Relative Entropy of Coherence Quantifies Performance in Bayesian Metrology

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The ability of quantum states to be in superposition is one of the key features that sets them apart from the classical world. This "coherence" is rigorously quantified by resource theories, which aim to understand how such properties may be exploited in quantum technologies. There has been much research on what the resource theory of coherence can reveal about quantum metrology, almost all of which has been from the viewpoint of Fisher information. We prove, however, that the relative entropy of coherence, and its recent generalization to positive operator-valued measures (POVMs), naturally quantify the performance of Bayesian metrology. In particular, we show how a coherence measure can be applied to an ensemble of states. We then prove that during parameter estimation, the ensemble relative entropy of coherence (C) is equal to the difference between the optimal Holevo information (X), and the mutual information attained by a measurement (I). We call this relation the CXI equality. The ensemble coherence lets us visualize how much information is locked away in superposition and hence is inaccessible with a given measurement scheme and quantifies the advantage that would be gained by using a joint measurement on multiple states. Our results hold regardless of how the parameter is encoded in the state, encompassing unitary, dissipative, and discrete settings. We consider both projective measurements and general POVMs. This work suggests new directions for research in coherence, provides a novel operation interpretation for the relative entropy of coherence and its POVM generalization, and introduces a new tool to study the role of quantum features in metrology.

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I. INTRODUCTION

Superposition is one of the most fundamental and unique aspects of quantum physics. This is rigorously quantified by the resource theory of coherence [1–3], which studies how the amount of superposition over a given basis relates to physical properties of a state and its ability to perform useful tasks. In recent years, there has been intensive investigation into the role of coherence in areas including thermodynamics [4–6], quantum information processing [7–10], and quantum correlations such as entanglement [11–14].

Classical systems can only exist as a statistical mixture of orthogonal states—such mixtures are termed *incoherent*. In resource theories, we quantify the *coherence* of a quantum system by measuring its distance from the set of incoherent states. There are many different notions of distance, leading to a variety of coherence measures [2]. The oldest, and one of the simplest, is the relative entropy of coherence, defined using the quantum relative entropy as the distance measure [1]. It is an outstanding question which coherence measures are more fundamental or relevant to different problems [2,15]. Answering this question requires finding applications and operational interpretations for these coherence measures.

One such application of great interest to quantum technologies is metrology [16–18]. It has long been known that quantum mechanics can be used to measure physical parameters more efficiently than is classically possible [19,20]. Precisely which quantum properties allow for this, and how they may best be exploited, is an area of active research [16,19]. Depending on how a quantum system is measured, some information is usually "locked away," inaccessible to the measurement scheme [21]. It is intuitive that this inaccessible information has something to do with superposition. For example, projective measurement in some basis is insensitive to phase information between the basis coefficients. It is thus natural to ask what insight the resource theory of coherence can provide in metrology [22–24].

Broadly speaking, there are two lenses through which parameter estimation may be viewed. The widely used *Fisher information* quantifies the smallest fluctuation in the parameter that may be detected around some operating point [18]. Often, however, we do not know the parameter accurately but instead have a probability distribution over a large range of values. Moreover, the unrealistic assumption of an operating point known with infinite precision can sometimes imply performance that is not attainable in practice [25,26]. Fisher information also requires continuous evolution of the system with respect to the parameter, which excludes cases such as state discrimination. These issues are addressed by the Bayesian approach [18,26,27], which instead quantifies the average error in our estimate of the parameter. Fisher information can generally be computed using derivatives of the quantum state, which naturally capture the change due to an infinitesimal fluctuation in the parameter. Analyzing Bayesian metrology, however, is generally more complicated [28–32], as we must consider all possible values of the parameter and all possible measurement results in each case. Quantum information theory is increasingly proving itself to be a useful tool for this [33–35].

The majority of research into coherence and metrology has focused on Fisher information. Detailed studies have shown that the quantum Fisher information is related to the second derivative of the relative entropy of coherence, though there are additional factors in the relation that are still not well understood [36]. Other authors have also found complex relationships between the two [37, 38]. The relationship between quantum Fisher information and coherence in general dynamics has been studied in Refs. [39–41] and there have even been formulations that define coherence using Fisher information [42–44].

In contrast to the wealth of research investigating the connection between coherence and Fisher information, the role of coherence in Bayesian metrology has received relatively little attention. In Refs. [45,46], the authors have studied unitary or displacement encoding of the parameter and a particular detection scheme analogous to measuring the phase evolution along a "ruler." They have defined a notion of coherence analogous to the classical Wiener-Khintchine theorem and related this to the measurement resolution. Another example is Ref. [33], where the relative entropy of coherence (termed "G-asymmetry" by the authors) has been used to study the limits of nonlinear metrology with unitary parameter encodings.

In this work, we show that the relative entropy of coherence [1] and its recent generalization to positive operator-valued measures (POVMs) [47,48] are fundamentally related to Bayesian metrology. We assume nothing about how the parameter is encoded in the state, even allowing for situations in which the encoding is not continuous. Our results apply to both projective measurement and general POVMs. In this general case, we show that the relative entropy of coherence naturally emerges as the quantifier of information gain from the measurement.

The paper is structured as follows. We begin in Sec. II with an overview of some necessary concepts from Bayesian estimation and information theory. In Sec. III, we show how the relative entropy of coherence may be applied to an ensemble of states. We then prove that the ensemble coherence (C) is equal to the difference between the optimal Holevo information assuming infinite resources (X), and the mutual information attained by a particular projective measurement (I), a relation we call the *CXI equality*. In Sec. IV, we generalize the CXI equality to POVM measurements. We then apply this in Sec. V to a simple example of state discrimination on the Bloch sphere (the code for which is available in Ref. [75]). Finally, in Sec. VI, we discuss promising directions for future research.

II. INFORMATION THEORY AND BAYESIAN METROLOGY

Quantum metrology exploits properties such as interference and entanglement for high-precision sensing. There is a veritable zoo of possible schemes but the majority follow the general procedure of initializing a probe state ρ_0, interacting this with a parameter ϕ of interest resulting in a state ρ_ϕ, and then measuring ρ_ϕ to gain information about the parameter [16]. Typically, many measurements are performed [32,49,50] and the results are then processed to make an estimate of the parameter [51]. Much work has been done to optimize each of these steps [18,32]. Finding fundamental bounds on performance is thus a complex task [16,18,52,53] but in recent years quantum information theory [54] has shown itself to be a powerful tool for addressing this problem [33–35].

Here, we will follow standard probability conventions [55] and use a capital letter such as Φ to represent a general probabilistic event and a lowercase letter ϕ for a specific outcome. For example, Φ may be the outcome of a dice roll, which can take values $\phi = 1$ through to $\phi = 6$. Each outcome has an associated probability $p_\Phi(\phi)$. In information theory, the uncertainty in the value of Φ is represented by the entropy:

$$H(\Phi) = -\sum_\phi p_\Phi(\phi) \log p_\Phi(\phi). \tag{1}$$

Entropy has a geometric interpretation in that when exponentiated, it gives the volume of possible values that Φ may take [56, § 3]. For example, for a fair dice, $p_\Phi(\phi) = 1/6$, so that $H(\Phi) = \log 6$, representing $e^{\log 6} = 6$ possible outcomes. However, if you learn via measurement that the value of the dice was 3, the entropy becomes 0, representing a single value $e^0 = 1$. If the outcome were known to be either 2 or 5 with equal probability, the entropy would be $\log 2$, corresponding to two possible outcomes. The case of continuous Φ is analogous with sums replaced by integrals and the exponentiated entropy then represents the volume of possible values [56].

Suppose that Φ represents an unknown parameter that we wish to estimate, such as the strength of a magnetic field. In Bayesian metrology, the probability distribution p_Φ then describes our initial knowledge about the value of Φ. For example, for physical reasons, we may know it to be Gaussian distributed about some ϕ_0. To apply the Bayesian framework in the case of no prior knowledge, one can take p_Φ to be the probability distribution with maximum entropy that satisfies physical constraints, which is typically the uniform distribution.

We begin our estimation process by interacting a probe state ρ_0 with Φ. This results in an ensemble $\mathcal{E}_\Phi$ of states ρ_ϕ with probability $p_\Phi(\phi)$. The ensemble state is then the average over the possible parameter values:

$$\rho_\Phi = \sum_\phi p_\Phi(\phi)\rho_\phi. \tag{2}$$

We choose a basis and perform a measurement on ρ_Φ, the outcome of which is a random value M. After observing a measurement result, we can use Bayes' theorem to combine this with our prior information to construct the posterior probability distribution conditioned on our measurement $p_{\Phi|M}$. Note that M can represent either a single measurement or a sequence of measurement results in different bases. Finally, we make an estimate $\hat{\phi}$ of ϕ that is some function of the final distribution $p_{\Phi|M}$, such as the mean or median [18,51].

To quantify the performance of Bayesian metrology, we look at the expected value of $(\phi - \hat{\phi})^2$. This is called the average mean-squared error (AMSE). Computing the AMSE can be quite challenging, since we must take into account the possible values of ϕ, all possible sequences of measurement results, and whatever algorithm is used to extract $\hat{\phi}$ from the final probability distribution. We will see that we can bound this error using information theory.

The entropy of the final distribution $p_{\Phi|M}$ is denoted by $H(\Phi|M)$. This will necessarily be less than or equal to $H(\Phi)$ [54,56], since measurement can only reduce uncertainty in the parameter. This decrease is quantified by the mutual information

$$I(\Phi; M) = H(\Phi) - H(\Phi|M), \tag{3}$$

where the ";" is to emphasize that I is symmetric in its arguments [56]. The mutual information provides a lower bound on the AMSE [33] and repeatedly maximizing mutual information optimizes performance for multiround measurement schemes [57].

Different measurement schemes will provide more or less information about the parameter. A natural question to ask, given a particular ensemble $\mathcal{E}_\Phi$, is what is the maximum mutual information that could be obtained from the optimum choice of measurement? The answer to this is provided by the Holevo information [54]. The entropy of a quantum state ρ, termed the von Neumann entropy, is defined as

$$S(\rho) = -\mathrm{tr}\{\rho \log \rho\}. \tag{4}$$

The Holevo information of the ensemble $\mathcal{E}_\Phi$ is then

$$\chi(\mathcal{E}_\Phi) = S(\rho_\Phi) - \sum_\phi p_\Phi(\phi) S(\rho_\phi). \tag{5}$$

The Holevo information provides an upper bound on the mutual information: $I(M; \Phi) \le \chi(\mathcal{E})$. It thus quantifies the maximum information that can be extracted per probe state. However, in general, to attain the Holevo information, we must obtain N identical probes ρ_ϕ, apply an entangling unitary operation, and then perform the optimum multipartite measurement. In the limit $N \to \infty$, the information gained approaches $N\chi(\mathcal{E})$, so the Holevo information is the information per probe, assuming infinite resources and technology.

It is thus natural to ask about the difference between the optimum Holevo information χ and the mutual information I actually attained from a given measurement scheme. In Sec. III, we will show that this is given by the relative entropy of coherence [1] and its generalization to POVMs [47].

III. COHERENCE AND BAYESIAN METROLOGY

In this section, we will derive the CXI equality for the case of projective measurement, which shows that the relative entropy of coherence quantifies the difference between the Holevo and mutual information. In our derivations, we will take the parameter to be discrete, the continuous case being analogous with sums replaced by integrals. We will denote our parameter as Φ and let $\mathcal{E}_\Phi$ denote the ensemble of states $\{(\rho_\phi, p_\Phi(\phi))\}$. Our measurement result will be M and we will refer to the basis of the projective measurement as the "basis of M."

We wish to study the difference $\chi(\mathcal{E}_\Phi) - I(\Phi; M)$. The information lost by a projective measurement is related to the "amount of superposition" of the probe, ρ_ϕ, over the basis of M. If ρ_ϕ is a measurement basis state, then the measurement will return the exact state with no information loss. However if the probe is in a superposition of many basis states, then a measurement can only return part of the information in the state. We will need to measure multiple times to recover the weights of the superposition and choose different bases to gain phase information. Thus it seems intuitive there should be a relationship between the mutual information $I(\Phi; M)$ and the coherence of the ensemble states in the basis of M.

In this work, we will use the relative entropy of coherence, referred to from now on as the *coherence*—the simplest and most widely used measure [2]. Let the measurement M be projection onto an orthogonal basis $\{|m\rangle\}$. For a quantum state ρ, let $\Delta_M[\rho]$ represent the same state decohered in the basis of M:

$$\Delta_M[\rho] = \sum_m \langle m|\rho|m\rangle \, |m\rangle\langle m|. \tag{6}$$

In other words, $\Delta_M[\rho]$ sets off-diagonal elements to zero in the matrix of ρ in the basis of M. Then, the coherence of ρ with respect to this basis is defined as

$$C_M(\rho) = S(\Delta_M[\rho]) - S(\rho), \tag{7}$$

where $S(\rho) = -\mathrm{tr}\{\rho \log \rho\}$ is the von Neumann entropy. It can be shown that $C_M(\rho)$ is equal to the quantum relative entropy between ρ and $\Delta_M[\rho]$ [2, §III.C.1]:

$$C_M(\rho) = \mathrm{tr}\{\rho \log \rho - \rho \log \Delta_M[\rho]\}. \tag{8}$$

Let us consider the information that is lost when we projectively measure an ensemble $\mathcal{E}_\Phi$. Our first thought may be to look at the average information loss upon measurement: $\sum_\phi p_\Phi(\phi) C_M(\rho_\phi)$. However, this represents loss of information, both about the parameter and also the quantum state itself. It is only the former that is of interest in parameter estimation; thus we must subtract the coherence of the ensemble state. This leads us to define the *ensemble coherence* of $\mathcal{E}_\Phi$ as

$$\begin{aligned} C_M(\mathcal{E}_\Phi) &= \sum_\phi p_\Phi(\phi) C_M(\rho_\phi) - C_M\left(\sum_\phi p_\Phi(\phi)\rho_\phi\right) \\ &= \sum_\phi p_\Phi(\phi) C_M(\rho_\phi) - C_M(\rho_\Phi). \end{aligned} \tag{9}$$

We note that this is analogous to the definition of the Holevo information in Eq. (5), replacing entropy with coherence. Coherence decreases under the mixing of quantum states [2, §III.C.1]; thus $C_M(\mathcal{E}_\Phi)$ is always positive. To the best of our knowledge, this notion of the coherence of an ensemble is novel.

Our primary result is to show that the ensemble coherence is equal to the difference between the Holevo and mutual information.

Theorem 1 (CXI equality for projective measurements). Let Φ be a parameter with probability distribution $p_\Phi(\phi)$, with a corresponding ensemble $\mathcal{E}_\Phi$ of states ρ_ϕ. If we perform a single projective measurement on the ensemble, the CXI equality holds:

$$C_M(\mathcal{E}_\Phi) = \chi(\mathcal{E}_\Phi) - I(\Phi; M), \tag{10}$$

where $C_M(\mathcal{E}_\Phi)$ is the ensemble coherence as defined in Eq. (9) and the coherence measure is the relative entropy of coherence.

This does not assume any particular form of the interaction between the probe and parameter that generates the states ρ_ϕ. Thus Eq. (10) holds for unitary encodings such as $\rho_\phi = e^{-iG\phi}\rho_0 e^{iG\phi}$ for some Hermitian generator G, as well as dissipative evolution, and discontinuous settings where a separate ρ_ϕ is specified for each value of ϕ. It also applies to both continuous and discrete parameters and for quantum states in finite and infinite dimensions. As we will discuss at the end of the section, the latter requires the additional condition that the prior distribution has finite entropy, $H(\Phi) < \infty$, which is satisfied by all physical distributions [58].

Let us now prove Theorem 1. We estimate a parameter ϕ encoded in a probe state ρ_ϕ by making a projective measurement M. Let M have eigenbasis $\{|m\rangle\}$ with corresponding orthonormal projectors Π_m. We will abbreviate the probability $p_M(j)$ of observing the jth result as p_j:

$$p_j = p_M(j). \tag{11}$$

Then, if ρ is the state being measured, we have $p_m = \mathrm{tr}\,\{\Pi_m \rho\}$. The entropy of M is

$$H(M) = -\sum_m p_m \log p_m. \tag{12}$$

We will first show that this is equal to the entropy of ρ decohered in the orthogonal basis of M:

$$H(M) = S(\Delta_M[\rho]). \tag{13}$$

To see this, we expand the right-hand side as

$$S(\Delta_M[\rho]) = -\mathrm{tr}\,\{\Delta_M[\rho] \log \Delta_M[\rho]\}. \tag{14}$$

The decohered state $\Delta_M[\rho]$ is a diagonal matrix in the basis of M, where the elements of the diagonal are the probabilities, p_m, of measurement outcome m. The entropy is then

$$\begin{aligned} S(\Delta_M[\rho]) &= -\mathrm{tr}\left\{\begin{pmatrix} p_1 & & \\ & \ddots & \\ & & p_{N_M}\end{pmatrix} \times \log \begin{pmatrix} p_1 & & \\ & \ddots & \\ & & p_{N_M}\end{pmatrix}\right\} \\ &= -\mathrm{tr}\left\{\begin{pmatrix} p_1 \log p_1 & & \\ & \ddots & \\ & & p_{N_M} \log p_{N_M}\end{pmatrix}\right\}, \end{aligned} \tag{15}$$

which is equal to $H(M)$. Note that in Eq. (15), off-diagonal elements of the matrices are zero.

We are now prepared to prove Theorem 1.

Proof. We begin by expanding the right-hand side of Eq. (10) in terms of the entropies of quantum states. The Holevo information is defined as

$$\chi(\mathcal{E}_\Phi) = S(\rho_\Phi) - \sum_\phi p_\Phi(\phi) S(\rho_\phi), \tag{16}$$

where ρ_Φ is the ensemble state defined in Eq. (2). To calculate the mutual information, we will use the expression $I(\Phi; M) = H(M) - H(M|\Phi)$ [54]. We have shown earlier that $H(M) = S(\Delta_M[\rho_\Phi])$, while for the conditional entropy,

$$\begin{aligned} H(M|\Phi) &= \sum_\phi p_\Phi(\phi) H(M|\phi) \\ &= \sum_\phi p_\Phi(\phi) S(\Delta_M[\rho_\phi]). \end{aligned} \tag{17}$$

The right-hand side of Eq. (10) is thus

$$\begin{aligned} &\left(S(\rho_\Phi) - \sum_\phi p_\Phi(\phi) S(\rho_\phi)\right) \\ &\quad - \left(S(\Delta_M[\rho_\Phi]) - \sum_\phi p_\Phi(\phi) S(\Delta_M[\rho_\phi])\right) \\ &= \sum_\phi p_\Phi(\phi) \left(S(\Delta_M[\rho_\phi]) - S(\rho_\phi)\right) \\ &\quad - (S(\Delta_M[\rho_\Phi]) - S(\rho_\Phi)) \\ &= C_M(\mathcal{E}_\Phi), \end{aligned} \tag{18}$$

where in the last line we have recalled the definition of the ensemble coherence, given in Eq. (9). ■

Let us discuss the validity of our proof in infinite-dimensional Hilbert spaces. In finite dimensions, entropy is always finite; thus all of the above sums converge and we do not have to worry about expressions such as $\infty - \infty$. However, many common quantum systems require an infinite-dimensional Hilbert space, optics being a prominent example. Let us first consider the parameter ϕ. Continuous parameters, and discrete parameters with an infinite number of values, can have infinite entropy. However, such distributions are highly pathological and are unlikely to occur in a natural context [58]. We will therefore add the requirement that the parameter ϕ must have finite entropy.

This requirement is sufficient to ensure that all other quantities are finite. For the mutual information, we have [54]

$$I(\Phi; M) = H(\Phi) - H(\Phi|M) \le H(\Phi), \tag{19}$$

while an ensemble $\mathcal{E}_\Phi$ encoding a parameter $p_\Phi(\phi)$ cannot contain more information than $p_\Phi(\phi)$ itself [59, Eq. (1)]:

$$\chi(\mathcal{E}_\Phi) \le H(\Phi). \tag{20}$$

Since all other quantities defined in the proof may be expressed in terms of the above, all sums are guaranteed to be finite. Thus the CXI equality also holds for continuous parameters and in infinite-dimensional Hilbert spaces.

From the CXI equality, we can see that the optimal projective measurement is the one that minimizes the ensemble coherence. Most of the time there does not exist a basis such that $C_M(\mathcal{E}_\Phi) = 0$, meaning that some information will always be "locked away" in the coherences. Accessing this information will require a collective measurement on multiple probes. Precisely how much advantage such a scheme would provide is quantified exactly by the minimum value of the ensemble coherence.

IV. GENERAL POVM MEASUREMENTS

Projective measurements do not describe all measurement schemes. A common example is photodetection, which does not project the system into an eigenstate. More generally, measurements can be modeled as entangling our state with an ancilla and then performing a projective measurement on the combined system. Such measurements are described by the framework of POVMs [60]. In this section, we will generalize Theorem 1 to POVMs, using the recent extension of the relative entropy of coherence [47,48].

A POVM is described by a set of positive semidefinite operators $\{M_j\}$ such that

$$\sum_j M_j^\dagger M_j = I, \tag{21}$$

where I is the identity operator. The probability of the jth measurement outcome is given by

$$p_j = \mathrm{tr}\{\rho M_j^\dagger M_j\}. \tag{22}$$

The positivity of the measurement operators ensures that the probabilities are positive and from Eq. (21), the probabilities sum to unity. If the jth measurement outcome is detected, the postmeasurement state is

$$\rho_j = M_j \rho M_j^\dagger / p_j. \tag{23}$$

If $\{M_j\}$ is a set of orthogonal projectors, then $M_j^\dagger M_j = M_j$ and we recover the usual expressions for a projective measurement.

We will now describe the Naimark dilation, which lets us represent every POVM as a projective measurement on a larger Hilbert space [61]. Let $\mathcal{H}$ be the Hilbert space of our system and let $\mathcal{A}$ be an ancilla space of dimension equal to the number N of measurement operators in the POVM. We define the map $V : \mathcal{H} \to \mathcal{H} \otimes \mathcal{A}$ as

$$V = \sum_j M_j \otimes |j\rangle. \tag{24}$$

This lifts our state $\rho \in \mathcal{H}$ to $V\rho V^\dagger$ in the larger Hilbert space $\mathcal{H} \otimes \mathcal{A}$. Going forward, we will use tildes to refer to quantities in our expanded Hilbert space:

$$\tilde{\rho} = V\rho V^\dagger. \tag{25}$$

Equation (21) implies that V is an isometry, meaning that for all states $|\psi\rangle, |\phi\rangle \in \mathcal{H}$, we have $\langle\psi|\phi\rangle = \langle\tilde{\psi}|\tilde{\phi}\rangle = \left(\langle\psi|V^\dagger\right)\left(V|\phi\rangle\right)$. To see this, we expand as follows:

$$\begin{aligned} \langle\psi|V^\dagger V|\phi\rangle &= \langle\psi| \sum_{jk} M_j^\dagger M_k \langle k|j\rangle |\phi\rangle \\ &= \langle\psi| \sum_j M_j^\dagger M_j |\phi\rangle \\ &= \langle\psi|\phi\rangle, \end{aligned} \tag{26}$$

where in the second-to-last line we have used Eq. (21). Thus $\tilde{\rho}$ has the same eigenvalues as ρ. Since entropy is a function of the eigenvalues of a state, the dilation $\tilde{\rho}$ has the same entropy as the original state ρ.

In the dilated Hilbert space, the projection operator corresponding to M_j is

$$\tilde{M}_j = I_{\mathcal{H}} \otimes |j\rangle\langle j|_{\mathcal{A}}, \tag{27}$$

where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$ and $|j\rangle_{\mathcal{A}}$ is the jth basis element of $\mathcal{A}$. We will omit the Hilbert-space subscript from now on. The probability of the jth measurement is given by Eq. (22):

$$\begin{aligned} \tilde{p}_j &= \mathrm{tr}\{\tilde{\rho}\tilde{M}_j\} \\ &= \mathrm{tr}\left\{(V\rho V^\dagger)(I \otimes |j\rangle\langle j|)\right\} \\ &= \sum_{kl} \mathrm{tr}\left\{M_k \rho M_l^\dagger \otimes (|k\rangle\langle l|j\rangle\langle j|)\right\} \\ &= \mathrm{tr}\{\rho M_j^\dagger M_j\} \\ &= p_j. \end{aligned} \tag{28}$$

Thus the dilated measurement has the same statistics as the POVM.

Naimark's dilation can be used to generalize the relative entropy of coherence to POVMs [47]. Given a state ρ, the coherence relative to a POVM $M = \{M_j\}$ is defined as the coherence of $\tilde{\rho}$ relative to the projective operators $\{\tilde{M}_j\}$:

$$\begin{aligned} C_M(\rho) &= S\left(\Delta_{\tilde{M}}[\tilde{\rho}]\right) - S(\tilde{\rho}) \\ &= S\left(\sum_j \tilde{M}_j \tilde{\rho} \tilde{M}_j^\dagger\right) - S(\tilde{\rho}). \end{aligned} \tag{29}$$

It is shown in Ref. [47] that this is well defined and satisfies the requirements of a coherence measure. When M is a projective measurement, this reduces to the relative entropy of coherence; thus we are justified in using the same symbol C_M for both. Going forward, we will refer to this as the *POVM coherence*.

We are now prepared to generalize the CXI equality to POVMs.

Theorem 2 (CXI equality for POVMs). Let Φ be a parameter with probability distribution $p_\Phi(\phi)$, with a corresponding ensemble $\mathcal{E}_\Phi$ of states ρ_ϕ. If we perform a single POVM measurement on the ensemble, the CXI equality holds:

$$C_M(\mathcal{E}_\Phi) = \chi(\mathcal{E}_\Phi) - I(\Phi; M), \tag{30}$$

where $C_M(\mathcal{E}_\Phi)$ is the ensemble coherence as defined in Eq. (9) and the coherence measure is the POVM coherence defined in Eq. (29).

Proof. Let $\tilde{\mathcal{E}}_\Phi$ be the dilated ensemble consisting of states $\tilde{\rho}_\phi$ with probability $p_\Phi(\phi)$. Since $\tilde{M} = \{\tilde{M}_j\}$ is a projective measurement of $\tilde{\mathcal{E}}_\Phi$, the CXI equality holds for the dilated measurement:

$$C_{\tilde{M}}(\tilde{\mathcal{E}}_\Phi) = \chi(\tilde{\mathcal{E}}_\Phi) - I(\tilde{M}, \Phi). \tag{31}$$

The term on the left-hand side is the POVM coherence $C_M(\mathcal{E}_\Phi)$. Let us thus consider the right-hand side.

The Holevo information is given by

$$\chi(\tilde{\mathcal{E}}_\Phi) = S\left(\int \tilde{\rho}_\phi p_\Phi(\phi) \mathrm{d}\phi\right) - \int S(\tilde{\rho}_\phi) p_\Phi(\phi) \mathrm{d}\phi. \tag{32}$$

Since the dilation preserves entropy, for the second term we have $S(\tilde{\rho}_\phi) = S(\rho_\phi)$. Thus let us examine the integral in brackets:

$$\begin{aligned}\int \tilde{\rho}_\phi p_\Phi(\phi) \mathrm{d}\phi &= \int \left(V\rho_\phi V^\dagger\right) p_\Phi(\phi) \mathrm{d}\phi \\ &= V\left(\int \rho_\phi p_\Phi(\phi) \mathrm{d}\phi\right) V^\dagger.\end{aligned} \tag{33}$$

Since the dilation V does not change the entropy, we find that the Holevo information of the dilated ensemble is the same as the original ensemble:

$$\chi(\tilde{\mathcal{E}}_\Phi) = \chi(\mathcal{E}_\Phi). \tag{34}$$

Now let us consider the mutual information:

$$I(\tilde{M}, \Phi) = H(\tilde{M}) - H(\tilde{M}|\Phi), \tag{35}$$

where H is the Shannon entropy. The first term, $H(\tilde{M})$, is the entropy of the probability distribution of measurement results. Since the dilation has the same measurement statistics, we immediately have $H(\tilde{M}) = H(M)$.

Now let us consider the second term:

$$H(\tilde{M}|\Phi) = \sum_\phi p_\Phi(\phi) H(\tilde{M}|\phi). \tag{36}$$

To calculate this, we need to find the measurement probabilities of the dilated conditional state $\tilde{\rho}_\phi$. These are

$$\begin{aligned}\tilde{p}_{j|\phi} &= \mathrm{tr}\left\{\tilde{M}_j \tilde{\rho}_\phi\right\} \\ &= \mathrm{tr}\left\{(I \otimes |j\rangle\langle j|)\left(V\rho_\phi V^\dagger\right)\right\} \\ &= \mathrm{tr}\left\{(I \otimes |j\rangle\langle j|)\left(\sum_{kl} M_k \rho_\phi M_l^\dagger \otimes |k\rangle\langle l|\right)\right\} \\ &= \mathrm{tr}\left\{M_j \rho_\phi M_j^\dagger\right\} \\ &= p_{j|\phi}.\end{aligned} \tag{37}$$

Since the probability distributions are the same, we thus have $H(\tilde{M}|\Phi) = H(M|\Phi)$. Thus we find that the mutual information between our measurement and the parameter is the same for both the original and the dilated ensemble:

$$I(\tilde{M}, \Phi) = I(M, \Phi). \tag{38}$$

Putting all of the above together gives the CXI equality of Eq. (30). ■

Thus using the POVM coherence, we find that the CXI equality holds for general quantum measurements.

V. EXAMPLE

Let us now consider a brief example to illustrate the ensemble coherence and CXI equality. *Mathematica* code for all computations in this section can be downloaded (see Ref. [75]). Suppose that we wish to discriminate between two pure states on the Bloch sphere. We take the first state to lie along the σ_x axis:

$$\rho_0 = |\uparrow_x\rangle. \tag{39}$$

The second state, ρ_θ, is rotated an angle θ from ρ_0 about the σ_z axis:

$$\rho_\theta = \cos(\theta/2)|\uparrow_x\rangle + \sin(\theta/2)|\downarrow_x\rangle. \tag{40}$$

In this problem of state discrimination, the parameter ϕ that we are estimating can take two values, 0 and θ, corresponding to the two possible states ρ_0 and ρ_θ. The two states are assumed to have equal probability. We will consider three different scenarios, with θ equal to $\pi/10$, $\pi/2$, and π.

Let us first study discrimination via projective measurement. A projective measurement on the Bloch sphere is described by a pair of antipodal points representing the basis states of the measurement. From the CXI equality, the ensemble coherence with respect to this basis tells us how efficient measurement in this basis is at discriminating between the two states. The greater the coherence, the more information is being "lost" by the measurement. We note, however, that the three scenarios will have differing Holevo information. For the perfectly distinguishable case of $\theta = \pi$, the Holevo information is $\chi = \log 2 \approx 0.69$. For $\theta = \pi/2$, we have $\chi \approx 0.42$, and for the small separation $\theta = \pi/10$, the Holevo information is $\chi \approx 0.04$ [62]. When evaluating the efficacy of a measurement basis, we must compare the ensemble coherence with the total Holevo information available.

In Fig. 1, we plot the ensemble coherence in each possible measurement basis on the Bloch sphere, normalized by the Holevo information. We can see that measurements the outcomes of which do not discriminate between the two states have coherence equal to the Holevo information, indicating that they provide no information. Meanwhile, the bases with minimum coherence make an angle $\theta/2 + \pi/2$ with the σ_x axis. This corresponds to the Helstrom measurement from the theory of minimum-error quantum state discrimination [53,63].

Now let us study the ensemble coherence for a POVM. In unambiguous state discrimination, we consider a three-element measurement, with outcomes corresponding to ρ_0, ρ_θ, and "unsure" [63,64]. The first two outcomes correspond, respectively, to measurement operators:

$$\begin{aligned} M_0 &= c(I - \rho_\theta) = c\rho_{\theta+\pi}, \\ M_\theta &= c(I - \rho_0) = c\rho_\pi. \end{aligned} \tag{41}$$

Here, c is a constant to be determined, which is the same for both M_0 and M_θ, since the problem is symmetric with regard to the two states. The measurement operator corresponding to the "unsure" outcome is then

$$\begin{aligned} M_? &= I - M_0 - M_\theta \\ &= I - c\left(\rho_{\theta+\pi} + \rho_\pi\right). \end{aligned} \tag{42}$$

The probability of this outcome is

$$p_? = \mathrm{tr}\left\{M_? \frac{1}{2}\left(\rho_0 + \rho_\theta\right)\right\} \tag{43}$$

$$= \frac{1}{2}\mathrm{tr}\left\{\left(\rho_0 + \rho_\theta\right) - c\left(\rho_{\theta+\pi}\rho_0 + \rho_\pi\rho_\theta\right)\right\} \tag{44}$$

$$= 1 - c\sin^2\left(\frac{\theta}{2}\right). \tag{45}$$

To minimize Eq. (45), we must make c as large as possible while preserving positivity of the operator $M_?$ in Eq. (42). Thus we find that $c = 1/\lambda$, where λ is the largest eigenvalue of $\rho_\pi + \rho_{\theta+\pi}$.

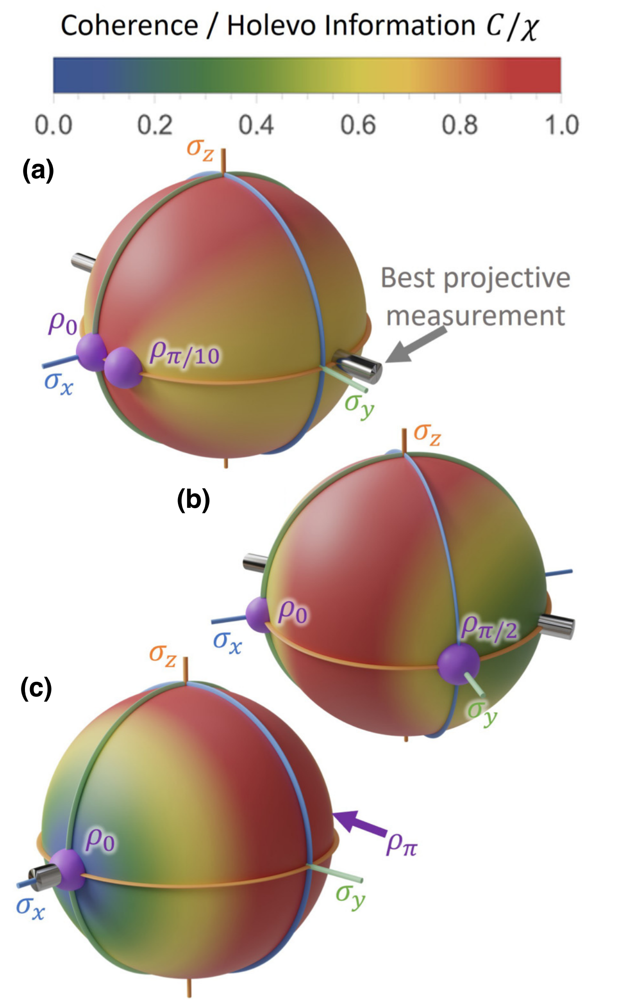

FIG. 1. Ensemble coherence in state discrimination. The states ρ_0 and ρ_θ are separated by an angle (a) $\pi/10$, (b) $\pi/2$, and (c) π. Points on the Bloch sphere are colored by the ensemble coherence of a projective measurement in the corresponding basis, normalized by the Holevo information. From the CXI equality, a larger coherence indicates more lost information, which we can see correlates with bases that do not distinguish between the two states. The silver bar denotes the basis with minimum ensemble coherence, which coincides with the basis given by minimum-error state-discrimination theory. As the states grow more distinguishable, the coherence in the optimum basis increases, quantifying the advantage that could be gained from a multipartite measurement.

In Fig. 2(a), we show the ensemble coherences for the POVM with measurement operators $\{M_0, M_\theta, M_?\}$, and compare these with the optimal projective measurement. We can see that apart from the perfectly distinguishable case of $\theta = \pi$, when both are zero, the coherence of unambiguous state discrimination is larger. Thus we will, on average, require more measurements to distinguish the states using unambiguous state discrimination than by the optimal projective measurement.

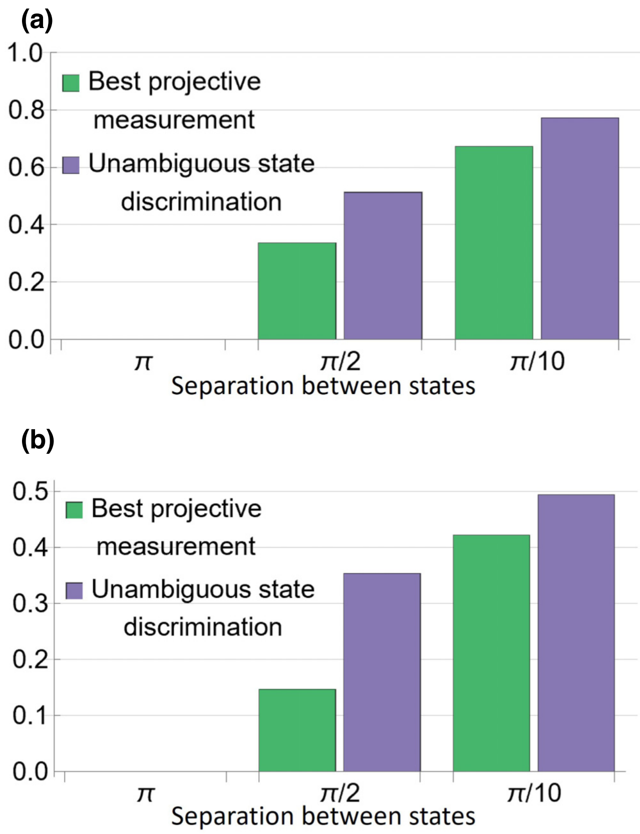

FIG. 2. (a) A comparison of the ensemble coherence of the best projective measurement with the ensemble coherence of the POVM for unambiguous state discrimination. The coherences are normalized with respect to the Holevo information. When the states are separated by an angle of π, both coherences are zero, since the measurements can discriminate perfectly between the states. As the states grow less distinguishable, however, these measurements lose information, as quantified by the increasing coherence. In general, the POVM has larger coherence, indicating it will take more measurements on average to successfully discriminate between the states. (b) The error probability if we attempt to discriminate between the two states based on a single measurement result. We can see that an increase in the ensemble coherence correlates with an increase in the error probability.

Let us consider the magnitudes of the coherence values in Fig. 2(a). For $\theta = \pi$, where ρ_0 and ρ_θ are orthogonal, the coherence is zero for both projective measurements and the POVM that we have considered. Thus no better measurement is possible than projective measurement on a single system. As the states grow increasingly indistinguishable, however, the coherence increases to a sizable fraction of the Holevo information. This means that a protocol that performed a collective measurement on multiple probes could obtain a substantial information gain per probe. Roughly speaking, this is possible because the states $\rho_0^{\otimes n}$ and $\rho_\theta^{\otimes n}$ are "more orthogonal" than ρ_0 and ρ_θ, making it easier to discriminate between them [65].

As a demonstration that ensemble coherence does indeed correspond to lost information, we will explicitly compute the success probabilities when performing state discrimination based on a single measurement. In the projective case, the two measurement operators for the basis with minimum coherence are

$$\begin{aligned} \Pi_0 &= \rho_{\theta/2-\pi/2}, \\ \Pi_\theta &= \rho_{\theta/2+\pi/2}. \end{aligned} \tag{46}$$

We projectively measure our state in this basis and estimate the parameter to be 0 if we observe Π_0 or θ if we observe Π_θ. The success probability, p_s, is then

$$\begin{aligned} p_s &= p_\Phi(0) p_{M|\Phi}(0|0) + p_\Phi(\theta) p_{M|\Phi}(\theta|\theta) \\ &= \frac{1}{2}\left(\mathrm{tr}\left\{\Pi_0\rho_0\right\} + \mathrm{tr}\left\{\Pi_\theta\rho_\theta\right\}\right). \end{aligned} \tag{47}$$

For the case of unambiguous state discrimination, as before we estimate a parameter value of 0 or θ if we observe M_0 or M_θ, respectively. If we observe the third outcome $M_?$, we gain no information and our best strategy is to randomly guess either 0 or θ, in which case we will be correct half the time. Our success probability is then

$$\begin{aligned} p_s &= p_\Phi(0)\left(p_{M|\Phi}(0|0) + \frac{1}{2} p_{M|\Phi}(?|0)\right) \\ &\quad + p_\Phi(\theta)\left(p_{M|\Phi}(\theta|\theta) + \frac{1}{2} p_{M|\Phi}(?|\theta)\right) \\ &= \frac{1}{2}\left(\mathrm{tr}\left\{\left(M_0 + \frac{1}{2}M_?\right)\rho_0\right\} + \mathrm{tr}\left\{\left(M_\theta + \frac{1}{2}M_?\right)\rho_\theta\right\}\right). \end{aligned} \tag{48}$$

Since coherence represents lost information, it is appropriate to compare this with the error probability $1 - p_s$. We plot this in Fig. 2(b) for both projective measurement and the POVM. We can see that unambiguous state discrimination is more likely to fail than projective measurement, as expected by its larger ensemble coherence. The probability of error also increases as the separation between the states decreases, again in line with the increasing coherence.

This example demonstrates that the ensemble coherence does indeed represent information "locked away in the coherences" and inaccessible to the chosen measurement. Plotting this allows us to visualize and compare different measurement schemes. In the Supplemental Material [66] (see also Refs. [67,68] contained therein), we study a more complex example of adaptive measurement. All code and data for the results in this section and in the Supplemental Material are available for download (see Ref. [75]).

VI. CONCLUSIONS

This work establishes a fundamental connection between coherence and Bayesian metrology. The information-theoretic tools provide a general proof, which holds regardless of how the parameter is encoded in the state and for both projective and POVM measurements. In particular, the CXI equality applies even to discontinuous situations such as state discrimination, where the quantum Fisher information is not applicable.

An obvious direction for generalization of this work would be to study multiparameter estimation [26]. It would also be fruitful to see what light existing results from the resource theory of coherence could shed on metrology. For example, given the current interest in metrology for fundamental tests of physics [69,70], coherence-based measures of macroscopicity could help us to understand how quantum effects could be observed on large scales [71–73].

There is an intuitive reason for why the coherence measure in the CXI relation is the relative entropy of coherence. Classically, the relative entropy between two probability distributions, p and q, can be thought of as the information lost if q is used to approximate p [74, § 2.1]. A projective measurement in the basis M effectively approximates ρ with the decohered state $\Delta_M[\rho]$, since all information contained in the off-diagonal elements is lost. Moreover, the appearance of the POVM coherence [47] when we generalize the CXI equality shows that it is indeed a natural generalization of the standard relative entropy and provides a novel operational interpretation. However, there are many other measures of coherence [2]. It would be interesting to study their meanings when applied to ensembles along the lines of Eq. (9).

All code and data used in this paper are available in Ref. [75].

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Correction: Errors in the wording of the data availability statement and associated statements in text were introduced during the proof cycle and have been fixed. The citations associated with the Supplemental Material and references cited therein were set incorrectly during the proof cycle and have been fixed. The "Corresponding author" preface was presented incorrectly and has been rectified for all footnotes.