is the case, then a separability transition from an SRE mixed state to an LRE mixed state is equivalent to the transition from success to failure in preparing the ensemble corresponding to the mixed state. One may similarly characterize a transition from a sym-SRE state to a sym-LRE state by putting symmetry constraints on the local gates that form the circuit.

Perhaps a more practical implication of our results is that they may allow efficient classical simulation of a class of mixed states. For example, in the context of the Gibbs state of the quantum Ising model, we argued that it admits a convex decomposition in terms of SRE pure states at any nonzero temperature if one does not impose any symmetry constraint on the pure states. Since SRE states are typically easier to study, such a representation facilitates the task of simulating the corresponding mixed state. In contrast, if one tries to prepare the Gibbs state of the quantum Ising model starting with a *product state* (assisted with ancillae), then long-range correlations below T_c imply that one necessarily requires a deep quantum circuit [96]. We note that the decompositions we study generically involve an exponentially large number of pure states, which may lead to another difficulty in preparation. We can imagine at least two distinct ways to address this. Firstly, if a mixed state is SRE, and does not contain any classical long-range correlations, then it is reasonable to expect that it can be purified into an SRE pure state (with use of auxiliary degrees of freedom). This equivalence between an SRE mixed state and SRE purification is discussed for Gibbs states in Ref. [96] and is suggested more generally in Ref. [2] although we are not aware of an explicit proof or construction in the non-Gibbs case. If one can indeed find an SRE purification, then an SRE mixed state can be prepared by a finite-depth unitary circuit acting on system and auxiliary degrees of freedom. An alternative route that is more generally available is to sample from the ensemble of SRE states that enter a given decomposition (assuming that the density matrix is SRE) with use of Monte Carlo algorithms, instead of one preparing each and every SRE state that enters the SRE decomposition. While the sampling task may still be generally hard even in classical mechanics [97,98], the decomposition we have provided clearly simplifies the "quantum hardness" of simulating a mixed state, analogous to the METTS algorithm [56].

On a different note, one way to prepare an ensemble of pure states that may show a mixed-state separability phase transition is via a judicious combination of unitaries and measurements [21,34,35,89–92,99]. For example, Refs. [34,35] provide a construction of mixed states that are closely related to the mixed states discussed in Sec. II, and which have also been implemented in a recent experiment [100]. It will be interesting to design experiments that probe the phase diagram in Fig. 1 using similar ideas, although we suspect it may be comparatively more challenging as it requires measuring nonlocal observables supplemented with an appropriate decoding scheme [34].

ACKNOWLEDGMENTS

The authors thank Dan Arovas, Tim Hsieh, John McGreevy, and Bowen Shi for helpful discussions, and

Tsung-Cheng Lu, Shengqi Sang, and William Witczak-Krempa for helpful comments on the draft. T.G. is supported by the National Science Foundation under Grant No. DMR-1752417. This research was supported in part by the National Science Foundation under Grant No. NSF PHY-1748958.

APPENDIX A: DETAILS OF THE STRING ORDER PARAMETER FOR A 1D CLUSTER STATE

This appendix provides details of evaluating the string order parameter for a 1D cluster state with respect to each ρ_{Q_a,Q_b}, i.e., $\mathrm{tr}(\rho_{Q_a,Q_b}S_{a/b})/\mathrm{tr}(\rho_{Q_a,Q_b})$. We first compute the denominator in this expression, namely, the trace of ρ_{Q_a,Q_b} [corresponds to probability of sector with charge (Q_a, Q_b)], by inserting the complete basis $\{|x_{\mathbf{a,b}}\rangle\}$ and $\{|z_{\mathbf{a,b}}\rangle\}$:

$$\mathrm{tr}(\rho_{Q_a,Q_b}) \propto \sum_{x_{\mathbf{a,b}},z_{\mathbf{a,b}}} \langle x_{\mathbf{a,b}}|\rho_{Q_a}|z_{\mathbf{a,b}}\rangle\langle z_{\mathbf{a,b}}|\rho_{Q_b}|x_{\mathbf{a,b}}\rangle \propto \sum_{x_{\mathbf{a,b}}\in Q_{a,b},z_{\mathbf{a,b}}} \langle x_{\mathbf{a,b}}|\rho_a|z_{\mathbf{a,b}}\rangle\langle z_{\mathbf{a,b}}|\rho_b|x_{\mathbf{a,b}}\rangle, \quad \text{(A1)}$$

where $\sum_{x_{\mathbf{a,b}}\in Q_{a,b}}$ denotes summation over all possible $x_{\mathbf{a,b}}$ in the Q_a and Q_b sectors, i.e., $\prod_j(x_{a,j}) = (-1)^{Q_a}$ and $\prod_j(x_{b,j}) = (-1)^{Q_b}$. Now,

$$\langle x_{\mathbf{a,b}}|\rho_a|z_{\mathbf{a,b}}\rangle \propto e^{-\beta_a\sum_j z_{b,j-1}x_{a,j}z_{b,j+1}}\langle x_{\mathbf{a,b}}|z_{\mathbf{a,b}}\rangle \propto e^{-\beta_a\sum_j z_{b,j-1}x_{a,j}z_{b,j+1}}. \quad \text{(A2)}$$

Similarly, $\langle z_{\mathbf{a,b}}|\rho_b|x_{\mathbf{a,b}}\rangle \propto e^{-\beta_b\sum_j z_{a,j}x_{b,j}z_{a,j+1}}$. It follows that $\mathrm{tr}(\rho_{Q_a,Q_b}) \propto \sum_{x_{\mathbf{a}}\in Q_a}\mathcal{Z}_{\text{1D Ising},x_{\mathbf{a}}}\sum_{x_{\mathbf{b}}\in Q_b}\mathcal{Z}_{\text{1D Ising},x_{\mathbf{b}}}$, where $\mathcal{Z}_{\text{1D Ising},x_{\mathbf{a}}} = \sum_{z_{\mathbf{b}}} e^{\beta_a\sum_j x_{a,j}z_{b,j-1}z_{b,j}}$ is the partition function of the 1D Ising model with the Ising interaction determined by $x_{\mathbf{a}}$ (the expression for $\mathcal{Z}_{\text{1D Ising},x_{\mathbf{b}}}$ is analogously obtained by one interchanging a and b). For a system with periodic boundary conditions, one can parameterize all $x_{\mathbf{a}} \in Q_a$ by performing the transformation $x_{a,j} \to x_{a,j}s_{b,j-1}s_{b,j}, s_{b,j} = \pm 1$ from any $x_{\mathbf{a}}$ that belongs to Q_a. Besides, one can easily verify that $\mathcal{Z}_{\text{1D Ising},x_{\mathbf{a}}}$ is invariant under the transformation by changing the dummy variables $z_{b,j-1} \to s_{b,j-1}z_{b,j-1}$. In other words, $\mathcal{Z}_{\text{1D Ising},x_{\mathbf{a}}} = \mathcal{Z}_{\text{1D Ising},Q_a}$ depends only on the charge Q_a. Therefore,

$$\mathrm{tr}(\rho_{Q_a,Q_b}) \propto \mathcal{Z}_{\text{1D Ising},Q_a}\mathcal{Z}_{\text{1D Ising},Q_b}, \quad \text{(A3)}$$

which is the product of the partition functions of the 1D Ising model in the Q_a and Q_b sectors. The evaluation of the numerator of the string order can be done in a similar way. The only term that does not cancel out with the denominator is associated with $\langle x_{\mathbf{a,b}}|\rho S_a(j,k)|z_{\mathbf{a,b}}\rangle$, which can be evaluated as

$$\langle x_{\mathbf{a,b}}|\rho S_a(j,k)|z_{\mathbf{a,b}}\rangle \propto e^{-\beta_a\sum_j z_{b,j-1}x_{a,j}z_{b,j+1}} z_{b,j-1}\left(\prod_{l=j}^{k}x_{a,l}\right)z_{b,k}\langle x_{\mathbf{a,b}}|z_{\mathbf{a,b}}\rangle \propto \left(\prod_{l=j}^{k}x_{a,l}\right)e^{-\beta_a\sum_j z_{b,j-1}x_{a,j}z_{b,j+1}}z_{b,j-1}z_{b,k}. \quad \text{(A4)}$$

It follows that

$$\frac{\mathrm{tr}(\rho_{Q_a,Q_b}S_a(j,k))}{\mathrm{tr}(\rho_{Q_a,Q_b})} = \left(\prod_{l=j}^{k}x_{a,l}\right)\langle z_{j-1}z_k\rangle_{\text{1D Ising},x_{\mathbf{a}}}\Big|_{x_{\mathbf{a}}\in Q_a}, \quad \text{(A5)}$$

where $\langle z_{j-1}z_k\rangle_{\text{1D Ising},x_{\mathbf{a}}\in Q_a}$ is the spin-spin correlation function of the 1D Ising model with the Ising interaction determined by any $x_{\mathbf{a}}$ belonging to the Q_a sector. Note that $(\prod_{l=j}^{k}x_{a,l})\langle z_{j-1}z_k\rangle_{\text{1D Ising},x_{\mathbf{a}}\in Q_a}$ is invariant under the transformation $x_{a,j} \to x_{a,j}s_{b,j-1}s_{b,j},\ s_{b,j} = \pm 1$, and thus Eq. (A5) is independent of the choice for any $x_{\mathbf{a}} \in Q_a$. For example, one can choose $x_{a,j} = 1$ for all j if $Q_a = 0$. On the other hand, if $Q_a = 1$, one can choose $x_{a,j} = 1$ for all $j \neq N$ and $x_{a,N} = -1$. It follows that the string order of ρ_{Q_a,Q_b} for both $Q_a = 0$ and $Q_a = 1$ can be mapped to $\langle z_{j-1}z_k\rangle_{\text{1D Ising}}$, the spin-spin correlation function of the 1D ferromagnetic Ising model. The results for S_b are similar. Since $\langle z_{j-1}z_k\rangle_{\text{1D Ising}}$ decays exponentially with $|j-k|$ for any $\beta < \infty$, we conclude that ρ_{Q_a,Q_b} has no string order as long as $p_a, p_b > 0$.

APPENDIX B: DETAILS OF CALCULATIONS FOR CHIRAL FERMIONS SUBJECTED TO DECOHERENCE

1. Covariance matrix under a channel linear in fermion operators

We are interested in subjecting the ground state $\rho_0 = |\psi_0\rangle\langle\psi_0|$ of a Gaussian fermionic Hamiltonian H to the composition of the following single-Majorana channel on all sites:

$$\mathcal{E}_j[\rho] = (1-p)\rho + p\gamma_j\rho\gamma_j. \quad \text{(B1)}$$

The goal in this section is to show that the resulting covariance matrix is $\mathcal{E}(M) = (1-2p)^2M$ and the resulting density matrix $\rho \propto e^{-i\beta\sum_j \xi_{2j-1}\xi_{2j}}$, where $\tanh\beta = (1-2p)^2$ and $\xi_j = (O^T)_{jk}\gamma_k$ are the Majorana operators that block-diagonalize the original Hamiltonian H. We note that one can also pair up two Majorana fermions to get regular fermions through $\alpha_j = (\xi_{2j-1} + i\xi_{2j})/2$, and the density matrix takes the form $\rho \propto e^{-2\beta\sum_j \alpha_j^\dagger\alpha_j}$ mentioned in Sec. V B.

To proceed, we note that Eq. (B1) will map a Gaussian state to a Gaussian state. A Gaussian fermionic state ρ, whether pure or mixed, is fully specified by the covariance matrix M, defined as

$$M_{jk} = -i\mathrm{tr}(\rho(\gamma_j\gamma_k - \delta_{jk})). \tag{B2}$$

Therefore, to determine the evolution of the density matrix under the channel, it suffices to determine how the covariance matrix evolves, which we denote as $\mathcal{E}(M)$. Using $\gamma_l(\gamma_j\gamma_k)\gamma_l = (-1)^{\delta_{jl}+\delta_{kl}}\gamma_j\gamma_k$, one can easily compute the element of the resulting covariance matrix $[\mathcal{E}_l(M)]_{jk}$:

$$\begin{aligned}
[\mathcal{E}_l(M)]_{jk} &= -i\mathrm{tr}(\mathcal{E}_l[\rho](\gamma_j\gamma_k - \delta_{jk}))\\
&= (1-p)M_{jk} + (-1)^{\delta_{jl}+\delta_{kl}}pM_{jk}\\
&= [(1-p) + (-1)^{\delta_{jl}+\delta_{kl}}p]M_{jk}\\
&= \begin{cases} M_{jk} & \text{for } j \neq l \text{ and } k \neq l,\\ (1-2p)M_{jk} & \text{for } j = l \text{ or } k = l. \end{cases}
\end{aligned} \tag{B3}$$

It follows that the composition of the channel $\mathcal{E}_l$ on all sites gives

$$\mathcal{E}(M) = (1-2p)^2 M. \tag{B4}$$

To see how one can use Eq. (B4) to deduce the resulting decohered mixed state, let us first explicitly write down the relation between a general density matrix ρ and its covariance matrix M. Let us write the density matrix as $\rho \propto e^{-H_\rho}$. Since ρ is Gaussian, H_ρ can be written as a sum of Majorana bilinears:

$$H_\rho = \frac{i}{2}\sum_{j,k=1}^{2N}\gamma_j K_{jk}\gamma_k, \tag{B5}$$

where K is a $2N\times 2N$ antisymmetric matrix and we denote the number of Majorana modes as $2N$. To see how the matrix M is related to the matrix K, we begin by block-diagonalizing K:

$$K = O(K_d \otimes (iY))O^T, \tag{B6}$$

where K_d is an $N\times N$ diagonal matrix and Y is the Pauli Y matrix (i.e., $[0\ \ -i; i\ \ 0]$). If we denote $\xi_j = (O^T)_{jk}\gamma_k$, the density matrix takes the following form:

$$\rho \propto \prod_j [I - \tanh(K_d)_{j,j}(i\xi_{2j-1}\xi_{2j})]. \tag{B7}$$

Using $-i\mathrm{tr}(\rho\xi_{2j-1}\xi_{2j}) = \tanh(K_d)_{j,j}$ and the relation between ξ_j and γ_j, one can obtain the covariance matrix as

$$M = O(\tanh K_d \otimes (iY))O^T. \tag{B8}$$

Now let us determine the matrix O and the relation between $\tanh K_d$ and p using the property of the initial pure state ρ_0 and Eq. (B4). Since $\rho_0 = |\psi_0\rangle\langle\psi_0|$ is the ground state of the Hamiltonian H, the matrix O at $p=0$ is precisely the one that diagonalizes H. Besides, using $\rho_0^2 = \rho_0$, one finds $\tanh K_d = I$ when $p=0$, and thus $M_0 = O(I\otimes(iY))O^T$. Equation (B4) then gives $M(p) = O[(1-2p)^2 I \otimes (iY)]O^T$. This implies that O remains unchanged and $\tanh(K_d)_{j,j} = (1-2p)^2$ is independent of j. Therefore, the entanglement Hamiltonian takes the form

$$H_\rho(p) = i\beta\sum_j \xi_{2j-1}\xi_{2j}, \tag{B9}$$

where $\tanh\beta = (1-2p)^2$ and $\xi_j = (O^T)_{jk}\gamma_k$ are the Majorana operators that block-diagonalize the original Hamiltonian H.

2. Double-state formalism for fermions

In this section, we derive the double-state formalism for fermions. As a warm-up, we first derive how the bosonic density matrices and channels are mapped to pure states and operators, respectively, under

$$|\rho\rangle = \rho\otimes I|\Phi\rangle. \tag{B10}$$

We later derive the similar correspondence for fermions using Grassmann algebra.

For a general bosonic density matrix $\rho = \sum_{p,q}\rho_q^p|p\rangle\langle q|$, one can compute $|\rho\rangle$ using Eq. (B10):

$$\begin{aligned}
|\rho\rangle &= \sum_{p,q}\rho_q^p|p\rangle\langle q|\Big(\sum_r |r\rangle\otimes|r\rangle\Big)\\
&= \sum_{p,q,r}\rho_q^p\delta_r^q|p\rangle\otimes|r\rangle = \sum_{p,q}\rho_q^p|p,q\rangle,
\end{aligned} \tag{B11}$$

which is tantamount to flipping the bra vector of ρ in the original Hilbert space $\mathcal{H}$ to the ket vector in $\bar{\mathcal{H}}$. This intuition can be visualized by our expressing Eq. (B10) using a tensor network [see Fig. 5(a)]. In this sense, one can regard the maximally entangled states as a tool to transform the bra (ket) spaces to ket (bra) spaces, and such a trick is called "Choi-Jamiołkowski isomorphism." We note that CJ isomorphism was originally used to map quantum channels $\mathcal{E}$ (superoperators) to quantum states σ (density matrices) [42,43],

$$\sigma_{\mathcal{E}} = \mathcal{E}\otimes I[|\Phi\rangle\langle\Phi|], \tag{B12}$$

which can be visualized in Fig. 5(b). However, while both Eq. (B12) and Eq. (B10) map quantum channels to operators, the operators they map quantum channels to are different in general. Specifically, consider the channel $\mathcal{E}[\cdot] = \sum_\alpha K_\alpha(\cdot)K_\alpha^\dagger$ and denote its corresponding operator

Eq. (B10) maps to as $\mathcal{N}$, i.e., $|\mathcal{E}[\rho]\rangle = \mathcal{N}|\rho\rangle$. The operator $\mathcal{N}$ can be obtained by the direct evaluation of $|\mathcal{E}[\rho]\rangle$ for any ρ:

$$\begin{aligned}
|\mathcal{E}[\rho]\rangle &= \sum_{\alpha} K_\alpha \rho K_\alpha^\dagger \otimes I|\Phi\rangle \\
&= \sum_{\alpha,} K_\alpha \left(\sum_q |q\rangle\langle q|\right)\rho\left(\sum_r |r\rangle\langle r|\right)K_\alpha^\dagger \\
&\quad \times \left(\sum_p |p\rangle \otimes |p\rangle\right) \\
&= \sum_\alpha \sum_{p,q,r} K_\alpha |q\rangle\langle r|K_\alpha^\dagger|p\rangle\langle q|\rho|r\rangle|p\rangle \\
&= \sum_\alpha \sum_{q,r} K_\alpha|q\rangle\left(\sum_p |p\rangle\langle p|\right)\bar{K}|r\rangle\langle q|\rho|r\rangle \\
&= \sum_\alpha \sum_{q,r} K_\alpha \otimes \bar{K}_\alpha |q,r\rangle\langle q,r|\rho\rangle \\
&= \sum_\alpha K_\alpha \otimes \bar{K}_\alpha|\rho\rangle. \qquad \text{(B13)}
\end{aligned}$$

Therefore, one finds

$$\mathcal{N}_{\mathcal{E}} = \sum_\alpha K_\alpha \otimes \bar{K}_\alpha. \qquad \text{(B14)}$$

On the other hand, the corresponding operator $\sigma_{\mathcal{E}}$ for the channel $\mathcal{E}$ is evaluated with use of Eq. (B12), and one finds

$$\sigma_{\mathcal{E}} = \sum_\alpha |K_\alpha\rangle\langle K_\alpha|, \qquad \text{(B15)}$$

where $|K_\alpha\rangle = K_\alpha \otimes I|\Phi\rangle$. It is then obvious that $\mathcal{N}_{\mathcal{E}}$ and $\sigma_{\mathcal{E}}$ are different. For example, if $\mathcal{E}[\cdot] = K(\cdot)K^\dagger$ consists of only a single Kraus operator K, then $\sigma_{\mathcal{E}}$ is necessarily a projector, while $N_{\mathcal{E}}$ is not in general.

Now we turn to the CJ map for fermions, i.e., the analogue of Eq. (B13) for fermions. Note that the derivation of $\mathcal{N}_{\mathcal{E}}$ in Eq. (B13) required the insertion of a complete basis. For fermions, this can be achieved by use of Grassmann algebra. To build intuition, we consider the mixed-state density matrix ρ with a single fermionic mode with creation/annihilation operators $\mathbf{c}^\dagger/\mathbf{c}$ (which act on the Hilbert space $\mathcal{H}$ in our notation), i.e., $\rho = \rho(\mathbf{c}, \mathbf{c}^\dagger)$. The maximally entangled state in the double Hilbert space for fermions can then be defined as

$$|\Phi\rangle \equiv (I + e^{i\theta}\mathbf{c}^\dagger\mathbf{d}^\dagger)|00\rangle. \qquad \text{(B16)}$$

Here $\mathbf{d}^\dagger$ denotes the fermionic creation operators in the Hilbert space $\overline{\mathcal{H}}$, $|00\rangle$ is the vacuum defined by $\mathbf{c}|00\rangle = \mathbf{d}|00\rangle = 0$, and $\theta \in [0, 2\pi)$ is an arbitrary phase that we will set to zero for convenience. To derive a compact form for $|\rho\rangle$, we make use of the coherent state $|c,d\rangle = e^{-c\mathbf{c}^\dagger}e^{-d\mathbf{d}^\dagger}|00\rangle$, where c and d are Grassmann numbers. The maximally entangled state in the coherent-state basis can be easily computed:

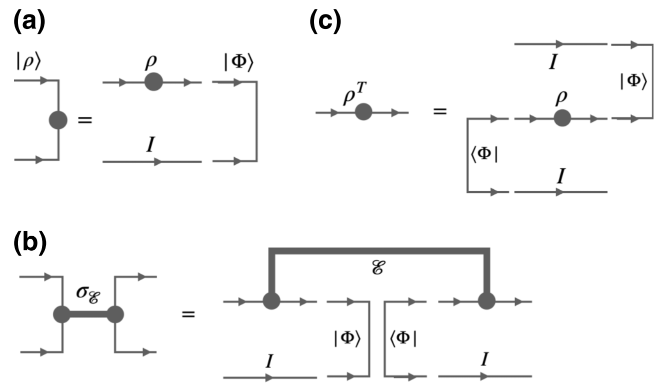

FIG. 5. Tensor network representations of CJ isomorphisn for (a) $|\rho\rangle = \rho \otimes I|\Phi\rangle$, (b) $\sigma_{\mathcal{E}} = \mathcal{E} \otimes I[|\Phi\rangle\langle\Phi|]$, and (c) $\langle m'|\rho^T|m\rangle = (\langle m'| \otimes \langle\Phi|)I \otimes \rho \otimes I(|\Phi\rangle \otimes |m\rangle)$.

$$\left\langle \bar{c}\bar{d}\middle|\Phi\right\rangle = \langle\bar{c}\bar{d}|I + \mathbf{c}^\dagger\mathbf{d}^\dagger|00\rangle = (1 + \bar{c}\bar{d})\left\langle\bar{c}\bar{d}\middle|00\right\rangle = e^{\bar{c}\bar{d}}. \qquad \text{(B17)}$$

Similarly, we can compute $|\rho\rangle$ in the coherent-state basis:

$$\begin{aligned}
\langle\bar{c}\bar{d}|\rho\rangle &= \int \mathcal{D}\bar{\alpha}\mathcal{D}\alpha\langle\bar{c}|\rho|\alpha\rangle e^{-\bar{\alpha}\alpha}\langle\bar{\alpha}\bar{d}|\Phi\rangle \\
&= \int \mathcal{D}\bar{\alpha}e^{\bar{\alpha}(\bar{d}-\alpha)}\mathcal{D}\alpha\langle\bar{c}|\rho|\alpha\rangle \qquad \text{(B18)} \\
&= \int \mathcal{D}\alpha(\alpha - \bar{d})\langle\bar{c}|\rho|\alpha\rangle = \langle\bar{c}|\rho|\bar{d}\rangle.
\end{aligned}$$

In the final line, we use the fact that $(\alpha - \bar{d}) = \delta(\bar{d} - \alpha)$. Therefore, we arrive at the following conclusion: given the density matrix in the coherent-state basis $\langle\bar{c}|\rho|c\rangle$, the corresponding double state in the coherent-state basis $\langle\bar{c}\bar{d}|\rho\rangle$ can be obtained simply by one substituting $c \to \bar{d}$. For example, if $\rho_0 = |\Psi_0\rangle\langle\Psi_0|$ is the density matrix of the pure state, then

$$\begin{aligned}
\langle\bar{c}\bar{d}|\rho_0\rangle &= \langle\bar{c}|\Psi_0\rangle\langle\Psi_0|\bar{d}\rangle = \langle\bar{c}|\Psi_0\rangle\langle\bar{d}|\Psi_0\rangle^* \\
&= \langle\bar{c}\bar{d}|\Psi_0, \Psi_0^*\rangle, \qquad \text{(B19)}
\end{aligned}$$

which is consistent with our intuition on bosonic fields. We emphasize that the left-hand side of Eq. (B18) is defined in the double Hilbert space spanned by the Fock basis $\{|00\rangle, \mathbf{c}^\dagger|00\rangle, \mathbf{d}^\dagger|00\rangle, \mathbf{c}^\dagger\mathbf{d}^\dagger|00\rangle\}$. On the other hand, the right-hand side of Eq. (B18) is defined in the original Hilbert space spanned by $\{|0\rangle, \mathbf{c}^\dagger|0\rangle\}$.

We are now ready to work out the corresponding operator $\mathcal{N}_{\mathcal{E}}$ for the channel $\mathcal{E}[\cdot] = \sum_\alpha K_\alpha(\cdot)K_\alpha^\dagger$

under Eq. (B10). As mentioned in the main text, since Eq. (B10) is linear, one can consider each $K_\alpha(\cdot)K_\alpha^\dagger$ individually. Using $|K_\alpha\rho K_\alpha^\dagger\rangle = (K_\alpha\rho K_\alpha^\dagger)\otimes I|\Phi\rangle = K_\alpha(\rho K_\alpha^\dagger \otimes I|\Phi\rangle) = K_\alpha|\rho K_\alpha^\dagger\rangle$, one finds K_α is unchanged under Eq. (30). Besides, since one can always write $K_\alpha^\dagger$ as a function of $\mathbf{c}$ and $\mathbf{c}^\dagger$, it suffices to consider how to express $|\rho\mathbf{c}\rangle$ and $|\rho\mathbf{c}^\dagger\rangle$ as an operator applying to $|\rho\rangle$. Using Eq. (B18) for $|\rho\mathbf{c}\rangle$, we find

$$\begin{aligned}\langle\bar{c}\bar{d}|\rho\mathbf{c}\otimes I|\Phi\rangle &= \langle\bar{c}|\rho\mathbf{c}|\bar{d}\rangle = \langle\bar{c}|\rho|\bar{d}\rangle\bar{d}\\ &= \bar{d}\langle\bar{c}\bar{d}|\rho\rangle = \langle\bar{c}\bar{d}|\mathbf{d}^\dagger|\rho\rangle.\end{aligned}\tag{B20}$$

In the second line, we use the fact that ρ preserves the fermionic parity for ρ. It is then obvious that $|\rho\mathbf{c}\rangle = \mathbf{d}^\dagger|\rho\rangle$.

Similarly, using Eq. (B18) for $\rho\mathbf{c}^\dagger$, we get $\langle\bar{c}\bar{d}|\rho\mathbf{c}^\dagger\otimes I|\Phi\rangle = \langle\bar{c}|\rho\mathbf{c}^\dagger|\bar{d}\rangle$. However, the evaluation of $\langle\bar{c}|\rho\mathbf{c}^\dagger|\bar{d}\rangle$ is not as straightforward as that of $\langle\bar{c}|\rho\mathbf{c}|\bar{d}\rangle$ since $\langle\bar{c}|\rho\mathbf{c}^\dagger|\bar{d}\rangle$ is not normally ordered. To proceed, we insert the identity between ρ and $\mathbf{c}^\dagger$:

$$\begin{aligned}\langle\bar{c}|\rho I\mathbf{c}^\dagger|\bar{d}\rangle &= \int\mathcal{D}\bar{\alpha}\mathcal{D}\alpha\langle\bar{c}|\rho|\alpha\rangle e^{-\bar{\alpha}\alpha}\langle\bar{\alpha}|\mathbf{c}^\dagger|\bar{d}\rangle\\ &= \int\mathcal{D}\bar{\alpha}\mathcal{D}\alpha\bar{\alpha}e^{\bar{\alpha}(\bar{d}-\alpha)}\langle\bar{c}|\rho|\alpha\rangle.\end{aligned}\tag{B21}$$

In the second line, we use the fact that $\langle\bar{\alpha}|\mathbf{c}^\dagger|\bar{d}\rangle = \bar{\alpha}\langle\bar{\alpha}|\bar{d}\rangle = \bar{\alpha}e^{\bar{\alpha}\bar{d}}$. Now we change the variable α ($\bar{\alpha}$) as $\bar{\beta}$ ($-\beta$) so that we can substitute $\langle\bar{c}\bar{\beta}|\rho\rangle$ for $\langle\bar{c}|\rho|\alpha\rangle$:

$$\begin{aligned}\langle\bar{c}|\rho\mathbf{c}^\dagger|\bar{d}\rangle &= \int\mathcal{D}(-\beta)\mathcal{D}\bar{\beta}(-\beta)e^{-\beta(\bar{d}-\bar{\beta})}\langle\bar{c}\bar{\beta}|\rho\rangle\\ &= \int\mathcal{D}\beta\mathcal{D}\bar{\beta}(\beta e^{\bar{d}\beta})e^{-\bar{\beta}\beta}\langle\bar{c}\bar{\beta}|\rho\rangle\\ &= \int\mathcal{D}\beta\mathcal{D}\bar{\beta}\langle\bar{d}|\mathbf{d}|\beta\rangle e^{-\bar{\beta}\beta}\langle\bar{c}\bar{\beta}|\rho\rangle\\ &= \int\mathcal{D}\bar{\beta}\mathcal{D}\beta\langle\bar{d}|\mathbf{d}|\beta\rangle e^{-\bar{\beta}\beta}\langle\bar{c}\bar{\beta}|\rho\rangle\\ &= \langle\bar{c}\bar{d}|-\mathbf{d}|\rho\rangle.\end{aligned}\tag{B22}$$

In the fifth line, the minus sign is attributed to the exchange of $\mathcal{D}\beta$ and $\mathcal{D}\bar{\beta}$. It follows that $|\rho\mathbf{c}^\dagger\rangle = -\mathbf{d}|\rho\rangle$.

Interestingly, treating the CJ map as a general way to transform the bra (ket) space to ket (bra) space leads to other useful applications for fermionic problems. For example, the fermionic transpose can be naturally defined with use of the CJ map, and we find that this definition is consistent with the fermionic time reversal, which was proposed in Ref. [101] to resolve the issue that the conventional definition of the fermionic transpose fails to capture the entanglement negativity due to the formation of the edge Majorana fermions. Specifically, the fermionic transpose can be defined as follows:

$$\langle m'|\rho^T|m\rangle = (\langle m'|\otimes\langle\Phi|)I\otimes\rho\otimes I(|\Phi\rangle\otimes|m\rangle),\tag{B23}$$

where $\{|m\rangle\}$ is an arbitrary complete basis for fermions. One may show that this definition makes sense by expressing Eq. (B23) in terms of a tensor network [see Fig. 5(c)]. We now show that this definition coincides with the ones proposed in Ref. [101] using fermionic time reversal. Noting that $\langle\bar{c}\bar{d}|\Phi\rangle = e^{\bar{c}\bar{d}}$ and $\langle\Phi|cd\rangle = e^{-cd}$, one can evaluate ρ in the coherent-state basis:

$$\begin{aligned}\langle\bar{c}|\rho^T|c\rangle &= \int\mathcal{D}\bar{\alpha}\mathcal{D}\alpha\mathcal{D}\bar{\beta}\mathcal{D}\beta\langle\Phi|\alpha c\rangle e^{-\bar{\alpha}\alpha}\langle\bar{\alpha}\rho|\beta\rangle e^{-\bar{\beta}\beta}e^{\bar{c}\bar{\beta}}\\ &= \int\mathcal{D}\bar{\alpha}\mathcal{D}\alpha\mathcal{D}\bar{\beta}\mathcal{D}\beta e^{\alpha(\bar{\alpha}-c)}\langle\bar{\alpha}\rho|\beta\rangle e^{-\bar{\beta}(\beta+\bar{c})}\\ &= \langle c|\rho|-\bar{c}\rangle.\end{aligned}\tag{B24}$$

Therefore, one can obtain $\langle\bar{c}|\rho^T|c\rangle$ by simply substituting $c\to-\bar{c}$ and $\bar{c}\to c$ in $\langle\bar{c}|\rho|c\rangle$.

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