


Equilibrium Dynamics of Infinite-Range Quantum Spin Glasses in a Field

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We determine the low-energy spectrum and Parisi replica-symmetry-breaking function for the spin-glass phase of the quantum Ising model with infinite-range random exchange interactions and transverse and longitudinal (h) fields. We show that, for all h , the spin-glass state has full replica symmetry breaking and the local spin spectrum is gapless, with a spectral density that vanishes linearly with frequency. These results are obtained using an action functional—argued to yield exact results at low frequencies—that expands in powers of a spin-glass order parameter, which is bilocal in time, and a matrix in replica space. We also present the exact solution of the infinite-range spherical quantum p -rotor model at nonzero h : here, the spin-glass state has one-step replica symmetry breaking and gaplessness only appears after imposition of an additional marginal stability condition. Possible connections to experiments on random arrays of trapped Rydberg atoms are noted.

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I. INTRODUCTION

Modern advances in the development and control of programmable quantum simulators [1–4] have led to remarkable implementations of the idea of solving classical optimization problems by quantum tunnelling [5,6]. For instance, in a recent experiment, Ebadi *et al.* [1] have used a two-dimensional Rydberg atom array to investigate quantum optimization algorithms and have demonstrated a superlinear quantum speed-up in finding exact solutions.

In such a setup, each atom can be either in the atomic ground state or in a highly excited Rydberg state (with a large principal quantum number), thus realizing a quantum two-level system, which can be represented by the eigenstates of the Pauli operator Z_i on atom i . The laser-induced Rabi flipping is then described by the operator $g \sum_i X_i$, while the laser detuning is given by $\Delta \sum_i Z_i$. The long-ranged van der Waals interactions between two atoms are active only when both are in the Rydberg state, so an interaction J_{ij} between atoms i and j , $J_{ij}(1 + Z_i)(1 + Z_j)/4$, provides a route to implementing pairwise constraints on

the optimization problems [1,7–10]. The atoms, which are trapped in optical tweezers, can be arranged in arbitrary geometries, and positioning them on randomly site-diluted lattices—as in Ref. [1]—introduces an element of spatial disorder in the Hamiltonian.

Inspired by the success of random infinite-range models in understanding classical optimization problems [11], here, we will examine the equilibrium dynamics of the infinite-range Ising spin glass in a field, h , with the Hamiltonian

$$H = \sum_{i<j} J_{ij} Z_i Z_j - g \sum_i X_i - h \sum_i Z_i, \quad (1.1)$$

where $i, j = 1, \dots, N$ denote the lattice sites and the interactions between them, J_{ij} , are taken as independent random numbers drawn from the probability distribution

$$P(J_{ij}) \propto \exp \left[-\frac{N J_{ij}^2}{2 J^2} \right]. \quad (1.2)$$

This model has been much studied in the quantum spin-glass literature [12–28] but only a few results have been obtained for the model with a nonzero field, $h \neq 0$ [18,26,28–30], which is an essential ingredient in the optimization toolbox of Rydberg quantum simulators. Here, we shall provide exact results for the equilibrium

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long-time dynamics of the $N = \infty$ model with nonzero h . Our results here are a prelude to the study of the experimental case where the couplings are time dependent but nonequilibrium results will not be presented here.

The equilibrium solution for the $N = \infty$ quantum Ising model with independent random interactions requires the self-consistent solution of a (0+1)-dimensional replicated Ising model with long-range interactions [26,28]. Such a model is not exactly solvable, so a full solution at all times requires Monte Carlo simulations. However, it has been argued in Ref. [18] that a Landau-theory-like strategy of expanding the quantum action functional in powers of an appropriately subtracted Z autocorrelation function (which is also a matrix in replica space) can provide the exact form of the long-time correlations in the $N = \infty$ model. Here, we shall implement this strategy for the $h \neq 0$ model and obtain the low-frequency dynamic spin spectrum, along with the Parisi spin-glass order parameter with full replica symmetry breaking.

Given the difficulty in obtaining the exact solution of the $N = \infty$ quantum Ising model at all times, we will also study a cousin of the quantum Ising model, which has been the focus of some attention in the literature but only at zero field: this is the “spherical quantum p -rotor model” [20,31–35]. Modern quantum simulation experiments have begun to examine multiple-qubit interactions [36] and so it is of interest to also study these models with $p > 2$. Moreover, it is for $p > 2$ that there are no efficient classical algorithms for low-energy states of p -spin classical models, so there is greater scope for impact from quantum simulation [37].

In the literature, this model has been referred to as a “ p -spin” model rather than a “ p -rotor” model. While the distinction between spins and rotors is not important for classical systems, it is crucial for quantum systems. “Spin” usually refers to a quantum degree of freedom the components of which do not commute with each other, while the components of a rotor all commute. Accordingly, we will use the “ p -rotor” terminology in this paper.

As an aside, we also note studies of the Heisenberg spin-glass model [38–48], which will not be considered in the present paper. In the Heisenberg model, the states on each site i have a twofold degeneracy, unlike both the Ising and rotor models considered here. An important consequence is that there is no trivial paramagnetic state in the Heisenberg model. In contrast, the Ising and rotor models have a gapped paramagnet with a nondegenerate ground state at large g .

We define the spherical quantum p -rotor model here by an imaginary time (τ) path integral over continuous real rotor and/or spin coordinates σ_i , which are the analogues of the discrete $Z_i = \pm 1$ Ising spins. The partition function of this model at an inverse temperature $\beta = 1/T$ and in a field h is

$$Z[J_{i_1 \dots i_p}] = \int \mathcal{D}\sigma_i(\tau) \exp \left[- \int_0^\beta d\tau \left(\frac{1}{2g} \dot{\sigma}_i(\tau) \dot{\sigma}_i(\tau) + \sum_{i_1 < \dots < i_p} J_{i_1 \dots i_p} \sigma_{i_1}(\tau) \dots \sigma_{i_p}(\tau) - h \sum_i \sigma_i(\tau) \right) \right], \quad (1.3)$$

with repeated indices summed over, and $\dot{\sigma}_i(\tau) \equiv d\sigma_i/d\tau$. Importantly, in order to keep the energy finite, we equip the rotors with a spherical constraint

$$\sum_{i=1}^N \sigma_i(\tau) \sigma_i(\tau) = N, \quad (1.4)$$

i.e., the rotor degrees of freedom lie on an N -dimensional sphere of radius \sqrt{N} . These rotors interact with p -rotor couplings $J_{i_1 \dots i_p}$ and all of these couplings are taken to be independent random variables with distribution

$$P(J_{i_1 \dots i_p}) \propto \exp \left[- \frac{N^{p-1} J_{i_1 \dots i_p}^2}{p! J^2} \right], \quad (1.5)$$

where the width of the distribution over couplings is set by the scale J . The factor of N^{p-1} ensures that the Hamiltonian is of order N and, accordingly, enforces an extensive scaling of the energy and free energy.

Owing to the global nature of the constraint in Eq. (1.4), the solution of the spherical p -rotor model is simpler than that of the Ising model with a local constraint $Z_i = \pm 1$ on every site i . On the other hand, the global constraint also makes the p -rotor model a less physical generalization of the Rydberg experiments. The solution of the p -rotor model requires analysis of a closed set of self-consistent Schwinger-Dyson equations, in which the self-energies are written as polynomials of the Green’s functions. Such Schwinger-Dyson equations have been numerically studied earlier at $h = 0$ and in imaginary time and frequency [31,33]; here, we will extend the numerical solution to $h \neq 0$ and obtain the full dynamic rotor spectrum by direct solution in real-frequency space. We will also compute the Parisi spin-glass order parameter and find that, as in the spherical classical p -rotor model [49–53] [obtained by taking the $g = 0$ limit of Eq. (1.3)], there is only one-step replica symmetry breaking.

A. Main results

Our results are expressed in terms of a field $Q_{ab}(\tau_1, \tau_2)$ that is bilocal in imaginary time and a matrix in replica space with indices $a, b = 1, \dots, n$. This field is related to

the spin and/or rotor autocorrelation functions via

$$Q_{ab}(\tau_1, \tau_2) \sim \begin{cases} \frac{1}{N} \sum_i Z_{ia}(\tau_1) Z_{ib}(\tau_2), & \text{Ising model,} \\ \frac{1}{N} \sum_i \sigma_{ia}(\tau_1) \sigma_{ib}(\tau_2), & p\text{-rotor model,} \end{cases} \quad (1.6)$$

where a, b are replica indices. Time-translational symmetry requires that the $N = \infty$ solution takes the form

$$Q_{ab}(\tau_1, \tau_2) = \frac{1}{\beta} \sum_{\nu_n} Q_{ab}(i\nu_n) e^{i\nu_n(\tau_1 - \tau_2)}, \quad (1.7)$$

where ν_n is a bosonic Matsubara frequency. We choose the following ansatz for Q_{ab} :

$$Q_{ab}(i\nu_n) = \begin{cases} Q_r(i\nu_n) + \beta q_{\text{EA}} \delta_{\nu_n, 0}, & a = b, \\ \beta q_{ab} \delta_{\nu_n, 0}, & a \neq b. \end{cases} \quad (1.8)$$

The replica off-diagonal terms in Q_{ab} are chosen to be time independent, because there is no correlation between the time evolution of the spins in distinct replicas [18]. We have parametrized these off-diagonal terms in terms of a Parisi matrix q_{ab} ; as in the classical spin-glass theory [54], this matrix has an ultrametric structure that is characterized by the Parisi function $q(u)$, with $q(1) \equiv q_{\text{EA}}$, the Edwards-Anderson order parameter at $T = 0$. We have included an additive factor of βq_{EA} in the replica-diagonal term of Eq. (1.8) for convenience. We will find that this ensures that the solution for $Q_r(\tau)$ vanishes as $\tau \rightarrow \infty$ at $T = 0$. Also, the diagonal components q_{aa} do not appear in the above and we use this freedom to choose $q_{aa} = 0$.

For the Ising model studied in Sec. II, we find the low-frequency (ω) spectrum in the replica-symmetry-breaking phase with $h \neq 0$ is the same as that obtained earlier [18, 23, 28] at $h = 0$:

$$\text{Im } Q_r(\omega) \sim \omega. \quad (1.9)$$

Note that a gapless spectrum for $h \neq 0$ is more surprising than at $h = 0$ because there is no $Z_i \rightarrow -Z_i$ symmetry that can be broken in the spin-glass phase. The replica symmetry breaking is characterized by the Parisi function $q(u)$ shown in Fig. 1. This function has the same form as that found for the classical Ising model with $g = 0$ [55]. The function $q(u)$ is nonanalytic at $u = x$ and the value of x vanishes linearly with T as $T \rightarrow 0$ as [see Eq. (2.27)]

$$x = \frac{2y}{\kappa} T q_{\text{EA}}, \quad (1.10)$$

where y and κ are couplings of order unity in the Landau action [see Eq. (2.8)]. Thus, although the breakpoint of the

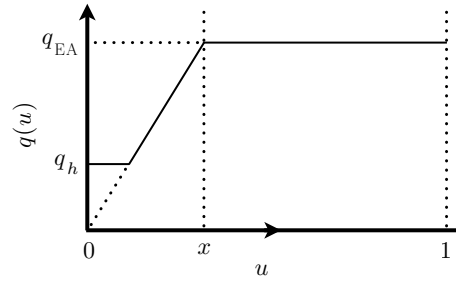


FIG. 1. The Parisi spin-glass order parameter for the Ising model at $h \neq 0$.

replica symmetry breaking vanishes as $T \rightarrow 0$, it is nevertheless important that the T - and h -dependent structure in Fig. 1 be included to obtain the gapless spectrum at $T = 0$ in Eq. (1.9). The smaller u plateau in Fig. 1 is at [18, 55]

$$q_h = \left(\frac{3h^2}{8y} \right)^{1/3}. \quad (1.11)$$

The h dependence of Eq. (1.11) is the only h dependence in the solution for the spin-glass state; the dynamic spectrum in Eq. (1.9) is h independent. This happens because there is a decoupling between the saddle-point equations for the static and dynamic components of the spin-glass order parameter and only the value of q_{EA} is present in both the static and dynamic equations. The replica-symmetric paramagnetic solution does have an energy gap and all aspects of its spectrum are h dependent.

The results for the spherical quantum p -rotor model, $p \geq 3$, studied in Sec. III, do have differences from the Ising model tied to the one-step replica symmetry breaking in the former, shown in Fig. 2, which is in turn tied to the nonlocal rotor constraint in Eq. (1.4). This $q(u)$ is characterized by a breakpoint at $u = x$ and the large- N saddle-point equations leave the value of x undetermined. The mathematical solution of the classical problem at $g = 0$ [53] implies that we should compute the free energy \mathcal{F} and then use the solution of $\partial \mathcal{F} / \partial x = 0$ to determine x .

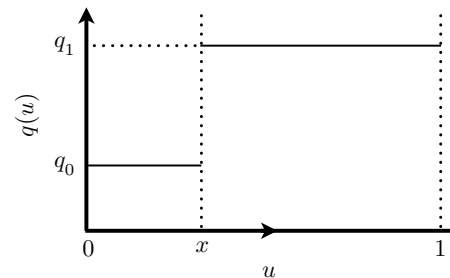


FIG. 2. The Parisi spin-glass order parameter for the spherical quantum p -rotor model for $p \geq 3$.

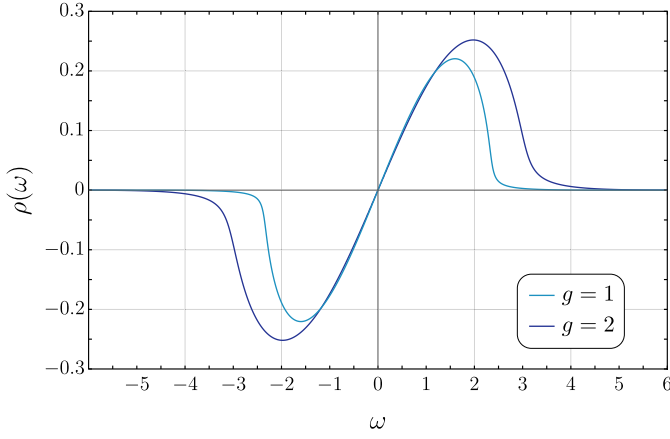


FIG. 3. The behavior of the zero-temperature spectral function $\rho(\omega) = \text{Im}Q_r(\omega)/\pi$, as determined by Eqs. (3.49)–(3.56), for the one-step replica-symmetry-breaking solution of the $p = 3$ spherical quantum p -rotor model at different values of the transverse field g (with $J = 1$). The marginal stability condition in Eq. (3.45) is imposed. The spectrum is independent of h as long as we are in the replica-symmetry-breaking phase in Fig. 10.

At such an x —and, indeed, at all generic values of x —the spectrum of $Q_r(\omega)$ turns out to have a gap. Earlier work [31,32,56] has advocated use of a “marginal-stability” condition, which leads to precisely the value of x for which the spectrum is gapless. We find that such a gapless spectrum obeys Eq. (1.9) as in the Ising model and we show a plot of $\text{Im}Q_r(\omega)$ for all frequencies in Fig. 3. Also, as in the Ising model, the spectrum in the spin-glass phase is h independent, and for similar reasons. The lower plateau at q_0 in Fig. 2 vanishes as $h \rightarrow 0$ as $q_0 \sim h^2$, unlike Eq. (1.11) for the Ising model. The value of x still vanishes linearly with T , as in the Ising model. We note that the $p = 2$ spherical model has no replica symmetry breaking and the rotor spectrum is always gapped for $h \neq 0$.

II. ISING MODEL

A. Landau action

We motivate the structure of the Landau action for the Ising model by recalling the solution of the spherical model for the $p = 2$ case [17], in which the interactions have the same form as in the Ising model but the local Ising constraint has been replaced by the global constraint in Eq. (1.4). At $h = 0$, the large- N solution in the paramagnetic phase is of the form (see Appendix C and Chap. 33 in Ref. [57])

$$Q_{ab}(iv_n) = \frac{2g\delta_{ab}}{v_n^2 + (\Delta^2 + \Lambda^2)/2 + [(v_n^2 + \Delta^2)(v_n^2 + \Lambda^2)]^{1/2}}, \quad (2.1)$$

where v_n is a Matsubara frequency, Δ is a small energy gap, and Λ is a high cutoff frequency. In the spin-glass phase of the spherical model, the solution is gapless and replica symmetric:

$$Q_{ab}(iv_n) = \beta q_{\text{EA}}\delta_{v_n,0} + \frac{2g\delta_{ab}}{v_n^2 + \Lambda^2/2 + |v_n| [v_n^2 + \Lambda^2]^{1/2}}, \quad (2.2)$$

where q_{EA} is the Edwards-Anderson order parameter.

We shall be interested in extending this solution beyond the spherical limit to the Ising model—at low-frequency scales in the vicinity of the quantum critical point, where $|v_n|, \Delta, T \ll \Lambda$, and q_{EA} is small—and allow for replica symmetry breaking. In this regime, we can approximate the paramagnetic solution of the spherical model by

$$Q_{ab}(iv_n) = \delta_{ab} \left[A - B\sqrt{v_n^2 + \Delta^2} + \dots \right], \quad (2.3)$$

where A and B are positive constants, while the spin-glass solution is

$$Q_{ab}(iv_n) = \beta q_{\text{EA}}\delta_{v_n,0} + \delta_{ab} [A - B|v_n|] + \dots \quad (2.4)$$

We now note that if we shift Q by a frequency-independent and replica-diagonal constant,

$$Q_{ab}(iv_n) \rightarrow Q_{ab}(iv_n) - A\delta_{ab}, \quad (2.5)$$

then the shifted $Q_{ab}(iv_n)$ becomes small at the relevant low-frequency scales on both sides of the quantum critical point. This makes the shifted $Q_{ab}(iv_n)$ a suitable field in which to carry out the Landau expansion of the Ising model. In terms of the original bilocal field, the shift is

$$Q_{ab}(\tau_1, \tau_2) \rightarrow Q_{ab}(\tau_1, \tau_2) - A\delta_{ab}\delta(\tau_1 - \tau_2) \quad (2.6)$$

and the constant A characterizes nonuniversal short-time physics that is not of interest to us.

Working with this shifted field, the important low-order terms in the Landau expansion for the action of the Ising model are [18]

$$\mathcal{A}_h = \mathcal{A} - \frac{h^2}{2} \sum_{ab} \int d\tau_1 d\tau_2 Q_{ab}(\tau_1, \tau_2) - \frac{\beta}{2} \chi_{hb} h^2, \quad (2.7)$$

where h is the longitudinal field, χ_{hb} is a background contribution to the linear spin susceptibility, and the $h = 0$

Landau action is

$$\begin{aligned} \mathcal{A} = & \frac{1}{\kappa} \int d\tau \sum_a \left[\frac{\partial}{\partial \tau_1} \frac{\partial}{\partial \tau_2} + r \right] Q_{aa}(\tau_1, \tau_2) \Big|_{\tau_1=\tau_2=\tau} \\ & - \frac{\kappa}{3} \int d\tau_1 d\tau_2 d\tau_3 \\ & \times \sum_{abc} Q_{ab}(\tau_1, \tau_2) Q_{bc}(\tau_2, \tau_3) Q_{ca}(\tau_3, \tau_1) \\ & + \frac{U}{2} \int d\tau \sum_a Q_{aa}(\tau, \tau) Q_{aa}(\tau, \tau) \\ & - \frac{y}{6} \int d\tau_1 d\tau_2 \sum_{ab} [Q_{ab}(\tau_1, \tau_2)]^4 \end{aligned} \quad (2.8)$$

(for a derivation of Eq. (2.8) from Eq. (1.1), see Appendix A of Ref. [18]). Here, r is the parameter that tunes across the spin-glass transition at $h = 0$: it is analogous to the coupling g in the Ising Hamiltonian in Eq. (1.1) with $r \sim g - g_c$ and with $r = 0$ at the $g = g_c$ quantum critical point. The cubic term κ is analogous to the cubic term in Parisi's original theory of classical spin glasses [54,58]. The U term only involves a single replica and is a quantum self-interaction of the soft Ising order parameter. The y term is analogous to a quartic term in the classical case [54], where it is the term responsible for full replica symmetry breaking in the spin-glass phase; however, the dynamic quantum effects of y are weak and can be treated perturbatively.

Note that Eq. (2.8) does not contain the allowed quadratic term $\int d\tau_1 d\tau_2 [Q_{ab}(\tau_1, \tau_2)]^2$. We have removed such a term by exploiting the shift in Eq. (2.6). We will see below that such a choice is equivalent to the requirement for the validity of the Landau theory that the full function $Q_{ab}(iv_n)$ is small near the quantum critical point. This shift strategy is analogous to that followed for the critical theory of the Yang-Lee edge singularity [59,60].

Our analysis of the physics of the action \mathcal{A}_h will examine the behavior as a function of the tuning parameter across the spin-glass transition r , the longitudinal field h , and the temperature T . We will keep the couplings κ and U of order unity and obtain results for small y . We find that for $h > 0$, there are contributions nonanalytic in y that are important to include; however, analytic corrections in integer powers of y are not crucial and these are relegated to Appendix A.

The analysis of thermodynamic properties in the spin-glass phase in Ref. [18] has been carried out with a vanishing coefficient of the quartic term, $y = 0$: in this case, the order parameter has replica symmetry. Here, we will extend the solution to small $y \neq 0$ and show that the solution has broken replica symmetry. We also show that the spin fluctuation spectrum remains gapless both at $y = 0$ and $y \neq 0$.

B. Free energy

Inserting the time-translational symmetric ansatz of Eq. (1.7) into Eqs. (2.7) and (2.8), we obtain the free energy

$$\mathcal{F}_h = \mathcal{F} - \frac{h^2}{2} \sum_{ab} Q_{ab}(iv_n = 0) - \frac{1}{2} \chi_{hb} h^2, \quad (2.9)$$

where the h -independent free energy is

$$\begin{aligned} \mathcal{F} = & \frac{1}{\beta\kappa} \sum_a \sum_{v_n} (v_n^2 + r) Q_{aa}(iv_n) \\ & - \frac{\kappa}{3\beta} \sum_{abc} \sum_{v_n} Q_{ab}(iv_n) Q_{bc}(iv_n) Q_{ca}(iv_n) \\ & + \frac{U}{2} \sum_a \left[\frac{1}{\beta} \sum_{v_n} Q_{aa}(iv_n) \right]^2 \\ & - \frac{y}{6\beta^3} \sum_{ab} \sum_{v_n, v'_n, v''_n} Q_{ab}(iv_n) Q_{ab}(iv'_n) Q_{ab}(iv''_n) \\ & \times Q_{ab}(-iv_n - iv'_n - iv''_n). \end{aligned} \quad (2.10)$$

We now insert Eq. (1.8) into Eq. (2.10) and obtain the free energy

$$\frac{\mathcal{F}_h}{n} = \mathcal{F}_{\text{sg},h} + \mathcal{F}_Q, \quad (2.11)$$

which consists of two pieces: a ‘‘spin glass’’ and a ‘‘quantum’’ component.

The first ‘‘spin-glass’’ component in Eq. (2.11) is given by

$$\begin{aligned} \mathcal{F}_{\text{sg},h} = & \mathcal{F}_{\text{sg}} - \frac{\beta h^2}{2n} \sum_{ab} q_{ab} - \frac{h^2}{2} Q_r(0) - \frac{\beta h^2}{2} q_{\text{EA}} \\ & - \frac{1}{2} \chi_{hb} h^2, \end{aligned} \quad (2.12)$$

where the h -independent terms are

$$\begin{aligned} \mathcal{F}_{\text{sg}} = & -R_1 \frac{1}{n} \text{Tr} q^2 - \frac{R_2}{3} \frac{1}{n} \text{Tr} q^3 - \frac{R_3}{6} \frac{1}{n} \sum_{ab} q_{ab}^4 \\ & + r \frac{q_{\text{EA}}}{\kappa} - \frac{\kappa}{3\beta} [(Q_r(0) + \beta q_{\text{EA}})^3 - Q_r(0)^3] \\ & - \frac{\beta y}{6} q_{\text{EA}}^4 - \frac{2y}{3} q_{\text{EA}}^3 Q_r(0), \end{aligned} \quad (2.13)$$

with

$$R_1 = \beta\kappa(Q_r(0) + \beta q_{\text{EA}}), \quad R_2 = \kappa\beta^2, \quad R_3 = \beta y. \quad (2.14)$$

The terms involving q_{ab} in Eq. (2.13) are identical to those of the Landau theory of the classical spin glass in Sec. 3.4 of Ref. [54].

The second ‘‘quantum’’ component in Eq. (2.11) is h independent:

$$\begin{aligned}
\mathcal{F}_Q &= \frac{1}{\beta\kappa} \sum_{v_n} (v_n^2 + r) Q_r(iv_n) - \frac{\kappa}{3\beta} \sum_{v_n} Q_r^3(iv_n) \\
&+ \frac{U}{2} \left[\frac{1}{\beta} \sum_{v_n} Q_r(iv_n) + q_{EA} \right]^2 \\
&- \frac{y q_{EA}^2}{\beta} \sum_{v_n} Q_r(iv_n) Q_r(-iv_n) \\
&- \frac{2y q_{EA}}{3\beta^2} \sum_{v_n, v'_n} Q_r(iv_n) Q_r(iv'_n) Q_r(-iv_n - iv'_n) \\
&- \frac{y}{6\beta^3} \sum_{v_n, v'_n, v''_n} Q_r(iv_n) Q_r(iv'_n) Q_r(iv''_n) \\
&\times Q_r(-iv_n - iv'_n - iv''_n). \tag{2.15}
\end{aligned}$$

We will solve the saddle-point equations for \mathcal{F}_{sg} exactly, while those for \mathcal{F}_Q can be solved order by order in y . We will present the y^0 solution below, while the y^1 solution is presented in Appendix A. It turns out that the solution of \mathcal{F}_{sg} contains terms nonperturbative in y for $h \neq 0$, so it is important to treat the spin-glass terms exactly.

C. Saddle-point equations

The saddle-point equations are most easily determined by taking the derivative of Eq. (2.10) with respect to $Q_{ab}(iv_n)$:

$$\begin{aligned}
\frac{\beta h^2}{2} \delta_{v_n,0} &= \frac{1}{\kappa} (v_n^2 + r) \delta_{ab} - \kappa \sum_c Q_{ac}(iv_n) Q_{cb}(iv_n) \\
&+ U \delta_{ab} \frac{1}{\beta} \sum_{v'_n} Q_{aa}(iv'_n) \\
&- \frac{2y}{3\beta^2} \sum_{v'_n, v''_n} Q_{ab}(iv'_n) Q_{ab}(iv''_n) \\
&\times Q_{ab}(-iv_n - iv'_n - iv''_n). \tag{2.16}
\end{aligned}$$

The replica off-diagonal equation of Eq. (2.16) is

$$\frac{\beta h^2}{2} = -2R_1 q_{ab} - R_2 \sum_c q_{ac} q_{cb} - \frac{2R_3}{3} q_{ab}^3, \tag{2.17}$$

while the replica-diagonal part gives

$$\begin{aligned}
0 &= \frac{1}{\kappa} (v_n^2 + r) - \kappa [Q_r(iv_n)]^2 + \frac{U}{\beta} \sum_{v'_n} Q_r(iv'_n) + u q_{EA} \\
&- \frac{2y}{3\beta^2} \sum_{v'_n, v''_n} Q_r(iv'_n) Q_r(iv''_n) Q_r(-iv_n - iv'_n - iv''_n) \\
&- \frac{2y}{\beta} q_{EA} \sum_{v'_n} Q_r(iv'_n) Q_r(iv_n - iv'_n) - 2y q_{EA}^2 Q_r(iv_n) \\
&+ \delta_{v_n,0} \left[-\kappa \beta^2 \sum_c q_{ac} q_{ca} - \kappa \beta^2 q_{EA}^2 - 2\kappa \beta q_{EA} Q_r(0) \right. \\
&\left. - \frac{2\beta y}{3} q_{EA}^3 - \frac{\beta h^2}{2} \right]. \tag{2.18}
\end{aligned}$$

D. Zero-field limit

In this section, we first rederive the results obtained earlier [18,42] at $h = 0$, which we will then contrast with the case for a nonzero field later in Sec. II E.

1. Quantum paramagnet

In the paramagnetic phase, with $q_{EA} = 0$ and $q_{ab} = 0$, the spin-glass free energy is $\mathcal{F}_{sg} = 0$. At $h = 0$, $\mathcal{F}_{sg,h} = 0$, as a result of which only the quantum component survives and Eq. (2.18) simplifies to an equation for $Q_r(iv_n)$ alone:

$$\begin{aligned}
0 &= \frac{1}{\kappa} (v_n^2 + r) - \kappa [Q_r(iv_n)]^2 + \frac{U}{\beta} \sum_{v'_n} Q_r(iv'_n) \\
&- \frac{2y}{3\beta^2} \sum_{v'_n, v''_n} Q_r(iv'_n) Q_r(iv''_n) Q_r(-iv_n - iv'_n - iv''_n). \tag{2.19}
\end{aligned}$$

This can be solved iteratively in powers of y . At order y^0 , the solution agrees with the form in Eq. (2.3),

$$Q_{r0}(iv_n) = -\frac{\sqrt{v_n^2 + \Delta^2}}{\kappa}, \tag{2.20}$$

where the gap Δ is given by the solution of

$$\Delta^2 = r - \frac{U}{\beta} \sum_{|v_n| < \Delta} \sqrt{v_n^2 + \Delta^2}, \tag{2.21}$$

and we have introduced a frequency cutoff to make the frequency summation finite. This cutoff dependence is inherent to the low-frequency expansion in the Landau theory. We will also need the cutoff at higher orders in y but the form of the low-frequency dynamics remains cutoff independent: we will verify this in Sec. III, where no cutoff dependence is needed, because the theory is

solved exactly without Landau expansion, and similar low-frequency behavior appears. The cutoff-dependent critical point $r = r_{c0}$ is obtained by setting $\Delta = 0$, and is given by

$$r_{c0} = \frac{U}{\beta} \sum_{|v_n| < \Delta} |v_n|. \quad (2.22)$$

The order- y^1 corrections to the saddle point appear in Appendix A 1 a.

a. Free energy. As mentioned above, for the case of the paramagnet, we have $\mathcal{F}_{\text{sg}} = 0$ for the spin-glass component of the free energy in Eq. (2.11). The free energy \mathcal{F}_Q in Eq. (2.11) at order y^0 is

$$\mathcal{F}_Q^0 = -\frac{2}{3\kappa^2\beta} \sum_{v_n} (v_n^2 + \Delta^2)^{3/2} - \frac{[\Delta^2 - r]^2}{2\kappa^2 u}, \quad (2.23)$$

where Δ is the solution of Eq. (2.21). The order- y^1 correction to \mathcal{F}_Q appears in Appendix A 1 a.

2. Spin glass

We begin by solving Eq. (2.17) at $h = 0$, which is the same equation of state as that obtained for the classical spin glass. In terms of the Parisi function $q(u)$ characterizing the $n \rightarrow 0$ limit of the replica matrix q_{ab} , we can write Eq. (2.17) at $h = 0$ as (see Appendix E)

$$\begin{aligned} 2R_1 q(u) + \frac{2R_3}{3} q^3(u) \\ = R_2 \left[2q(u) \int_0^1 q(v) dv + \int_0^u dv (q(u) - q(v))^2 \right]. \end{aligned} \quad (2.24)$$

This is the same as Eq. (3.74) in Ref. [54] for the classical case. A replica-symmetric solution with $q(u) = \text{constant}$ is possible but this is not the preferred solution at $h = 0$ (as in the classical case, and, as we will see in Sec. II E, for the quantum $h \neq 0$ case). So, we only consider the replica-symmetry-breaking solution here, as shown in Fig. 4:

$$q(u) = \begin{cases} q_{\text{EA}} u/x, & 0 < u < x, \\ q_{\text{EA}}, & x < u < 1, \end{cases} \quad (2.25)$$

with

$$\begin{aligned} q_{\text{EA}} &= \frac{R_2 - (R_2^2 - 4R_1 R_3)^{1/2}}{2R_3}, \\ x &= \frac{2R_3 q_{\text{EA}}}{R_2}. \end{aligned} \quad (2.26)$$

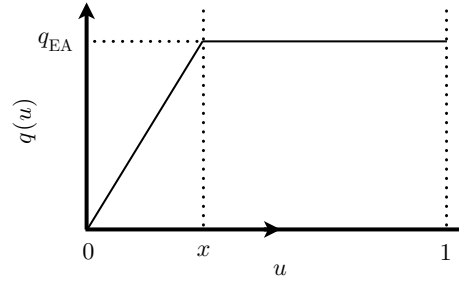


FIG. 4. The Parisi spin-glass order parameter for the Ising model at $h = 0$.

Using Eq. (2.14), we can exactly simplify Eq. (2.26) to

$$Q_r(0) = -\frac{y}{\kappa} q_{\text{EA}}^2, \quad x = 2yq_{\text{EA}}/(\beta\kappa). \quad (2.27)$$

It can now be verified that the term proportional to $\delta_{v_n,0}$ in Eq. (2.18) vanishes:

$$\begin{aligned} \kappa\beta^2 \int_0^1 du [q(u)]^2 - \kappa\beta^2 q_{\text{EA}}^2 - 2\kappa\beta q_{\text{EA}} Q_r(0) \\ - \frac{2\beta y}{3} q_{\text{EA}}^3 = 0. \end{aligned} \quad (2.28)$$

We can therefore conclude that the complete saddle-point equations for $Q_r(iv_n)$ and $q(u)$ reduce to the following three equations for $Q_r(iv_n)$, q_{EA} , and x :

$$\begin{aligned} 0 &= \frac{1}{\kappa} (v_n^2 + r) - \kappa [Q_r(iv_n)]^2 + \frac{U}{\beta} \sum_{v'_n} Q_r(iv'_n) + uq_{\text{EA}} \\ &\quad - \frac{2y}{3\beta^2} \sum_{v'_n, v''_n} Q_r(iv'_n) Q_r(iv''_n) Q_r(-iv_n - iv'_n - iv''_n) \\ &\quad - \frac{2y}{\beta} q_{\text{EA}} \sum_{v'_n} Q_r(iv'_n) Q_r(iv_n - iv'_n) \\ &\quad - 2yq_{\text{EA}}^2 Q_r(iv_n), \end{aligned} \quad (2.29)$$

$$yq_{\text{EA}}^2 = -\kappa Q_r(0), \quad (2.30)$$

$$\beta\kappa x = 2yq_{\text{EA}}. \quad (2.31)$$

Note that only the value of q_{EA} is needed from the spin-glass solution for the solution of the quantum equation for $Q_r(iv_n)$ in Eq. (2.29).

a. Gapless condition. Now, let us analytically continue Eq. (2.29) to real frequency and assume that

$$Q_r(\omega \rightarrow 0) = Q_r(0) + |\omega|^\alpha [a + ib \operatorname{sgn}(\omega)], \quad (2.32)$$

with $\alpha > 0$. Then, at $T = 0$, the terms of order Q_r^2 and Q_r^3 in Eq. (2.29) will have imaginary parts that vanish faster than $|\omega|^\alpha$ as $|\omega| \rightarrow 0$. Collecting all terms of order $|\omega|^\alpha$ in the imaginary part of Eq. (2.29), we obtain the condition

$$-2\kappa Q_r(0) - 2yq_{\text{EA}}^2 = 0, \quad (2.33)$$

which is automatically satisfied from Eq. (2.30). So, the spin dynamics are gapless at all values of y in the spin-glass phase with $h = 0$. We will see in the explicit solution below that the exponent $\alpha = 1$, so that $\operatorname{Im} Q_r(\omega \rightarrow 0) \sim \omega$.

b. Solution. Equations (2.29)–(2.31) can be solved iteratively in powers of y . At order y^0 , we have

$$\begin{aligned} q_{\text{EA}0} &= \frac{1}{\beta\kappa} \sum_{v_n} |v_n| - \frac{r}{\kappa U} \equiv \frac{1}{\kappa U} (r_{c0} - r), \\ Q_{r0}(iv_n) &= -\frac{|v_n|}{\kappa}, \\ x &= 0. \end{aligned} \quad (2.34)$$

The order- y^1 corrections to the saddle point appear in Appendix A 1 b.

c. Free energy. Inserting the solution in Eqs. (2.25)–(2.27) back into Eq. (2.13) and using the relation (see Appendix E)

$$\begin{aligned} & -R_1 \frac{1}{n} \operatorname{Tr} q^2 - \frac{R_2}{3} \frac{1}{n} \operatorname{Tr} q^3 - \frac{R_3}{6} \frac{1}{n} \sum_{ab} q_{ab}^4 \\ &= R_1 \int_0^1 du [q(u)]^2 - \frac{R_2}{3} \int_0^1 du [u[q(u)]^3 \\ &+ 3q(u) \int_0^u dv [q(v)]^2] + \frac{R_3}{6} \int_0^1 du [q(u)]^4, \end{aligned} \quad (2.35)$$

we obtain

$$\mathcal{F}_{\text{sg}} = \frac{rq_{\text{EA}}}{\kappa} + \frac{y^2 q_{\text{EA}}^5}{5\kappa}. \quad (2.36)$$

This is the full expression for \mathcal{F}_{sg} in Eq. (2.11), valid to all orders in y in terms of the exact q_{EA} . However, q_{EA} is known only to order y^1 in Eq. (A8).

As in the paramagnet, we will only determine \mathcal{F}_Q in Eq. (2.11) order by order in y . At order y^0 , the value of \mathcal{F}_Q is

$$\mathcal{F}_Q^0 = -\frac{rq_{\text{EA}0}}{\kappa} - \frac{2}{3\kappa^2\beta} \sum_{v_n} |v_n|^3 - \frac{r^2}{2U\kappa^2}. \quad (2.37)$$

The order- y^1 correction to \mathcal{F}_Q appears in Appendix A 1 b.

3. Phase diagram

At order y^0 , the system is the spin-glass phase for $r < r_{c0}$, where r_{c0} is given by Eq. (2.22). In the classical limit, $T \gg \Lambda$, we need only consider the $v_n = 0$ term, so the phase boundary is at $r = 0$. In the quantum limit, $T \ll \Lambda$, we can evaluate the summation over Matsubara frequencies using the identity

$$\frac{1}{\beta} \sum_{v_n} D(iv_n) = \int_0^\infty \frac{dv}{\pi} D(iv) + 2 \int_0^\infty \frac{d\Omega}{\pi} \frac{\operatorname{Im} D(\Omega)}{e^{\Omega/T} - 1} \quad (2.38)$$

for odd spectral functions $\operatorname{Im} D(\Omega) = -\operatorname{Im} D(-\Omega)$. Then, we obtain the phase boundary at

$$r = U \frac{\Lambda^2}{2\pi} - U \frac{\pi T^2}{3}. \quad (2.39)$$

The phase boundary depends on the value of the cutoff. However, the temperature-dependent part is cutoff independent.

E. Nonzero field

Having reviewed the results in the absence of a longitudinal field above, we now address the $h \neq 0$ case and determine its phase diagram.

1. Replica-symmetric solution

Due to the longitudinal magnetic field, there is an average moment on each site, so q_{EA} is always nonzero; therefore, the replica-symmetric solution is simply the paramagnet. Thus, q_{EA} is no longer an order parameter and we will rely on replica symmetry breaking to identify the spin-glass phase in Sec. II E 2. For $q_{ab} = q_{\text{EA}}(1 - \delta_{ab})$, Eq. (2.17), together with Eq. (2.14), reads

$$3h^2 + 12\kappa q_{\text{EA}} Q_r(0) + 4yq_{\text{EA}}^3 = 0. \quad (2.40)$$

Consequently, the term proportional to $\delta_{v_n,0}$ in Eq. (2.18) vanishes, so that the equation for $Q_r(iv_n)$ remains unchanged from that in Eq. (2.29). The gapless condition in Eq. (2.33) is *not* satisfied by Eq. (2.40), even though $q_{\text{EA}} \neq 0$. Thus, the solution for $Q_r(iv)$ will have a gap.

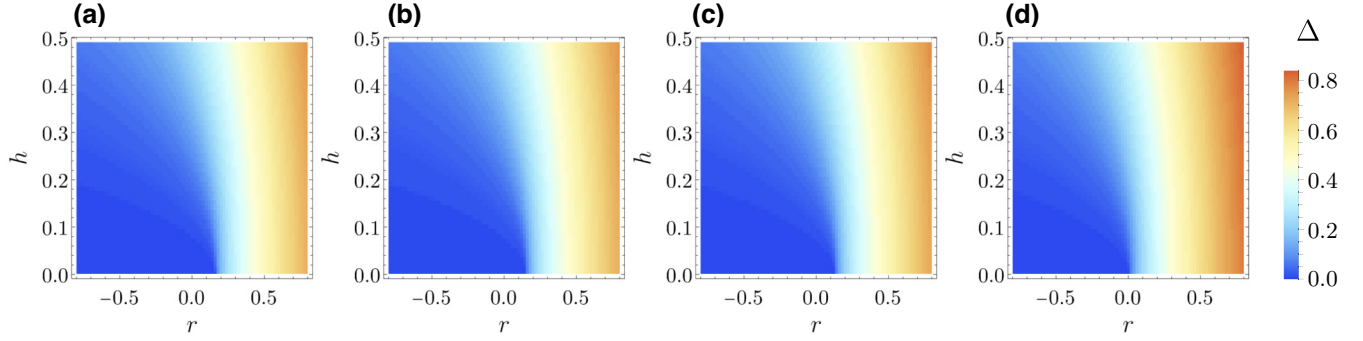


FIG. 5. The gap Δ of the replica-symmetric solution of the Ising model as a function of r and h for various temperatures, as determined from Eq. (2.41): (a) $T = 10^{-3}$; (b) $T = 10^{-2}$; (c) $T = 0.1$; (d) $T = 0.2$. The parameters U , κ , and Λ have all been set to unity in these calculations.

At order y^0 , the solution for $Q_{r0}(iv_n)$ from Eq. (2.18) has the same form as Eq. (2.20) but the equation for the gap Δ in Eq. (2.21) is now modified to [18]

$$\Delta^2 = r + \frac{U\kappa h^2}{4\Delta} - \frac{U}{\beta} \sum_{|v_n| < \Delta} \sqrt{v_n^2 + \Delta^2}, \quad (2.41)$$

while

$$q_{EA0} = \frac{h^2}{4\Delta}. \quad (2.42)$$

The solutions to Eq. (2.41) are shown in Fig. 5. Note that the gap Δ is h dependent. The order- y^1 corrections to the saddle point appear in Appendix A 2 a.

a. Free energy. Inserting the solution for $Q_r(0)$ in Eq. (2.40) into Eq. (2.12), we obtain the spin-glass component of the free energy:

$$\mathcal{F}_{\text{sg},h} = \frac{q_{EA}r}{\kappa} + \frac{y^2 q_{EA}^5}{9\kappa} + \frac{h^2 q_{EA}^2 y}{6\kappa} + \frac{h^4}{16q_{EA}\kappa} - \frac{1}{2} \chi_{hb} h^2. \quad (2.43)$$

Note that we need to insert the solution for q_{EA} in Eq. (A15) into Eq. (2.43).

For the quantum component, the order- y^0 contribution to the free energy in Eq. (2.15) is

$$\mathcal{F}_Q^0 = -\frac{2}{3\kappa^2\beta} \sum_{v_n} (v_n^2 + \Delta^2)^{3/2} - \frac{[\Delta^2 - r]^2}{2\kappa^2 U} - \frac{h^2(r - \Delta^2)}{4\kappa\Delta}, \quad (2.44)$$

which reduces to Eq. (2.23) at $h = 0$. The order- y^1 correction to \mathcal{F}_Q appears in Appendix A 2 a.

2. Replica symmetry breaking

From Eq. (2.17), Eq. (2.24) for the Parisi function is modified to

$$\begin{aligned} & \frac{\beta h^2}{2} + 2R_1 q(u) + \frac{2R_3}{3} q^3(u) \\ &= R_2 \left[2q(u) \int_0^1 q(v) dv + \int_0^u dv (q(u) - q(v))^2 \right]. \end{aligned} \quad (2.45)$$

The solution of Eq. (2.45) is modified from Eq. (2.25) to that shown in Fig. 1:

$$q(u) = \begin{cases} q_h, & 0 < u < (q_h/q_{EA})x, \\ q_{EA}u/x, & (q_h/q_{EA})x < u < x, \\ q_{EA}, & x < u < 1. \end{cases} \quad (2.46)$$

The function $q(u)$ is nonanalytic at $u = (q_h/q_{EA})x$ and $u = x$, but has no discontinuities. The values of q_{EA} and x are unchanged from those in Eq. (2.27) and the value of q_h is in Eq. (1.11). The result in Eq. (1.11) is the origin of the nonanalytic dependence on y .

It can now be verified that the term proportional to $\delta_{v_n,0}$ in Eq. (2.18) vanishes and the equations for $Q_r(iv_n)$, x , and q_{EA} remain unchanged from those in Eqs. (2.29)–(2.31). Indeed, the *only* change from the $h = 0$ solution in Sec. IID 2 is in the form of $q(u)$ for $x < (q_h/q_{EA})x$ in Eq. (2.46).

Hence, the gapless condition in Eq. (2.33) is now satisfied and the solutions for $Q_r(iv_n)$ and q_{EA} remain unchanged from that for the $h = 0$ gapless spectrum in Sec. IID 2 b.

a. Free energy. Inserting the solution Eqs. (2.46), (1.11) into Eq. (2.12), we obtain the extension of Eq. (2.36) to

nonzero h :

$$\mathcal{F}_{\text{sg},h} = \frac{q_{\text{EA}} r}{\kappa} + \frac{y^2 q_{\text{EA}}^5}{5\kappa} + \frac{3(9h^{10}y)^{1/3}}{40\kappa} - \frac{1}{2} \chi_{hb} h^2, \quad (2.47)$$

which agrees with Eq. (2.36) at $h = 0$.

As the introduction of h does not modify the values of $Q_r(i\nu_n)$ and q_{EA} , the free energy \mathcal{F}_Q remains the same as that in Sec. II D 2 c.

3. Phase diagram

To obtain the phase boundary between the replica-symmetric and replica-symmetry-breaking phases, we compare their free energies at leading nontrivial order in y .

For the gapped replica-symmetric phase, we have the free energy given by Eqs. (2.42)–(2.44):

$$\begin{aligned} \mathcal{F}_{\text{rs}} = \mathcal{F}_{\text{sg},h} + \mathcal{F}_Q^0 &= \frac{h^2 \Delta}{2\kappa} - \frac{2}{3\kappa^2 \beta} \sum_{|\nu_n| < \Delta} (v_n^2 + \Delta^2)^{3/2} \\ &\quad - \frac{[\Delta^2 - r]^2}{2\kappa^2 U} - \frac{1}{2} \chi_{hb} h^2, \end{aligned} \quad (2.48)$$

where the gap Δ is given by the solution of Eq. (2.41),

For the gapless replica-symmetry-breaking phase, we have, from Eqs. (2.37) and (2.47),

$$\begin{aligned} \mathcal{F}_{\text{rsb}} = \mathcal{F}_{\text{sg},h} + \mathcal{F}_Q^0 &= \frac{3(9h^{10}y)^{1/3}}{40\kappa} - \frac{2}{3\kappa^2 \beta} \sum_{|\nu_n| < \Delta} |\nu_n|^{3/2} \\ &\quad - \frac{r^2}{2\kappa^2 U} - \frac{1}{2} \chi_{hb} h^2. \end{aligned} \quad (2.49)$$

The summations in Eqs. (2.41), (2.48), and (2.49) can be evaluated for $T, \Delta \ll \Lambda$ using identities obtained from

Eq. (2.38):

$$\begin{aligned} &\frac{1}{\beta} \sum_{|\nu_n| < \Delta} (v_n^2 + \Delta^2)^{1/2} \\ &= \frac{\Lambda^2}{2\pi} + \frac{\Delta^2}{4\pi} (1 + \ln(4\Lambda^2/\Delta^2)) \\ &\quad - 2 \int_{\Delta}^{\infty} \frac{d\Omega}{\pi} \frac{(\Omega^2 - \Delta^2)^{1/2}}{e^{\Omega/T} - 1}, \\ &\frac{1}{\beta} \sum_{|\nu_n| < \Delta} (v_n^2 + \Delta^2)^{3/2} \\ &= \frac{\Lambda^4}{4\pi} + \frac{3\Lambda^2 \Delta^2}{4\pi} + \frac{3\Delta^4}{32\pi} (3 + \ln(16\Lambda^4/\Delta^4)) \\ &\quad + 2 \int_{\Delta}^{\infty} \frac{d\Omega}{\pi} \frac{(\Omega^2 - \Delta^2)^{3/2}}{e^{\Omega/T} - 1}. \end{aligned} \quad (2.50)$$

The phase diagram so obtained is shown in Fig. 6; note that due to the $n \rightarrow 0$ limit, we have to choose the phase with the maximum free energy [55]. We observe that the extent of the RSB phase in parameter space, which occurs for $r < 0$, shrinks with increasing h and as y is reduced. Note that Δ remains finite, albeit small, at the transition point for $h > 0$ (it vanishes at the transition for $h = 0$), indicating a first-order transition.

III. QUANTUM SPHERICAL p -ROTOR MODEL

We now turn to the analysis of the p -rotor model described in Eq. (1.3). The classical infinite-range spin glass with p -spin interactions, but without any spherical constraints, was originally introduced by Derrida [61,62]. Gross and Mézard [49] first studied the generalization of this model to nonzero magnetic fields, obtaining an exact solution for $p \rightarrow \infty$. The classical spherical model

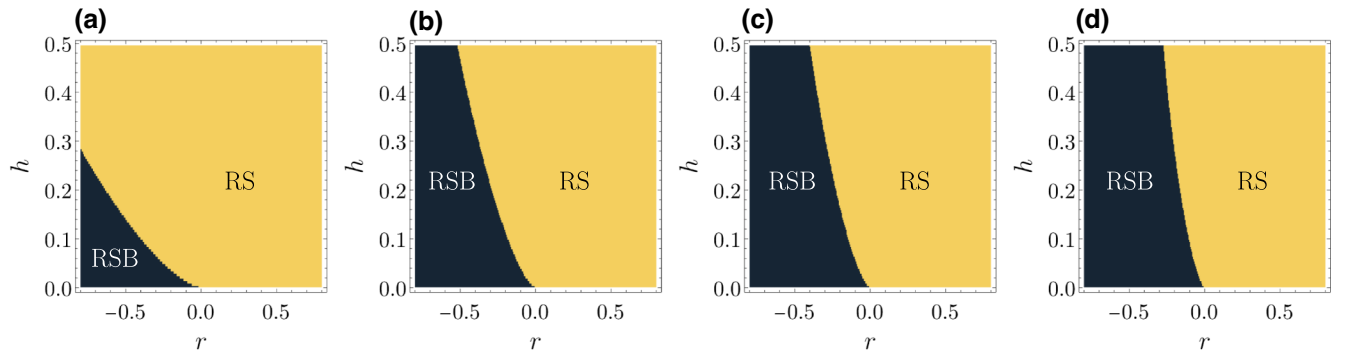


FIG. 6. The phase diagram of the Ising model in the r - h plane for various values of y at $T = 0.2$, illustrating the replica-symmetric (RS) and replica-symmetry-breaking (RSB) phases: (a) $y = 0.01$; (b) $y = 0.1$; (c) $y = 0.2$; (d) $y = 0.5$. Here, the Landau parameter r is interpreted as proportional to $g - g_c$, where g is the transverse field and g_c is the critical transverse field at $T = 0$ and $h = 0$. The value of Δ at each point in parameter space is obtained from Fig. 5, and U, κ , and Λ have been set to unity, as previously.

[52] can also be solved for any finite p , including in the presence of an external field [51,63]. Quantum extensions of these models, by adding a noncommuting transverse field, have been investigated both with (for $p = 3$) [31,33] and without (for $p \rightarrow \infty$) [64] the supplementary spherical constraint. However, the problem of the quantum model in a longitudinal field, which we examine next, has remained unexplored so far.

A. Effective action

To begin, we derive the effective action for the quantum spherical p -rotor model, which will then form the basis for our subsequent saddle-point calculations. In the path integral, the spherical constraint in Eq. (1.4) can be enforced using the exponential representation of the Dirac delta function,

$$\delta \left(\sum_{i=1}^N \sigma_i(\tau) \sigma_i(\tau) - N \right) = \int \mathcal{D}z \exp \left[i \int_0^\beta d\tau z(\tau) \left(\sum_{i=1}^N \sigma_i(\tau) \sigma_i(\tau) - N \right) \right], \quad (3.1)$$

which is then inserted into Eq. (1.3) at the expense of introducing an auxiliary field $z(\tau)$.

At this stage, since the disorder is quenched, we average the free energy $\log Z[J_{i_1 \dots i_p}]$ over disorder—using the replica trick—instead of the partition function $Z[J_{i_1 \dots i_p}]$ itself (which would correspond to the annealed average). The replicated partition function is given by

$$\begin{aligned} \overline{Z^n} = & \int dJ_{i_1 \dots i_p} P(J_{i_1 \dots i_p}) \int \mathcal{D}\sigma_i^a \mathcal{D}z^a \exp \left[i \int_0^\beta d\tau z^a(\tau) (\sigma_i^a(\tau) \sigma_i^a(\tau) - N) \right. \\ & \left. - \int_0^\beta d\tau \left(\frac{1}{2g} \dot{\sigma}_i^a(\tau) \dot{\sigma}_i^a(\tau) + \sum_{i_1 < \dots < i_p} J_{i_1 \dots i_p} \sigma_{i_1}^a(\tau) \dots \sigma_{i_p}^a(\tau) + h \sum_i \sigma_i^a(\tau) \right) \right], \end{aligned} \quad (3.2)$$

where $a = 1, \dots, n$ is the replica index. Now, we can perform the disorder average by evaluating simple Gaussian integrals to find

$$\begin{aligned} \overline{Z^n} = & \int \mathcal{D}\sigma_i^a \mathcal{D}z^a \exp \left[- \int_0^\beta d\tau \left(\frac{1}{2g} \dot{\sigma}_i^a(\tau) \dot{\sigma}_i^a(\tau) - iz^a(\tau) (\sigma_i^a(\tau) \sigma_i^a(\tau) - N) + h \sum_i \sigma_i^a(\tau) \right) \right. \\ & \left. + \frac{J^2}{4N^{p-1}} \int_0^\beta \int_0^\beta d\tau d\tau' \sum_{a,b=1}^n \left(\sum_{i=1}^N \sigma_i^a(\tau) \sigma_i^b(\tau') \right)^p \right]. \end{aligned} \quad (3.3)$$

Note that the overlap between two different replicas of the system,

$$Q_{ab}(\tau, \tau') \equiv \frac{1}{N} \sum_{i=1}^N \overline{\langle \sigma_i^a(\tau) \sigma_i^b(\tau') \rangle}, \quad (3.4)$$

where the overline denotes an average over disorder realizations, appears naturally in Eq. (3.3). We then insert the identity

$$\begin{aligned} 1 = & \int \mathcal{D}Q_{ab} \delta \left(N Q_{ab}(\tau, \tau') - \sum_{i=1}^N \sigma_i^a(\tau) \sigma_i^b(\tau') \right) \\ = & \int \mathcal{D}Q_{ab} \mathcal{D}\lambda_{ab} \exp \left[i \int_0^\beta \int_0^\beta d\tau d\tau' \lambda_{ab}(\tau, \tau') \left(N Q_{ab}(\tau, \tau') - \sum_{i=1}^N \sigma_i^a(\tau) \sigma_i^b(\tau') \right) \right] \end{aligned} \quad (3.5)$$

into the partition function, which, in terms of the collective variables Q_{ab} and λ_{ab} , reads

$$\begin{aligned} \overline{Z}^n = & \int \mathcal{D}\sigma_i^a \mathcal{D}z^a \mathcal{D}Q_{ab} \mathcal{D}\lambda_{ab} \exp \left[- \int_0^\beta d\tau \left(\frac{1}{2g} \dot{\sigma}_i^a(\tau) \dot{\sigma}_i^a(\tau) - iz^a(\tau) \sigma_i^a(\tau) \sigma_i^a(\tau) + h \sum_i \sigma_i^a(\tau) \right) \right. \\ & \left. - i \int_0^\beta \int_0^\beta d\tau d\tau' \sigma_i^a(\tau) \lambda_{ab}(\tau, \tau') \sigma_i^b(\tau') \right] \exp \left[-Ni \sum_{a=1}^n \int_0^\beta d\tau z^a(\tau) \right. \\ & \left. + N \int_0^\beta \int_0^\beta d\tau d\tau' \left(i\lambda_{ab}(\tau, \tau') Q_{ab}(\tau, \tau') + \frac{J^2}{4} \sum_{a,b=1}^n (Q_{ab}(\tau, \tau'))^p \right) \right]. \end{aligned} \quad (3.6)$$

Here, we have replaced bilinears of σ_i with Q_{ab} wherever possible and collected in the first two lines all the remaining terms involving the σ_i fields, which we will integrate out in the next step. Using, for concreteness, the convention

$$\begin{aligned} \int \mathcal{D}\mathbf{v}(x) \exp \left[-\frac{1}{2} \int \int dx dx' \mathbf{v}(x) \mathbf{M}(x, x') \mathbf{v}(x') + \int dx \mathbf{j}(x) \mathbf{v}(x) \right] \\ = \sqrt{\frac{(2\pi)^n}{\det \mathbf{M}}} \exp \left[\frac{1}{2} \int \int dx dx' \mathbf{j}(x) \mathbf{M}^{-1}(x, x') \mathbf{j}(x') \right] \end{aligned}$$

for n -dimensional vectors \mathbf{v}, \mathbf{j} and an $n \times n$ matrix \mathbf{M} , we obtain

$$\begin{aligned} \overline{Z}^n = & \int \mathcal{D}z^a \mathcal{D}Q_{ab} \mathcal{D}\lambda_{ab} \det^{-N/2} \left[-\frac{1}{\pi} \delta_{ab} \delta(\tau - \tau') \left(\frac{1}{2g} \partial_{\tau'}^2 + iz^a(\tau) \right) + \frac{i}{\pi} \lambda_{ab}(\tau, \tau') \right] \\ & \times \exp \left\{ \frac{Nh^2}{2} \int_0^\beta \int_0^\beta d\tau d\tau' \sum_{ab} (2\mathcal{G}(\tau, \tau') + 2i\lambda(\tau, \tau'))_{ab}^{-1} \right\} \\ & \times \exp \left[-Ni \sum_{a=1}^n \int_0^\beta d\tau z^a(\tau) + N \int_0^\beta \int_0^\beta d\tau d\tau' \left(i\lambda_{ab}(\tau, \tau') Q_{ab}(\tau, \tau') + \frac{J^2}{4} \sum_{a,b=1}^n (Q_{ab}(\tau, \tau'))^p \right) \right], \end{aligned} \quad (3.7)$$

$$\equiv \int \mathcal{D}z^a \mathcal{D}Q_{ab} \mathcal{D}\lambda_{ab} \exp(-S_{\text{eff}}), \quad (3.8)$$

with the matrix \mathcal{G} defined in replica and imaginary-time space as

$$\mathcal{G}_{ab}(\tau, \tau') \equiv -\delta_{ab} \delta(\tau - \tau') \left(\frac{1}{2g} \partial_{\tau'}^2 + iz^a(\tau) \right). \quad (3.9)$$

The replicated effective action can thus be extracted as

$$\begin{aligned} \frac{S_{\text{eff}}}{N} = & \frac{1}{2} \log \det \left[\frac{1}{\pi} (\mathcal{G}_{ab}(\tau, \tau') + i\lambda_{ab}(\tau, \tau')) \right] + i \sum_{a=1}^n \int_0^\beta d\tau z^a(\tau) \\ & - \int_0^\beta \int_0^\beta d\tau d\tau' \left[i\lambda_{ab}(\tau, \tau') Q_{ab}(\tau, \tau') + \frac{J^2}{4} \sum_{a,b=1}^n (Q_{ab}(\tau, \tau'))^p + \frac{h^2}{4} \sum_{a,b=1}^n (\mathcal{G}(\tau, \tau') + i\lambda(\tau, \tau'))_{ab}^{-1} \right]. \end{aligned} \quad (3.10)$$

One of the advantages of the replica approach that we are now positioned to harness is that since the effective action given in Eq. (3.10) is proportional to N , the saddle-point approximation is exact in the limit $N \rightarrow \infty$.

B. Saddle-point equations

We will only be interested in the saddle point of the theory Eq. (3.10), in which case we can assume that all fields are time-translation invariant and transform to Matsubara-frequency space, as in Eq. (1.7). Then, we have

$$\begin{aligned} \frac{S_{\text{eff}}}{N} = & \frac{1}{2} \sum_{\omega_n} \log \det \left[\frac{1}{\pi} \left(\delta_{ab} \left(\frac{1}{2g} \omega_n^2 - iz^a \right) + i\lambda_{ab}(\omega_n) \right) \right] + i\beta \sum_{a=1}^n z^a \\ & - \sum_{a,b=1}^n \sum_{\omega_n} i\lambda_{ab}(-\omega_n) Q_{ab}(\omega_n) - \frac{\beta h^2}{4} \left[-i\delta_{ab} z^a + i\lambda_{ab}(\omega_n = 0) \right]_{ab}^{-1} - \frac{J^2}{4} \int_0^\beta \int_0^\beta d\tau d\tau' (Q_{ab}(\tau - \tau'))^p. \end{aligned} \quad (3.11)$$

We now employ a slight change of notation to obtain expressions similar to those in Ref. [57], defining

$$-iz^a = \frac{\bar{\lambda}}{2g}, \quad i\lambda_{ab}(\omega_n) = -\frac{\Sigma_{ab}(\omega_n)}{2g}, \quad (3.12)$$

in terms of which, the effective action is

$$\begin{aligned} \frac{S_{\text{eff}}}{N} = & \frac{1}{2} \sum_{\omega_n} \log \det \left[\frac{1}{2\pi g} \left(\delta_{ab} \left(\omega_n^2 + \bar{\lambda} \right) - \Sigma_{ab}(\omega_n) \right) \right] - \beta \sum_{a=1}^n \frac{\bar{\lambda}}{2g} \\ & + \sum_{a,b=1}^n \left(\frac{1}{2g} \sum_{\omega_n} \Sigma_{ab}(-\omega_n) Q_{ab}(\omega_n) - \frac{\beta h^2 g}{2} \left[\bar{\lambda} \delta_{cd} - \Sigma_{cd}(\omega_n = 0) \right]_{ab}^{-1} - \frac{J^2}{4} \int_0^\beta \int_0^\beta d\tau d\tau' (Q_{ab}(\tau - \tau'))^p \right). \end{aligned} \quad (3.13)$$

Now, we use the identity in Eq. (E21) to absorb the h^2 term into the log det as

$$\begin{aligned} \frac{S_{\text{eff}}}{N} = & \frac{1}{2} \sum_{\omega_n} \log \det \left[\frac{1}{2\pi g} \left(\delta_{ab} \left(\omega_n^2 + \bar{\lambda} \right) - \Sigma_{ab}(\omega_n) - \beta h^2 g \delta_{\omega_n,0} \right) \right] - \beta \sum_{a=1}^n \frac{\bar{\lambda}}{2g} \\ & + \sum_{a,b=1}^n \left(\frac{1}{2g} \sum_{\omega_n} \Sigma_{ab}(-\omega_n) Q_{ab}(\omega_n) - \frac{J^2}{4} \int_0^\beta \int_0^\beta d\tau d\tau' (Q_{ab}(\tau - \tau'))^p \right). \end{aligned} \quad (3.14)$$

Then, the saddle-point equation with respect to $\Sigma_{ab}(\omega_n)$ is

$$Q_{ab}(\omega_n) = g \left[\delta_{ab} \left(\omega_n^2 + \bar{\lambda} \right) - \Sigma_{ab}(\omega_n) - \beta h^2 g \delta_{\omega_n,0} \right]_{ab}^{-1}. \quad (3.15)$$

Inserting this back into Eq. (3.14), we obtain an effective action just for Q_{ab} and $\bar{\lambda}$:

$$\begin{aligned} \frac{S_{\text{eff}}}{N} = & -\frac{1}{2} \sum_{\omega_n} \log \det Q_{ab}(\omega_n) - \beta \sum_{a=1}^n \frac{\bar{\lambda}}{2g} + \sum_{a,b=1}^n \left(\frac{1}{2g} \sum_{\omega_n} [(\omega_n^2 + \bar{\lambda}) \delta_{ab} - \beta h^2 g \delta_{\omega_n,0}] Q_{ab}(\omega_n) \right. \\ & \left. - \frac{J^2}{4} \int_0^\beta \int_0^\beta d\tau d\tau' (Q_{ab}(\tau - \tau'))^p \right). \end{aligned} \quad (3.16)$$

From this, the saddle-point equations for Q_{ab} and $\bar{\lambda}$ can be read off as

$$\begin{aligned} 1 = & \frac{1}{\beta} \sum_{\omega_n} Q_{aa}(\omega_n), \\ \Sigma_{ab}(\tau) = & \frac{pgJ^2}{2} (Q_{ab}(\tau))^{p-1}, \\ gQ_{ab}^{-1}(\omega_n) = & \delta_{ab}(\omega_n^2 + \bar{\lambda}) - \Sigma_{ab}(\omega_n) - \beta h^2 g \delta_{\omega_n,0}, \end{aligned} \quad (3.17)$$

with the Fourier transform defined by

$$Q_{ab}(\omega_n) = \int_0^\beta d\tau Q_{ab}(\tau) e^{i\omega_n \tau}, \quad (3.18)$$

and similarly for $\Sigma_{ab}(\omega_n)$. We numerically study these equations for the replica-symmetric and one-step replica-symmetry-breaking cases for specific values of p . In the discussion below, we focus on the $p = 3$ model since, as we show below, it exhibits an interesting and nontrivial phase diagram due to the competition between these solutions.

For completeness, the solution of the $p = 2$ model is presented in Appendix C. Note that with $p = 2$ and $h = 0$, Eqs. (3.17) reduce to Eqs. (33.32) and (33.33) in Ref. [57]. Moreover, for $p = 4$ and $h = 0$, Eqs. (3.17) are very similar to Eqs. (S50)–(S53) in Ref. [65], the main difference being that ω_n^2 is replaced by $i\omega_n$.

C. Replica-symmetric solution

In this section, we numerically determine the replica-symmetric (RS) solution of the $p = 3$ model as a function of transverse and longitudinal fields (g and h , respectively), while choosing the temperature to be close to zero.

The ansatz given in Eq. (1.8) for the replica overlap functions in the RS case necessarily has a replica off-diagonal component

$$\begin{aligned} Q_{ab}(\omega_n) &= \beta q \delta_{\omega_n, 0} + Q_r(\omega_n) \delta_{ab}, \\ \Sigma_{ab}(\omega_n) &= \beta \varrho \delta_{\omega_n, 0} + \Sigma_r(\omega_n) \delta_{ab}, \end{aligned} \quad (3.19)$$

since, for nonzero h , the static moment is always nonzero. Then, using the identities in Appendix E, the equations in Eq. (3.17) become (for $p = 3$)

$$1 = q + \frac{1}{\beta} \sum_{\omega_n} Q_r(\omega_n), \quad (3.20)$$

$$\Sigma_r(\tau) = \frac{3gJ^2}{2} (q + Q_r(\tau))^2 - \frac{3gJ^2}{2} q^2, \quad (3.21)$$

$$\varrho = \frac{3}{2} gJ^2 q^2, \quad (3.22)$$

$$\frac{g}{Q_r(\omega_n)} = \omega_n^2 + \bar{\lambda} - \Sigma_r(\omega_n), \quad (3.23)$$

$$\frac{gq}{Q_r(\omega_n = 0)^2} = \varrho + gh^2. \quad (3.24)$$

We solve these equations self-consistently in the imaginary-frequency domain for the order parameter q as a function of the transverse field g at low temperatures for several values of the longitudinal field, as shown in Fig. 7. In this model, the phase transition between the RS phase ($q \neq 0$) and the paramagnet (PM, $q = 0$) occurs only at zero longitudinal field. With even a slight increase in h ,

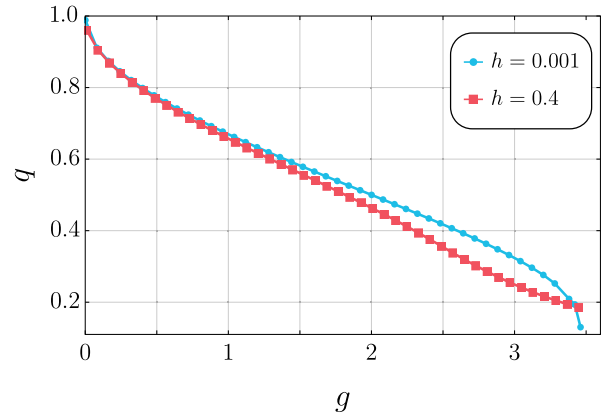


FIG. 7. The behavior of the order parameter q for the replica-symmetric solution of the quantum spherical $p = 3$ model as a function of the transverse field g for different values of longitudinal fields h (with $J = 1$ and $T = 0.005$). The phase transition between the replica-symmetric and paramagnetic solutions occurs only for $h = 0$.

the replica-symmetric solution becomes equivalent to the paramagnet for all values of g , as q is always nonzero [see Eq. (3.24)], as in Sec. II E 1.

Regardless of the value of the longitudinal field, we can show that the spectrum of the RS solution is gapped. To do so, we compute the dynamic spin susceptibility of the system, which requires analytic continuation of Eqs. (3.20)–(3.24) to real frequencies. Taking the zero-temperature limit, we obtain equations of the form

$$\rho(\omega) = \frac{g}{\pi} \frac{\Sigma_r''(\omega)}{(\Sigma_r''(\omega))^2 + (-\omega^2 + \bar{\lambda} - \Sigma_r'(\omega))^2}, \quad (3.25)$$

$$\Sigma_r''(\omega) = 3\pi gJ^2 \left(q\rho(\omega) + \frac{1}{2} \int_0^\omega d\omega_1 \rho(\omega_1) \rho(\omega - \omega_1) \right), \quad (3.26)$$

$$\Sigma_r'(\omega) = 2 \int_0^{+\infty} \frac{dv}{\pi} \frac{v \Sigma_r''(v) - \omega \Sigma_r''(\omega)}{v^2 - \omega^2}, \quad (3.27)$$

$$\varrho = \frac{3}{2} gJ^2 q^2, \quad (3.28)$$

$$q = \frac{g\varrho + g^2 h^2}{[\bar{\lambda} - \Sigma_r'(\omega = 0)]^2}, \quad (3.29)$$

$$1 = q + \int_0^{+\infty} d\omega \rho(\omega), \quad (3.30)$$

where the spectral representation of the regular component of the replica overlap $Q_{ab}(i\omega_n)$ is

$$Q_r(z) = \int_{-\infty}^{\infty} d\omega \frac{\omega \rho(\omega)}{z^2 + \omega^2}, \quad (3.31)$$

and $\Sigma_r'(\omega)$ and $\Sigma_r''(\omega)$ are the real and imaginary parts of $\Sigma_r(i\omega_n)$, respectively. The spectral function is related to the

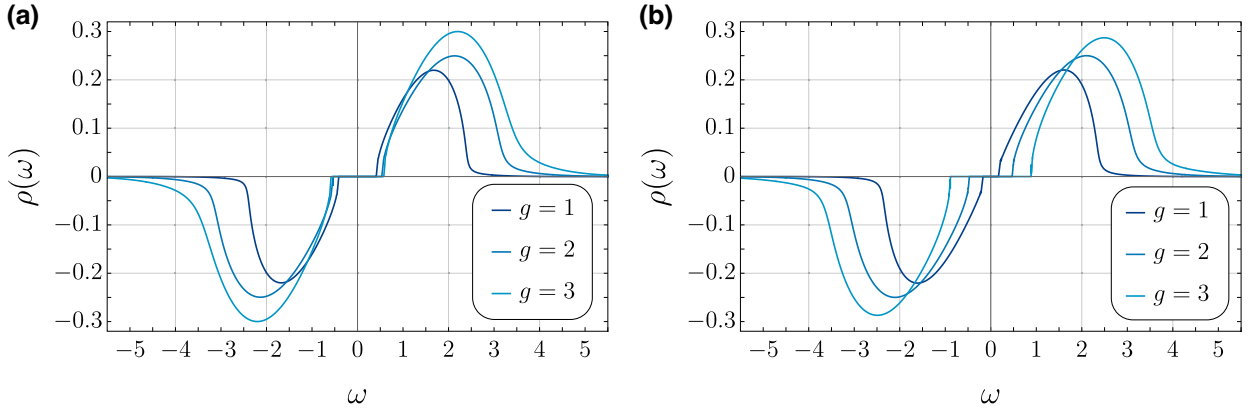


FIG. 8. The spectral functions of the quantum spherical $p = 3$ model for (a) $h = 0$ and (b) $h = 1$ at zero temperature (with $J = 1$). (a) The phase transition between the RS phase and the quantum paramagnet occurs at a finite $g > 3$ and the gap thus eventually saturates upon increasing g . (b) The replica-symmetric solution persists for all values of g with a gap that grows with increasing g .

retarded Green's function as $\rho(\omega) = \text{Im } Q_r(\omega)/\pi$. Taking the $z \rightarrow \infty$ limit above, the associated sum rule becomes $\int_0^{+\infty} d\omega \omega \rho(\omega) = g/2$.

We note that, strictly speaking, the above equations permit several solutions, one of which has a discontinuity at zero frequency given by $\rho(\omega = 0) = 1/(2\pi J \sqrt{6q})$. To avoid this, we initialize the system with a simple step function with a small gap and let the equations converge (defined as when the error reaches 10^{-20}). As a final step, we also check that the sum rule is satisfied.

The spectral functions obtained in this fashion are presented in Fig. 8. For $h = 0$, we observe that as g increases, the gap saturates to a finite value and eventually, after the transition point, the system becomes a quantum-disordered paramagnet, as shown in Appendix D. However, at a finite field, the gap increases with increasing g without a visible saturation, since the replica-symmetric solution and the paramagnet are one and the same.

D. One-step replica symmetry breaking

In this section, we now allow for nontrivial structure in the replica-off-diagonal space. In general, replacing Eq. (3.19), we can write

$$\begin{aligned} Q_{ab}(\omega_n) &= \beta q_{ab} \delta_{\omega_n, 0}, \quad a \neq b, \\ \Sigma_{ab}(\omega_n) &= \beta \varrho_{ab} \delta_{\omega_n, 0}, \quad a \neq b. \end{aligned} \quad (3.32)$$

The matrices q_{ab} and ϱ_{ab} are characterized by Parisi functions $q(u)$ and $\varrho(u)$ with $u \in [0, 1]$. For the diagonal components, we make the same ansatz as in Eq. (3.19):

$$\begin{aligned} Q_{aa}(\omega_n) &= \beta q_1 \delta_{\omega_n, 0} + Q_r(\omega_n), \\ \Sigma_{aa}(\omega_n) &= \beta \varrho_1 \delta_{\omega_n, 0} + \Sigma_r(\omega_n). \end{aligned} \quad (3.33)$$

We consider the one-step replica-symmetry-breaking (RSB) ansatz, for which (see Fig. 2)

$$\begin{aligned} q(u) &= \begin{cases} q_1, & x < u < 1, \\ q_0, & 0 < u < x, \end{cases} \quad \text{and} \\ \varrho(u) &= \begin{cases} \varrho_1, & x < u < 1, \\ \varrho_0, & 0 < u < x. \end{cases} \end{aligned} \quad (3.34)$$

Using the identities in Appendix E, Eqs. (3.17) now become

$$1 = q_1 + \frac{1}{\beta} \sum_{\omega_n} Q_r(\omega_n), \quad (3.35)$$

$$\varrho_1 + \Sigma_r(\tau) = \frac{pgJ^2}{2} (q_1 + Q_r(\tau))^{p-1}, \quad (3.36)$$

$$\varrho_1 = \frac{pgJ^2}{2} q_1^{p-1}, \quad (3.37)$$

$$\varrho_0 = \frac{pgJ^2}{2} q_0^{p-1}, \quad (3.38)$$

$$Q_r(\omega_n) = \frac{g}{\omega_n^2 + m^2 - \Sigma_r(\omega_n) + \Sigma_r(\omega_n = 0)}, \quad (3.39)$$

$$gh^2 + \varrho_1 = \frac{q_1 g + m^2 \beta x (q_1 - q_0)^2}{[g/m^2 + \beta x (q_1 - q_0)]^2}, \quad (3.40)$$

$$gh^2 + \varrho_0 = \frac{g q_0}{[g/m^2 + \beta x (q_1 - q_0)]^2}, \quad (3.41)$$

where we have simplified the equations by changing variables and introducing $m^2 = \bar{\lambda} - \Sigma_r(\omega_n = 0)$.

There are eight unknowns—namely, m^2 , Q_r , Σ_r , q_1 , q_0 , ϱ_1 , ϱ_0 , and βx —in the above equations. All but one of these variables can be determined by solving the seven equations given in Eqs. (3.35)–(3.41). However, the value of the breakpoint βx is undetermined and can be fixed

either by demanding a gapless solution, as in Appendix 3 of Ref. [65], or by requiring the free energy to be stationary with respect to x . The additional equation then makes the system complete and the solution can be obtained by solving the set of equations self-consistently. In the following, we derive this extra equation in both cases: by imposing the gapless constraint and by using the free energy. In general, the two methods lead to different solutions. It is known for the classical system that the gapless condition is chosen by dynamic preparation [32,43,66], while the free-energy stationarity is chosen by the thermodynamic equilibrium state [53].

1. Gapless condition

Let us first determine the condition for a gapless spectrum in the RSB case; our approach here is similar to the analysis in Ref. [65]. For a gapless spectrum, we expect the low-frequency expansion of Q_r , for real ω , to have the form

$$Q_r(\omega + i\eta) = \frac{g}{m^2} + |\omega|^\alpha [a + ib \operatorname{sgn}(\omega)] + \dots, \quad (3.42)$$

for some $\alpha > 0$ and $b > 0$. From Eq. (3.36), we find that the leading singularity in $\Sigma_r(\omega)$ is given by the linear-in- $Q_r(\omega)$ term in Eq. (3.36), so

$$\begin{aligned} \operatorname{Im} \Sigma_r(\omega + i\eta) &= \frac{p(p-1)gJ^2}{2} q_1^{p-2} \operatorname{Im} Q_r(\omega + i\eta), \\ &= \frac{p(p-1)gJ^2}{2} q_1^{p-2} b |\omega|^\alpha \operatorname{sgn}(\omega). \end{aligned} \quad (3.43)$$

However, from Eq. (3.39), for low frequencies, we also have

$$\begin{aligned} \operatorname{Im} \Sigma_r(\omega + i\eta) &= -g \operatorname{Im} [Q_r(\omega + i\eta)]^{-1} \\ &= \frac{m^4}{g} b |\omega|^\alpha \operatorname{sgn}(\omega). \end{aligned} \quad (3.44)$$

Therefore, comparing Eqs. (3.43) and (3.44), we arrive at the constraint on the variable m that has to be satisfied for the gapless solution to exist:

$$m^4 = \frac{p(p-1)g^2J^2}{2} q_1^{p-2}. \quad (3.45)$$

It is easy to check that for the special case of $p = 2$, this expression agrees precisely with Eq. (33.41) in Ref. [57] and is similar to Eq. (4.11) in Ref. [33].

It is important to note that upon solving the replica-symmetric equations self-consistently, we have not obtained a gapless solution. One way to understand this is to plug the gapless constraint in Eq. (3.45) into the equations for the replica-symmetric case given in Eqs. (3.20)–(3.24), which expressly demonstrates that the constraint cannot be satisfied for any values of h . Therefore, the replica-symmetric solution always has a gap.

2. Free energy

Another way to fix x is to determine the saddle point of the free energy with respect to x . To do so, we rewrite the effective action of the p -rotor model of Eq. (3.16) explicitly for the one-step replica-symmetry-breaking case:

$$\begin{aligned} \frac{S_{\text{eff}}}{Nn} &= -\frac{1}{2} \sum_{\omega_n} \log[Q_r(\omega_n)] - \frac{1}{2} \frac{\beta q_0}{Q_r(\omega_n = 0) + \beta x(q_1 - q_0)} + \frac{1}{2x} \ln \frac{Q_r(\omega_n = 0)}{Q_r(\omega_n = 0) + \beta x(q_1 - q_0)} \\ &\quad - \frac{\beta \bar{\lambda}}{2g} + \frac{1}{2g} \sum_{\omega_n} (\omega_n^2 + \bar{\lambda}) Q_r(\omega_n) + \frac{\bar{\lambda} \beta q_1}{2g} - \frac{\beta h^2}{2} [Q_r(\omega_n = 0) + \beta x(q_1 - q_0)] \\ &\quad - \frac{\beta J^2}{4} \int_0^\beta d\tau [(Q_r(\tau) + q_1)^p - q_1^p(1-x) - q_0^p x]. \end{aligned} \quad (3.46)$$

This action allows us to directly verify that the saddle-point equations are in fact correct: taking the derivative of Eq. (3.46) with respect to q_0 and $\bar{\lambda}$, we see that Eqs. (3.35) and (3.41) indeed hold after a change of variables to $m^2 = \bar{\lambda} - \Sigma_r(\omega_n = 0)$.

To obtain an equation for x , we differentiate the action with respect to the breakpoint and obtain

$$\begin{aligned} \frac{\partial S_{\text{eff}}}{\partial x} = 0 \quad \Rightarrow \quad &\frac{J^2 \beta^2}{4} (q_1^p - q_0^p) - \frac{1}{2x^2} \log \left[1 + \frac{\beta x(q_1 - q_0)}{Q_r(\omega_n = 0)} \right] \\ &+ \frac{\beta^2 h^2}{2} (q_1 - q_0) + \frac{\beta(q_1 - q_0)(Q_r(\omega_n = 0) + \beta q_1 x - 2\beta q_0 x)}{2x(Q_r(\omega_n = 0) + \beta x(q_1 - q_0))^2} = 0. \end{aligned} \quad (3.47)$$

Although such a saddle-point condition has been employed to obtain the equilibrium state for the classical model [53], we will not use Eq. (3.47) here. In what follows, we instead use the gapless constraint to determine the breakpoint x .

3. Spectral functions

As for the replica-symmetric case before, we are interested in the behavior of the spectral functions in the RSB case. To this end, we analytically continue the equations on the imaginary-frequency axis Eqs. (3.35)–(3.38) and Eqs. (3.39)–(3.41), focusing on the $p = 3$ case at zero temperature. After such analytic continuation, we obtain

$$1 = q_1 + \int_0^{+\infty} d\omega \rho(\omega), \quad (3.48)$$

$$\Sigma_r''(\omega) = 3\pi gJ^2 \left(q_1 \rho(\omega) + \frac{1}{2} \int_0^\omega d\omega_1 \rho(\omega_1) \rho(\omega - \omega_1) \right), \quad (3.49)$$

$$\Sigma_r'(\omega) = 2 \int_0^{+\infty} \frac{d\nu}{\pi} \frac{\nu \Sigma_r''(\nu) - \omega \Sigma_r''(\omega)}{\nu^2 - \omega^2}, \quad (3.50)$$

$$q_1 = \frac{3gJ^2}{2} q_1^2, \quad (3.51)$$

$$q_0 = \frac{3gJ^2}{2} q_0^2, \quad (3.52)$$

$$\rho(\omega) = \frac{g}{\pi} \frac{\Sigma_r''(\omega) - \Sigma_r''(\omega = 0)}{(\Sigma_r''(\omega) - \Sigma_r''(\omega = 0))^2 + (\omega^2 - m^2 + \Sigma_r'(\omega) - \Sigma_r'(\omega = 0))^2}, \quad (3.53)$$

$$gh^2 + q_1 = \frac{q_1 g + m^2 \beta x (q_1 - q_0)^2}{[g/m^2 + \beta x (q_1 - q_0)]^2}, \quad (3.54)$$

$$gh^2 + q_0 = \frac{gq_0}{[g/m^2 + \beta x (q_1 - q_0)]^2}. \quad (3.55)$$

For the final equation, as mentioned, we use the gapless constraint

$$m^4 = 3g^2 J^2 q_1, \quad (3.56)$$

derived above in Eq. (3.45) and this gapless behavior should also be reflected in the spectral functions. Therefore, we are allowed to make a linear ansatz

$$\rho(\omega) = \omega f(\omega). \quad (3.57)$$

We insert this ansatz in Eqs. (3.49)–(3.55) and solve the equations self-consistently for $f(\omega)$. It is quite challenging to obtain the self-consistent solution to these equations for all unknowns. Instead, we simplify this task by solving the equations on the imaginary-frequency axis and self-consistently obtain the value of the order parameter q_1 at a temperature close to zero with a high precision. Then, we use this result to obtain the behavior of the spectral functions. To ensure that the converged solution is meaningful, we check the sum rule, which, in this case, is given by $\int_0^{+\infty} d\omega \omega^2 f(\omega) = g/2$.

We sketch the behavior of the spectral functions $\text{Im} Q_r(\omega)$ in Fig. 3. From the above equations, it is clear

that spectral functions are independent of h , βx , and q_0 . Only two parameters, namely, q_0 and βx , depend on h . This is similar to the behavior of the Ising model in Sec. II E 2, where the only change from the $h = 0$ solution in Sec. II D 2 was in the form of $q(u)$ for $u < (q_h/q_{\text{EA}})x$ in Eq. (2.46). It is also instructive to determine the dependence of q_0 and βx on h in the $h \rightarrow 0$ limit, which can be calculated from Eqs. (3.54) and (3.55). We find that, to the lowest order, the dependence is simply

$$q_0 = \frac{4g^2}{m^4} h^2; \quad \beta x = \frac{g}{q_1 m^2} - \frac{4g^2}{3m^2 J^2 q_1^3} h^2, \quad (3.58)$$

which, as noted in Sec. I A, is in distinction to the $h^{2/3}$ scaling observed for the Ising spin glass.

E. Phase diagram

Finally, having assembled all the necessary ingredients, we now compute the phase diagram of the quantum spherical $p = 3$ rotor model in a longitudinal field at low temperatures. To do so, we first determine the boundary of stability of the RSB solution by self-consistently computing the spin-glass order parameter from Eqs. (3.35)–(3.41)

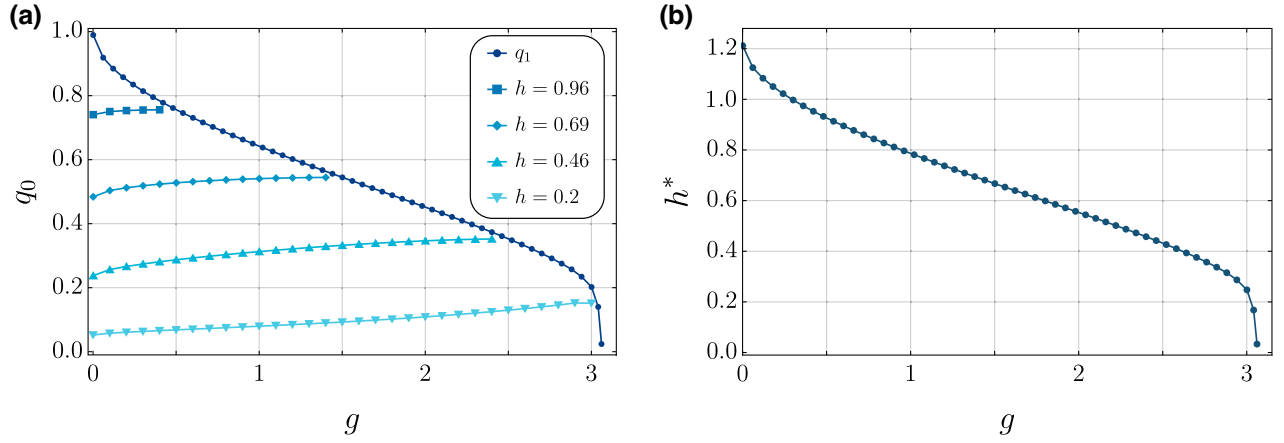


FIG. 9. (a) The behavior of q_0 and q_1 as a function of g for the spherical quantum $p = 3$ rotor model in a longitudinal field with $T = 0.005$ and $J = 1$. Note that q_1 is independent of h but q_0 is not. (b) The boundary of stability of the RSB phase as determined by setting $q_0(g, h^*) = q_1(g)$. For $h > h^*(g)$, the necessary condition $q_0 \leq q_1$ required for the RSB solution no longer holds. Note that both plots are extrapolated around $q_0 \approx 0$ and $h^* \approx 0$.

and comparing q_0 and q_1 . For the stability of the RSB solution, one has to require $q_0 \leq q_1$ [52] and the solution ceases to be physical when $q_0 > q_1$. Our numerical results in this regard are displayed in Fig. 9. In particular, in Fig. 9(a), we present q_0 and q_1 as a function of g for various values of h ; the RSB solution exists only for q_0 that lie below the line $q_1(g)$. Based on this information, in Fig. 9(b), we trace, in g - h space, the line $h^*(g)$ along which $q_0(g, h^*) = q_1(g)$.

Since this marks the boundary of stability of the RSB solution, the phase transition between the RSB and RS phases may occur at this line.

To check whether this is indeed the case, we compute the free energies of both the RS and RSB solutions. The free energy in the RS case can be written using Eq. (3.16) and we obtain

$$\begin{aligned} \frac{F_{\text{eff}}^{\text{RS}}}{Nn} = & -\frac{1}{2\beta} \sum_{\omega_n} \log[(\omega_n^2 + \Gamma^2)Q_r(\omega_n)/g] - \frac{1}{2} \frac{q}{Q_r(\omega_n = 0)} - \frac{\bar{\lambda}}{2g} + \frac{\bar{\lambda}q}{2g} + \frac{1}{2g\beta} \sum_{\omega_n} [(\omega_n^2 + \bar{\lambda})Q_r(\omega_n) - g] \\ & - \frac{h^2}{2} Q_r(\omega_n = 0) - \frac{J^2}{4} \int_0^\beta d\tau [(q + Q_r(\tau))^3 - q^3] + \frac{\Gamma}{2} + \frac{1}{\beta} \ln[1 - e^{-\beta\Gamma}], \end{aligned} \quad (3.59)$$

where we have inserted a factor of $(\omega_n^2 + \Gamma^2)/g$ into the argument of the logarithm to regularize the behavior at large $|\omega_n|$ and subtracted a constant to regulate the frequency summation in the term linear in Q_r . This is equivalent to normalizing the path integral with an oscillator of frequency Γ . In order to make the resulting expression independent of Γ , we have subtracted the free energy of this oscillator. For the RSB case, we use the expression for the free energy computed above, focusing on $p = 3$:

$$\begin{aligned} \frac{F_{\text{eff}}^{\text{RSB}}}{Nn} = & -\frac{1}{2\beta} \sum_{\omega_n} \log[(\omega_n^2 + \Gamma^2)Q_r(\omega_n)/g] - \frac{1}{2} \frac{q_0}{Q_r(\omega_n = 0) + \beta x(q_1 - q_0)} + \frac{1}{2\beta x} \ln \frac{Q_r(\omega_n = 0)}{Q_r(\omega_n = 0) + \beta x(q_1 - q_0)} \\ & - \frac{\bar{\lambda}}{2g} + \frac{1}{2g\beta} \sum_{\omega_n} [(\omega_n^2 + \bar{\lambda})Q_r(\omega_n) - g] + \frac{\bar{\lambda}q_1}{2g} - \frac{h^2}{2} [Q_r(\omega_n = 0) + \beta x(q_1 - q_0)] \\ & - \frac{J^2}{4} \int_0^\beta d\tau [(q_1 + Q_r(\tau))^3 - q_1^3(1-x) - q_0^3x] + \frac{\Gamma}{2} + \frac{1}{\beta} \ln[1 - e^{-\beta\Gamma}]. \end{aligned} \quad (3.60)$$

Here, we use the same regularization as for the free energy of the replica-symmetric solution. In both cases, Γ has to be much smaller than the Matsubara-frequency cutoff.

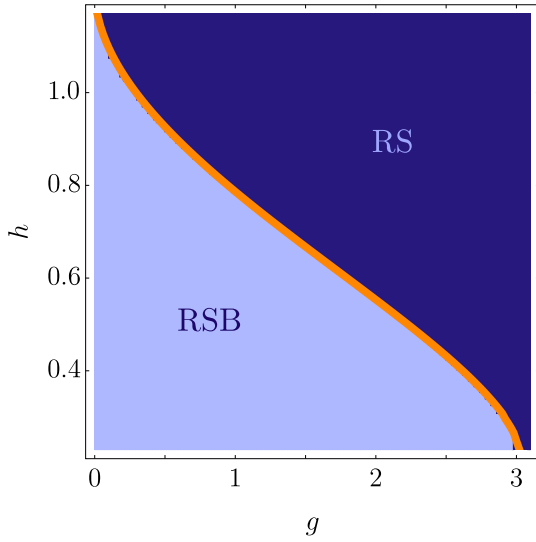


FIG. 10. The phase diagram of the spherical quantum $p = 3$ rotor model in a longitudinal field with $T = 0.005$ and $J = 1$. The phase diagram is evaluated by calculating the difference between the free energies of the replica-symmetric [RS, Eq. (3.59)] and replica-symmetry-breaking [RSB, Eq. (3.60)] solutions. The orange line on the phase boundary traces the function $h^*(g)$ plotted in Fig. 9(b).

We evaluate the difference in free energies between the RSB and RS solutions to find the parameter regions in which each phase is dominant. Following the discussion in Ref. [55], we pick the ground state as the one that has a larger free energy. The resulting phase diagram is shown in Fig. 10; we find that the phase boundary obtained by comparing the free energies coincides (to a precision of $|F_{\text{eff}}^{\text{RSB}} - F_{\text{eff}}^{\text{RS}}|/(Nn) < 10^{-5}$) with the line $h^*(g)$ demarcating the stability of the RSB solution in Fig. 9(b).

IV. CONCLUSIONS

Motivated by the quantum simulation of spatially disordered long-ranged interacting systems for solving combinatorial optimization problems [1,67], we have studied the equilibrium dynamic properties of two quantum spin-glass models with infinite-range interactions: the Ising model in Eq. (1.1) and the spherical p -rotor model in Eqs. (1.3) and (1.4). In addition to random couplings between spins and rotors, both models have an applied longitudinal field h and do not break any global symmetry in any phase. A coupling g tunes the strength of the quantum fluctuations and, at large g , we obtain a gapped paramagnetic phase. Our primary interest has been in the small- g regime, where replica symmetry breaking leads to a quantum spin-glass phase.

For the Ising model, we have employed the Landau-theory approach of Ref. [18] to show that the spin-glass phase has full replica symmetry breaking, with the Parisi function sketched in Fig. 1. We have computed the spin

autocorrelation function in this spin-glass phase and shown that it is generically gapless, with a spectral density that vanishes linearly with frequency [see Eq. (1.9) and Secs. IID 2 b and IIE 2].

In contrast, the spherical p -rotor model ($p \geq 3$) has a different behavior in the spin-glass phase. There is only one-step replica symmetry breaking (see Fig. 2) and the breakpoint x remains undetermined by the saddle-point equations for the matrix spin-glass order parameter. Rigorous mathematical work on the classical $g = 0$ model [53] has shown that the equilibrium value of x is determined by the stationarity of the free energy with respect to x . For this x , we have shown that the spin-glass state has an energy gap. In contrast, if a marginal stability condition [31,32,56] is used to determine x , we have found a gapless spectrum with a linear frequency dependence at small frequency (see Fig. 3), similar to that of the Ising model.

Our results on the nature of the spin-glass phase also find relevance in the context of quantum optimization problems since the properties of the gap—or the lack thereof—directly inform the feasibility of preparing low-energy states via adiabatic algorithms [68–70].

Looking ahead, it would be useful to obtain the results described here without the replica method, using a path integral on the Schwinger-Keldysh contour. This has already been studied for the spherical p -rotor model [32, 43,66], where the breakpoint of one-step replica symmetry breaking, x , has been connected to an effective temperature of the aging dynamics. Moreover, the equation for a gapless spectrum in Eq. (3.45) is the same as Eq. (6.24) obtained in Ref. [32] from aging dynamics. However, the aging dynamics of the quantum Ising model [20] with the crucial y quartic term have not been studied so far and it would be interesting to relate that to the full replica symmetry breaking characterized by Fig. 1. Such a computation would be the analogue of earlier studies [71–75] of aging dynamics in classical spin glasses described by Langevin equations, and we expect that there would be significant differences [75,76] between the Ising model and the spherical model in the quantum problem too.

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APPENDIX A: TERMS OF ORDER y^1 IN \mathcal{F}_Q FOR THE ISING MODEL

This appendix contains the $\mathcal{O}(y)$ corrections to the free energy and saddle-point values from the ‘‘quantum’’ contribution to the free energy, \mathcal{F}_Q .

1. Zero longitudinal field

As in the main text, we first consider the $h = 0$ case of Sec. IID.

a. Paramagnet

Here, we continue the analysis of Sec. IID 1.

For the saddle point at order y^1 , we write

$$Q_r(iv_n) = -\frac{\sqrt{v_n^2 + \Delta^2 + y\Delta_1^2 - y\Sigma(iv)}}{\kappa}, \quad (\text{A1})$$

where we can choose $\Sigma(0) = 0$ without loss of generality by adjusting the value of Δ_1^2 . Then, from Eq. (2.19), we obtain

$$\begin{aligned} 0 = & -[y\Delta_1^2 - y\Sigma(iv_n)] - \frac{U}{\beta} \sum_{v'_n} \left[\sqrt{v_n'^2 + \Delta^2 + y\Delta_1^2 - y\Sigma(iv'_n)} - \sqrt{v_n'^2 + \Delta^2} \right] \\ & + \frac{2y}{3\beta^2\kappa^2} \sum_{v'_n, v''_n} \sqrt{v_n'^2 + \Delta^2} \sqrt{v_n''^2 + \Delta^2} \sqrt{(v_n - v'_n - v''_n)^2 + \Delta^2}. \end{aligned} \quad (\text{A2})$$

After subtracting from Eq. (A2) its value at $v_n = 0$, we determine

$$\Sigma(iv_n) = -\frac{2}{3\beta^2\kappa^2} \sum_{v'_n, v''_n} \sqrt{v_n'^2 + \Delta^2} \sqrt{v_n''^2 + \Delta^2} \left[\sqrt{(v_n - v'_n - v''_n)^2 + \Delta^2} - \sqrt{(v'_n + v''_n)^2 + \Delta^2} \right]. \quad (\text{A3})$$

Then, Δ_1^2 can be obtained from Eq. (A2) as the solution of

$$\begin{aligned} \Delta_1^2 = & \frac{2}{3\beta^2\kappa^2} \sum_{v_n, v'_n} \sqrt{v_n'^2 + \Delta^2} \sqrt{v_n'^2 + \Delta^2} \sqrt{(v'_n + v_n)^2 + \Delta^2} + \frac{U}{2\beta} \sum_{v_n} \frac{\Sigma(iv)}{\sqrt{v_n^2 + \Delta^2}} \\ & - \lim_{y \rightarrow 0} \frac{U}{y\beta} \sum_{v_n} \left[\sqrt{v_n^2 + \Delta^2 + y\Delta_1^2} - \sqrt{v_n^2 + \Delta^2} \right]. \end{aligned} \quad (\text{A4})$$

At the critical point, where $\Delta^2 + y\Delta_1^2 = 0$ and $r = r_{c0} + yr_{c1}$, using Eqs. (2.21) and (2.22), we can write the term proportional to u in Eq. (A2) as

$$y\Delta_1^2 + yr_{c1} + \frac{Uy}{2\beta} \sum_{v_n} \frac{\Sigma(iv_n)|_{\Delta=0}}{|v_n|}, \quad (\text{A5})$$

where $\Sigma(iv_n)$ is to be evaluated from Eq. (A3) at $\Delta = 0$. Then, Eq. (A2) at $v = 0$ yields the shift in the position of the critical point at order y as

$$r_{c1} = -\frac{2}{3\beta^2\kappa^2} \sum_{v_n, v'_n} |v'_n| |v_n| |v'_n + v_n| - \frac{U}{2\beta} \sum_{v_n} \frac{\Sigma(iv_n)|_{\Delta=0}}{|v_n|}. \quad (\text{A6})$$

For the free energy at order y^1 , with $\mathcal{F}_Q = \mathcal{F}_Q^0 + y\mathcal{F}_Q^1$, we can just insert the $\mathcal{O}(y^0)$ saddle-point values into the terms explicitly dependent upon y in Eq. (2.15):

$$\mathcal{F}_Q^1 = -\frac{1}{6\beta^3\kappa^4} \sum_{v_n, v'_n, v''_n} \sqrt{v_n^2 + \Delta^2} \sqrt{v_n'^2 + \Delta^2} \sqrt{v_n''^2 + \Delta^2} \sqrt{(v_n + v'_n + v''_n)^2 + \Delta^2}. \quad (\text{A7})$$

b. Spin glass

Here, we continue the analysis of Sec. IID 2.

For the saddle-point equations at order y^1 , we write

$$\begin{aligned} q_{EA} &= \frac{1}{\kappa U}(r_{c0} - r) + yq_{EA1}, \\ Q_r(iv_n) &= -\frac{|v_n|}{\kappa} + yQ_{r1}(iv_n). \end{aligned} \quad (\text{A8})$$

Then, Eqs. (2.29) and (2.30) yield

$$\begin{aligned} 0 &= 2|v_n|Q_{r1}(iv_n) + \frac{U}{\beta} \sum_{v'_n} Q_{r1}(iv'_n) + uq_{EA1} + \frac{2}{3\kappa^3\beta^2} \sum_{v'_n, v''_n} |v'_n||v''_n||v_n - v'_n - v''_n| \\ &\quad - \frac{2}{\kappa^2\beta} q_{EA0} \sum_{v'_n} |v'_n||v_n - v'_n| - 2Q_{r1}(0)|v_n|, \end{aligned} \quad (\text{A9})$$

$$q_{EA0}^2 = -\kappa Q_{r1}(0). \quad (\text{A10})$$

From Eqs. (A9) and (2.30), we determine that

$$q_{EA1} = -\frac{1}{\beta} \sum_{v_n} Q_{r1}(iv_n) - \frac{2}{3\kappa^3 U \beta^2} \sum_{v'_n, v''_n} |v'_n||v_n||v'_n + v_n| + \frac{2}{\kappa^2 U \beta} q_{EA0} \sum_{v_n} |v_n|^2 \quad (\text{A11})$$

and

$$\begin{aligned} Q_{r1}(iv_n) &= -\frac{q_{EA0}^2}{\kappa} - \frac{1}{3\kappa^3\beta^2|v_n|} \sum_{v'_n, v''_n} \left(|v'_n||v''_n||v_n - v'_n - v''_n| - |v'_n||v''_n||v'_n + v''_n| \right) \\ &\quad + \frac{1}{\beta\kappa^2|v_n|} q_{EA0} \sum_{v'_n} \left(|v'_n||v_n - v'_n| - |v'_n|^2 \right). \end{aligned} \quad (\text{A12})$$

At $T = 0$, we obtain, with a frequency cutoff $|v_n| < \Lambda$,

$$Q_{r1}(iv_n) = -\frac{q_{EA0}^2}{\kappa} - \frac{1}{6\pi\kappa^3} \left(\frac{2\Lambda^3}{3}|v_n| - \frac{\Lambda^2}{3}|v_n|^2 + \frac{1}{20}|v_n|^4 \right) + \frac{1}{6\pi\kappa^2} q_{EA0}|v_n|^2. \quad (\text{A13})$$

The shift in the quantum critical point is obtained by setting $r = r_{c0} + yr_{c1}$ in Eqs. (A11) and (A12) and this yields the same value of r_{c1} as that obtained from the vanishing-gap condition on the paramagnetic side in Eq. (A6).

For the free energy at order y^1 , we now obtain, from Eq. (2.15),

$$\mathcal{F}_Q^1 = -\frac{r q_{EA1}}{\kappa} - \frac{q_{EA0}^2}{\beta\kappa^2} \sum_{v_n} v_n^2 + \frac{2q_{EA0}}{3\kappa^3\beta^2} \sum_{v_n, v'_n} |v_n||v'_n||v_n + v'_n| - \frac{1}{6\beta^3\kappa^4} \sum_{v_n, v'_n, v''_n} |v_n||v'_n||v''_n||v_n + v'_n + v''_n|. \quad (\text{A14})$$

2. Nonzero longitudinal field

Next, we consider the case of $h \neq 0$ studied in Sec. IIE.

a. Replica-symmetric solution

This section continues the analysis of Sec. IIE 1.

For the saddle point at order y^1 , we write as in Eq. (A1)

$$Q_r(iv_n) = -\frac{\sqrt{v_n^2 + \Delta^2 + y\Delta_1^2 - y\Sigma(iv)}}{\kappa},$$

$$q_{EA} = q_{EA0} + q_{EA1}, \quad (\text{A15})$$

where we can again choose $\Sigma(0) = 0$ without loss of generality by adjusting the value of Δ_1^2 . Then, from Eq. (2.18), we obtain

$$0 = -[y\Delta_1^2 - y\Sigma(iv_n)] - \frac{U}{\beta} \sum_{v'_n} \left[\sqrt{v_n'^2 + \Delta^2 + y\Delta_1^2 - y\Sigma(iv'_n)} - \sqrt{v_n'^2 + \Delta^2} \right]$$

$$+ U\kappa y q_{EA1} + \frac{2y}{3\beta^2\kappa^2} \sum_{v'_n, v''_n} \sqrt{v_n'^2 + \Delta^2} \sqrt{v_n''^2 + \Delta^2} \sqrt{(v_n - v'_n - v''_n)^2 + \Delta^2}$$

$$- \frac{2y}{\beta\kappa} q_{EA0} \sum_{v'_n} \sqrt{v_n'^2 + \Delta^2} \sqrt{(v_n - v'_n)^2 + \Delta^2} + 2yq_{EA0}^2 \sqrt{v_n^2 + \Delta^2}. \quad (\text{A16})$$

After subtracting from Eq. (A16) its value at $v_n = 0$, we determine that

$$\Sigma(iv_n) = -\frac{2}{3\beta^2\kappa^2} \sum_{v'_n, v''_n} \sqrt{v_n'^2 + \Delta^2} \sqrt{v_n''^2 + \Delta^2} \left[\sqrt{(v_n - v'_n - v''_n)^2 + \Delta^2} - \sqrt{(v'_n + v''_n)^2 + \Delta^2} \right]$$

$$+ \frac{2}{\beta\kappa} q_{EA0} \sum_{v'_n} \left[\sqrt{v_n'^2 + \Delta^2} \sqrt{(v_n - v'_n)^2 + \Delta^2} - (v_n^2 + \Delta^2) \right] - 2q_{EA0}^2 \left[\sqrt{v_n^2 + \Delta^2} - \Delta \right]. \quad (\text{A17})$$

Then, Δ_1^2 obeys, from Eq. (A16),

$$\Delta_1^2 = U\kappa q_{EA1} + \frac{2y}{3\beta^2\kappa^2} \sum_{v'_n, v''_n} \sqrt{v_n'^2 + \Delta^2} \sqrt{v_n''^2 + \Delta^2} \sqrt{(v_n - v'_n - v''_n)^2 + \Delta^2}$$

$$- \frac{2y}{\beta\kappa} q_{EA0} \sum_{v'_n} \sqrt{v_n'^2 + \Delta^2} \sqrt{(v_n - v'_n)^2 + \Delta^2} + 2yq_{EA0}^2 \sqrt{v_n^2 + \Delta^2}$$

$$+ \frac{U}{2\beta} \sum_{v_n} \frac{\Sigma(iv)}{\sqrt{v_n^2 + \Delta^2}} - \lim_{y \rightarrow 0} \frac{U}{y\beta} \sum_{v_n} \left[\sqrt{v_n^2 + \Delta^2 + y\Delta_1^2} - \sqrt{v_n^2 + \Delta^2} \right]. \quad (\text{A18})$$

The determination of q_{EA1} and Δ_1 requires the solution of Eq. (A16), along with a second equation obtained from Eq. (2.40):

$$-3q_{EA0} \frac{\Delta_1^2}{2\Delta} - 3\Delta q_{EA1} + q_{EA0}^3 = 0. \quad (\text{A19})$$

For the free energy at order y^1 , we now obtain, from Eq. (2.15),

$$\mathcal{F}_Q^1 = -\frac{yq_{EA0}^2}{\beta\kappa^2} \sum_{v_n} (v_n^2 + \Delta^2) + \frac{2yq_{EA0}}{3\kappa^3\beta^2} \sum_{v_n, v'_n} \sqrt{v_n^2 + \Delta^2} \sqrt{v_n'^2 + \Delta^2} \sqrt{(v_n + v'_n)^2 + \Delta^2}$$

$$- \frac{y}{6\beta^3\kappa^4} \sum_{v_n, v'_n, v''_n} \sqrt{v_n^2 + \Delta^2} \sqrt{v_n'^2 + \Delta^2} \sqrt{v_n''^2 + \Delta^2} \sqrt{(v_n + v'_n + v''_n)^2 + \Delta^2}. \quad (\text{A20})$$

APPENDIX B: CLASSICAL SPHERICAL p -ROTOR MODEL

In the classical limit $g = 0$, all components of $Q_{ab}(\tau)$ and $\Sigma_{ab}(\tau)$ are independent of τ and the theory should reduce to that in Ref. [51]. So, we have $Q_{ab}(\omega_n) = \beta Q_{ab}(\tau) \delta_{\omega_n, 0} \equiv \beta Q_{ab} \delta_{\omega_n, 0}$ and similarly for Σ_{ab} and λ_{ab} . Then, the effective action in Eq. (3.11) becomes

$$\begin{aligned} \frac{S_{\text{eff}}}{N} &= \frac{1}{2} \log \det \left[\frac{1}{\pi} (-i\delta_{ab} z^a + i\beta\lambda_{ab}) \right] + i\beta \sum_{a=1}^n z^a \\ &\quad - \sum_{a,b=1}^n \left(\beta^2 i\lambda_{ab} Q_{ab} + \frac{\beta h^2}{4} \left[-i\delta_{ab} z^a + i\beta\lambda_{ab} \right]_{ab}^{-1} + \frac{\beta^2 J^2}{4} (Q_{ab})^p \right). \end{aligned} \quad (\text{B1})$$

Let us now change our notation a little to obtain expressions similar to those in Ref. [57]. We define

$$-iz^a = \bar{z}, \quad i\lambda_{ab} = -\sigma_{ab}. \quad (\text{B2})$$

Then, the effective action is

$$\frac{S_{\text{eff}}}{N} = \frac{1}{2} \log \det \left[\frac{1}{\pi} (\delta_{ab} \bar{z} - \beta\sigma_{ab}) \right] - \beta \sum_{a=1}^n \bar{z} + \sum_{a,b=1}^n \left(\beta^2 \sigma_{ab} Q_{ab} - \frac{\beta h^2}{4} \left[\bar{z} \delta_{cd} - \beta\sigma_{cd} \right]_{ab}^{-1} - \frac{\beta^2 J^2}{4} (Q_{ab})^p \right). \quad (\text{B3})$$

Now, as before, we use the identity Eq. (E21) to absorb the h^2 term into the log det:

$$\begin{aligned} \frac{S_{\text{eff}}}{N} &= \frac{1}{2} \log \det \left[\frac{1}{\pi} \left(\delta_{ab} \bar{z} - \beta\sigma_{ab} - \frac{\beta h^2}{2} \right) \right] \\ &\quad - \beta \sum_{a=1}^n \bar{z} + \sum_{a,b=1}^n \left(\beta^2 \sigma_{ab} Q_{ab} - \frac{\beta^2 J^2}{4} (Q_{ab})^p \right). \end{aligned} \quad (\text{B4})$$

The saddle-point equation with respect to σ_{ab} is

$$\beta Q_{ab} = \frac{1}{2} \left[\delta_{ab} \bar{z} - \beta\sigma_{ab} - \frac{\beta h^2}{2} \right]_{ab}^{-1}. \quad (\text{B5})$$

Inserting this back into Eq. (B4), we obtain an effective action just for Q_{ab} and \bar{z} :

$$\begin{aligned} \frac{S_{\text{eff}}}{N} &= -\frac{1}{2} \log \det Q - \beta \sum_{a=1}^n \bar{z} + \sum_{a,b=1}^n \left(\beta \bar{z} \delta_{ab} Q_{ab} \right. \\ &\quad \left. - \frac{\beta^2 h^2}{2} Q_{ab} - \frac{\beta^2 J^2}{4} (Q_{ab})^p \right). \end{aligned} \quad (\text{B6})$$

Then, the remaining saddle-point equations are

$$\begin{aligned} 1 &= Q_{aa}, \\ Q_{ab}^{-1} &= 2\beta \bar{z} \delta_{ab} - \beta^2 h^2 - \frac{p\beta^2 J^2}{2} (Q_{ab})^{p-1}. \end{aligned} \quad (\text{B7})$$

1. Replica-symmetric solution

Now, we use the replica-symmetric ansatz

$$\begin{aligned} Q_{ab} &= q + Q_r \delta_{ab}, \\ \sigma_{ab} &= \varrho + \sigma_r \delta_{ab}. \end{aligned} \quad (\text{B8})$$

Then, using the identities in Appendix E, the equations in Eq. (B7) become

$$1 = q + Q_r, \quad (\text{B9})$$

$$\sigma_r = \frac{pJ^2}{4} (q + Q_r)^{p-1} - \frac{pJ^2}{4} q^{p-1}, \quad (\text{B10})$$

$$\varrho = \frac{pJ^2}{4} q^{p-1}, \quad (\text{B11})$$

$$\frac{1}{\beta Q_r} = 2(\beta \bar{z} - \beta \sigma_r), \quad (\text{B12})$$

$$\frac{q}{Q_r^2} = \beta^2 h^2 + \frac{p\beta^2 J^2}{2} q^{p-1}. \quad (\text{B13})$$

Focusing on $p = 2$ first, we can simplify the above to

$$1 = q + Q_r, \quad (\text{B14})$$

$$\beta Q_r = \frac{1}{\bar{z} + \sqrt{\bar{z}^2 - J^2}}, \quad (\text{B15})$$

$$q = \frac{J^2 q + h^2}{(\bar{z} + \sqrt{\bar{z}^2 - J^2})^2}. \quad (\text{B16})$$

These equations are easily seen to be equivalent to the classical limit of Eqs. (C1), (C5), and (C7) derived for the

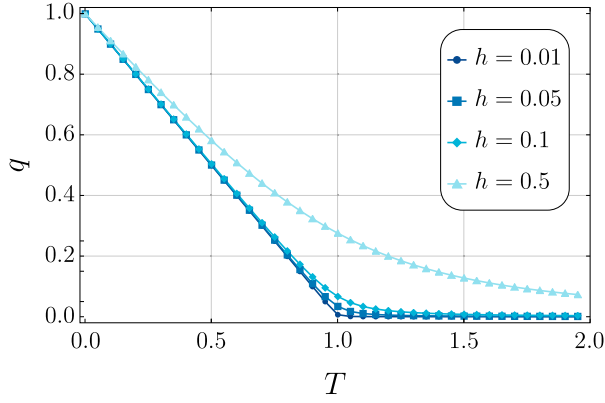


FIG. 11. The spin-glass order parameter q of the classical spherical $p = 2$ rotor model (with $J = 1$), as a function of the temperature T for several values of the longitudinal field h . For the replica-symmetric solution considered here, the spin-glass phase transition occurs at $T = 1$.

quantum model later. In Fig. 11, we showcase the temperature dependence of the order parameter q for nonzero h . For $p = 2$ and $h = 0$, it is known [77] that the replica-symmetric solution is stable and optimal and we expect that this statement continues to hold for any $h \neq 0$ as well [51].

2. One-step replica symmetry breaking

The classical spherical p -rotor model can be solved exactly for all p , temperatures, and fields and hosts a one-step RSB phase for any $p > 2$ [51] (akin to the model without the spherical constraint [50]). In this case, for the diagonal elements of Q_{ab} and σ_{ab} , we use the ansatz

$$\begin{aligned} Q_{aa} &= q_1 + Q_r, \\ \sigma_{aa} &= \varrho_1 + \sigma_r, \end{aligned} \quad (\text{B17})$$

while the off-diagonal elements are as in Eq. (3.34). The equations analogous to Eqs. (3.20)–(3.24) are

$$1 = q_1 + Q_r, \quad (\text{B18})$$

$$\sigma_r = \frac{pJ^2}{4}(q_1 + Q_r)^{p-1} - \frac{pJ^2}{4}q_1^{p-1}, \quad (\text{B19})$$

$$\varrho_1 = \frac{pJ^2}{4}q_1^{p-1}, \quad (\text{B20})$$

$$\varrho_0 = \frac{pJ^2}{4}q_0^{p-1}, \quad (\text{B21})$$

$$\frac{1}{\beta Q_r} = 2(\bar{z} - \beta\sigma_r), \quad (\text{B22})$$

$$\frac{q_1 Q_r + x(q_1 - q_0)^2}{Q_r(Q_r + x(q_1 - q_0))^2} = \beta^2 h^2 + \frac{p\beta^2 J^2}{2}q_1^{p-1}, \quad (\text{B23})$$

$$\frac{q_0}{(Q_r + x(q_1 - q_0))^2} = \beta^2 h^2 + \frac{p\beta^2 J^2}{2}q_0^{p-1}. \quad (\text{B24})$$

We are mainly interested in the $\beta \rightarrow \infty$ limit. From the above equations, we see that this limit exists if we write

$$q_1 = \hat{q} - \frac{\theta}{\beta}, \quad x = \frac{\mu}{\beta}, \quad (\text{B25})$$

where \hat{q} is a β -independent constant, and θ and μ are finite as $\beta \rightarrow \infty$. For the present classical theory, $\hat{q} = 1$ but we will see that $\hat{q} < 1$ in the $g \neq 0$ theory.

a. Gapless condition

The gapless condition, obtained in the $g \rightarrow 0$ limit of Eq. (3.45) is

$$\frac{1}{\beta^2 Q_r^2} = \frac{p(p-1)J^2}{2}q_1^{p-2}. \quad (\text{B26})$$

b. Free energy

The classical limit of the action in Eq. (3.46) is

$$\begin{aligned} \frac{S_{\text{eff}}}{Nn} &= -\frac{1}{2} \log[Q_r] - \frac{1}{2} \frac{q_0}{Q_r + x(q_1 - q_0)} \\ &+ \frac{1}{2x} \ln \frac{Q_r}{Q_r + x(q_1 - q_0)} \\ &- \beta\bar{z} + \beta\bar{z}(Q_r + q_1) - \frac{\beta^2 h^2}{2} [Q_r + x(q_1 - q_0)] \\ &- \frac{\beta^2 J^2}{4} [(Q_r + q_1)^p - q_1^p (1-x) - q_0^p x]. \end{aligned} \quad (\text{B27})$$

We can now confirm that Eqs. (B18)–(B24) are indeed the saddle-point equations obtained from Eq. (B27).

The stationarity condition with respect to the breakpoint yields

$$\begin{aligned} \frac{\partial S_{\text{eff}}}{\partial x} = 0 &\Rightarrow \frac{J^2 \beta^2}{4}(q_1^p - q_0^p) \\ &- \frac{1}{2x^2} \log \left[1 + \frac{x(q_1 - q_0)}{Q_r} \right] \\ &+ \frac{\beta^2 h^2}{2}(q_1 - q_0) + \frac{(q_1 - q_0)(Q_r + q_1 x - 2q_0 x)}{2x(Q_r + x(q_1 - q_0))^2} \\ &= 0. \end{aligned} \quad (\text{B28})$$

One can use either Eq. (B26) or Eq. (B28) to determine x .

APPENDIX C: REPLICA-SYMMETRIC SOLUTION FOR THE $h \neq 0$ QUANTUM SPHERICAL $p = 2$ MODEL

We now consider the quantum version of the spherical $p = 2$ rotor model in a longitudinal field h . As discussed in Ref. [17], for $h = 0$, the replica-symmetric solution is the optimal one. Anticipating the same conclusion to hold for the case of $h \neq 0$, in what follows, we focus on the replica-symmetric solution.

For $p = 2$, Eqs. (3.17), combined with the replica-symmetric ansatz in Eq. (3.19), reduce to

$$1 = q + \frac{1}{\beta} \sum_{\omega_n} Q_r(\omega_n), \quad (C1)$$

$$\varrho = gJ^2 q, \quad (C2)$$

$$Q_r(\omega_n) = \frac{g}{\omega_n^2 + \bar{\lambda} - gJ^2 Q_r(\omega_n)}, \quad (C3)$$

$$q = \frac{g\varrho + g^2 h^2}{[\bar{\lambda} - gJ^2 Q_r(\omega_n = 0)]^2}. \quad (C4)$$

Specifically, the equation for $Q_r(\omega_n)$ can be written as

$$Q_r(\omega_n) = \frac{2g}{\omega_n^2 + \bar{\lambda} + \sqrt{(\omega_n^2 + \bar{\lambda})^2 - 4g^2 J^2}}, \quad (C5)$$

$$Q_r(\omega_n = 0) = \frac{1}{2gJ^2} \left(\bar{\lambda} - \sqrt{\bar{\lambda}^2 - 4g^2 J^2} \right). \quad (C6)$$

Then, the order parameter is given by

$$q = 4g^2 \frac{J^2 q + h^2}{\left[\bar{\lambda} + \sqrt{\bar{\lambda}^2 - 4g^2 J^2} \right]^2}. \quad (C7)$$

From the above equation, we observe that the condition $\bar{\lambda} \geq 2gJ$ has to be satisfied for the order parameter to be a real number. Note that for $h = 0$, we find

$$\bar{\lambda}|_{h=0} = 2gJ, \quad (C8)$$

which is consistent with Eq. (33.41) of Ref. [57]. For $h \neq 0$, the equation for the order parameter can be formulated as

$$\begin{aligned} q &= \frac{4g^2 h^2}{\left[\bar{\lambda} + \sqrt{\bar{\lambda}^2 - 4g^2 J^2} \right]^2 - 4g^2 J^2} \\ &= \frac{2g^2 h^2}{\bar{\lambda}^2 - 4g^2 J^2 + \bar{\lambda} \sqrt{\bar{\lambda}^2 - 4g^2 J^2}}. \end{aligned} \quad (C9)$$

We can simplify the expression above further to

$$\begin{aligned} q &= -\frac{h^2}{2J^2} \frac{\bar{\lambda}^2 - 4g^2 J^2 - \bar{\lambda} \sqrt{\bar{\lambda}^2 - 4g^2 J^2}}{\bar{\lambda}^2 - 4g^2 J^2} \\ &= -\frac{h^2}{2J^2} \frac{\lambda^{*2} - \bar{\lambda} \lambda^*}{\lambda^{*2}} \\ &= -\frac{h^2}{2J^2} \frac{\lambda^* - \bar{\lambda}}{\lambda^*} = \frac{h^2}{2J^2} \left(\frac{\bar{\lambda}}{\lambda^*} - 1 \right), \end{aligned} \quad (C10)$$

where we introduce the notation

$$\lambda^{*2} = \bar{\lambda}^2 - 4g^2 J^2. \quad (C11)$$

We note that the order parameter has to be positive, $q > 0$, so $\bar{\lambda} > \lambda^*$. In addition, we require $\bar{\lambda} > 2gJ$ such that $\lambda^* > 0$.

Finally, we rewrite the correlator $Q_r(\omega_n)$ in terms of our new notation as

$$Q_r(\omega_n) = \frac{2g}{\omega_n^2 + \bar{\lambda} + \sqrt{\omega_n^4 + 2\omega_n^2 \bar{\lambda} + \lambda^{*2}}}. \quad (C12)$$

Since $\lambda^* > 0$, there is always a gap in the spectrum. The gap is given by the value of the frequency at which the square root in the Green's function (analytically continued to the real frequency) changes sign. This value is simply

$$\Delta = \sqrt{\bar{\lambda} - 2gJ}, \quad (C13)$$

which is always greater than zero in the presence of a nonzero field h . The resulting behavior of the gap as a function of the longitudinal field is shown in Fig. 12(a).

To find the value of the spin-glass order parameter, we need to find $\bar{\lambda}$ according to

$$q = \frac{h^2}{2J^2} \left(\frac{\bar{\lambda}}{\lambda^*} - 1 \right), \quad (C14)$$

$$Q_r(\omega_n) = \frac{2g}{\omega_n^2 + \bar{\lambda} + \sqrt{\omega_n^4 + 2\omega_n^2 \bar{\lambda} + \lambda^{*2}}}, \quad (C15)$$

$$1 = q + \frac{1}{\beta} \sum_{\omega_n} Q_r(\omega_n). \quad (C16)$$

The numerical solution for q as a function of g is shown in Fig. 12(b). For $h = 0$, one can also compute the critical point, g_c , analytically, to find

$$g_c = \frac{9\pi^2 J}{16} \approx 5.55J, \quad (C17)$$

which reproduces the result obtained in Ref. [57].

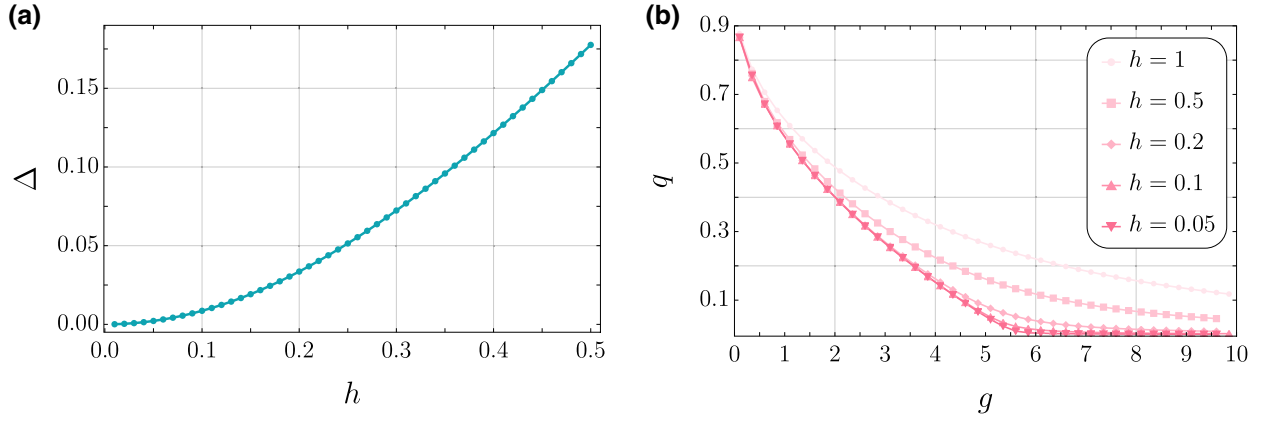


FIG. 12. (a) The gap in the spectrum Eq. (C13) as a function of the longitudinal field strength h for the quantum spherical $p = 2$ rotor model, taking $g = 1$. The system is gapless in the limit $h \rightarrow 0$. (b) The order parameter q as a function of the transverse magnetic field for several values of h , illustrating the absence of a sharp phase transition at any nonzero g and h .

APPENDIX D: REPLICA-SYMMETRIC SOLUTION FOR THE $h = 0$ QUANTUM SPHERICAL $p = 3$ MODEL AT LARGE g

In the limit of large longitudinal fields g , one can analytically solve Eqs. (3.20)–(3.24) at $h = 0$. We first note that in this limit, the order parameter q is zero, i.e., the system is in a paramagnetic phase, as shown in Fig. 7. To obtain the solution in this phase, we rewrite the saddle-point equations in the following form:

$$1 = Q_r(\tau = 0), \quad (\text{D1})$$

$$\Sigma_r(\tau) = \frac{3}{2} g J^2 Q_r^2(\tau), \quad (\text{D2})$$

$$Q_r(\omega_n) = \frac{g}{\omega_n^2 + \bar{\lambda} - \Sigma_r(\omega_n)}. \quad (\text{D3})$$

We numerically solve these equations for several values of g and infer that $\bar{\lambda}$ grows faster with g than Σ_r , which can be seen in Fig. 13(a). In addition, $\Sigma_r(\omega_n)$ is close to a constant in this limit, as evidenced by Fig. 13(b). We can therefore assume that, to leading order,

$$Q_r(\omega_n) = \frac{g}{\omega_n^2 + \bar{\lambda}}. \quad (\text{D4})$$

The resulting analytic solution yields

$$\bar{\lambda} = \frac{g^2}{4}, \quad Q_r(\omega_n) = \frac{g}{\omega_n^2 + \bar{\lambda}}, \quad \Sigma_r(\tau) = \frac{3g}{2} e^{-g|\tau|},$$

$$\Sigma_r(\omega_n) = \frac{3g^2}{g^2 + \omega_n^2} \approx \frac{3}{g^2} (g^2 - \omega_n^2). \quad (\text{D5})$$

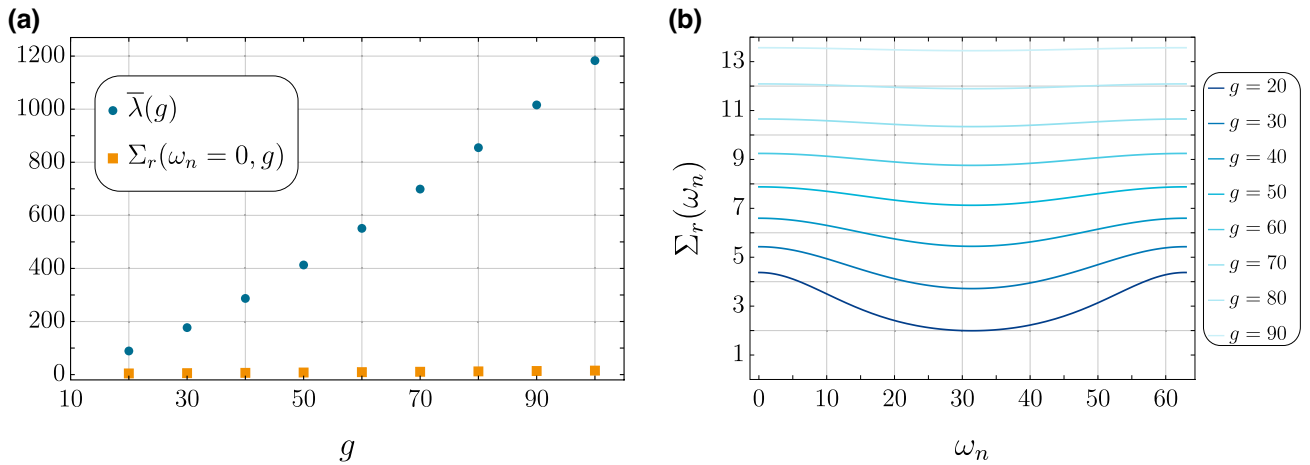


FIG. 13. (a) The behavior of $\bar{\lambda}$ and $\Sigma_r(\omega_n = 0)$ as a function of g for the quantum spherical $p = 3$ rotor model; $\bar{\lambda}$ grows significantly faster with g than Σ_r . In addition, $\Sigma_r(\omega_n = 0) \ll \bar{\lambda}$, which justifies the leading-order approximation made in Eq. (D4). (b) The self-consistent solution of the paramagnetic Eqs. (D1)–(D3) for the self-energy, Σ_r , for large values of g . The self-energy becomes a constant in the limit $g \rightarrow \infty$.

To improve upon our earlier approximation, we compute the Green's function to second order in $1/g$. At this order, we obtain the following result:

$$\bar{\lambda} = \frac{g^2}{4} + \frac{9}{4}, \quad (\text{D6})$$

$$Q_r(\omega_n) = \frac{g}{\omega_n^2(1 + 3/g^2) + (\bar{\lambda} - 3)}, \quad (\text{D7})$$

$$\Sigma_r(\tau) = \frac{3g^5}{2(g^4 - 9)} e^{-g|\tau| \sqrt{\frac{g^2-3}{g^2+3}}}, \quad (\text{D8})$$

$$\Sigma_r(\omega_n) = \frac{3g^6}{\sqrt{g^4 - 9}} \frac{1}{g^2(g^2 - 3) + \omega_n^2(g^2 + 3)}. \quad (\text{D9})$$

We therefore conclude that at large g , the spectral function has the form of a delta function,

$$\rho(\omega) = \frac{g^3}{g^2 + 3} \delta\left(\omega^2 - \frac{g^2 g^2 - 3}{g^2 + 3}\right), \quad (\text{D10})$$

with a finite gap. Thus, within this approximate framework, we expect the replica-symmetric solution to persist up to $g \approx \sqrt{3}$.

APPENDIX E: REPLICA IDENTITIES

We consider $n \times n$ replica matrices A_{ab} the off-diagonal elements of which are parametrized by the Parisi function $a(u)$, $u \in [0, 1]$, and the diagonal element $A_{aa} = \tilde{a}$ (and similarly for B_{ab} and C_{ab}). We find it useful to define

$$\langle a \rangle \equiv \int_0^1 du a(u), \quad (\text{E1})$$

and

$$[a](u) \equiv - \int_0^u dv a(v) + ua(u). \quad (\text{E2})$$

For the replica-symmetric case, we take

$$a(u) = a, \quad \tilde{a} = a + \sigma_a. \quad (\text{E3})$$

For the case of one-step replica symmetry breaking, we take

$$\begin{aligned} \tilde{a} &= a_1 + \sigma_a, \\ a(u) &= \begin{cases} a_1, & x < u < 1, \\ a_0, & 0 < u < x. \end{cases} \end{aligned} \quad (\text{E4})$$

Then, we have $\langle a \rangle = a_1 - x(a_1 - a_0)$ and

$$[a](u) = \begin{cases} x(a_1 - a_0), & x < u < 1, \\ 0, & 0 < u < x. \end{cases} \quad (\text{E5})$$

We now present some useful identities for matrix operations in the limit $n \rightarrow 0$.

1. Term-by-term sum

$$\frac{1}{n} \sum_{a,b=1}^n A_{ab} = \tilde{a} - \langle a \rangle. \quad (\text{E6})$$

2. Matrix product

For

$$C_{ab} = \sum_{c=1}^n A_{ac} B_{cb}, \quad (\text{E7})$$

we have [78]

$$\begin{aligned} \tilde{c} &= \tilde{a}\tilde{b} - \langle ab \rangle, \\ c(u) &= (\tilde{b} - \langle b \rangle)a(u) + (\tilde{a} - \langle a \rangle)b(u) \\ &\quad - \int_0^u dv (a(u) - a(v))(b(u) - b(v)). \end{aligned} \quad (\text{E8})$$

3. Matrix inverse

To determine $B = A^{-1}$, we set $C = 1$ in the matrix product to obtain the relations

$$\begin{aligned} 1 &= \tilde{a}\tilde{b} - \langle ab \rangle, \\ 0 &= (\tilde{b} - \langle b \rangle)a(u) + (\tilde{a} - \langle a \rangle)b(u) \\ &\quad - \int_0^u dv (a(u) - a(v))(b(u) - b(v)). \end{aligned} \quad (\text{E9})$$

With replica-symmetric matrices, we have

$$\sigma_b = \frac{1}{\sigma_a}, \quad b = -\frac{a}{\sigma_a^2}. \quad (\text{E10})$$

For the case of one-step replica symmetry breaking, we have

$$\begin{aligned} 1 &= \tilde{a}\tilde{b} - a_0 b_0 x - a_1 b_1 (1 - x), \\ 0 &= (\tilde{b} - b_0 x - b_1 (1 - x))a_0 + (\tilde{a} - a_0 x - a_1 (1 - x))b_0, \\ 0 &= (\tilde{b} - b_0 x - b_1 (1 - x))a_1 + (\tilde{a} - a_0 x - a_1 (1 - x))b_1 \\ &\quad - x(a_1 - a_0)(b_1 - b_0). \end{aligned} \quad (\text{E11})$$

Solving these equations, we obtain

$$\begin{aligned} \sigma_b &= \frac{1}{\sigma_a}, \\ b_1 &= -\frac{a_1 \sigma_a + x(a_1 - a_0)^2}{\sigma_a (\sigma_a + x(a_1 - a_0))^2}, \\ b_0 &= -\frac{a_0}{(\sigma_a + x(a_1 - a_0))^2}. \end{aligned} \quad (\text{E12})$$

4. Term-by-term inverse sum

If $A^{-1} = B$, then, from Eq. (E6),

$$\frac{1}{n} \sum_{ab} A_{ab}^{-1} = \tilde{b} - \langle b \rangle. \quad (\text{E13})$$

From Eq. (E9), we have

$$\begin{aligned} 1 &= \tilde{a}\tilde{b} - \langle ab \rangle, \\ 0 &= (\tilde{b} - \langle b \rangle)\langle a \rangle + (\tilde{a} - \langle a \rangle)\langle b \rangle - \langle ab \rangle + \langle a \rangle\langle b \rangle. \end{aligned} \quad (\text{E14})$$

Taking the difference, we obtain

$$\frac{1}{n} \sum_{ab} A_{ab}^{-1} = \frac{1}{\tilde{a} - \langle a \rangle}. \quad (\text{E15})$$

5. Term-by-term product

For the product

$$D_{ab} = \sum_{c,d=1}^n A_{ac} B_{db}, \quad (\text{E16})$$

we use the fact that $\sum_c A_{ac}$ is independent of the index a for replica matrices. This yields a replica-symmetric matrix D_{ab} , which is of independent a and b and using Eq. (E6), we have

$$\tilde{d} = d(u) = (\tilde{a} - \langle a \rangle)(\tilde{b} - \langle b \rangle). \quad (\text{E17})$$

For the replica-symmetric case, $\tilde{d} = d = \sigma_a \sigma_b$, and $\sigma_d = 0$. With one-step replica symmetry breaking, $\tilde{d} = d_0 = d_1 = (\sigma_a + x(a_1 - a_0))(\sigma_b + x(b_1 - b_0))$ and $\sigma_d = 0$.

6. Trace log

The trace log is needed in the computation of the free energy. We have [78]

$$\begin{aligned} \frac{1}{n} \text{Tr} \ln A &= \ln(\tilde{a} - \langle a \rangle) + \frac{a(0)}{\tilde{a} - \langle a \rangle} \\ &- \int_0^1 \frac{du}{u^2} \ln \frac{\tilde{a} - \langle a \rangle - [a](u)}{\tilde{a} - \langle a \rangle}. \end{aligned} \quad (\text{E18})$$

In the replica-symmetric case, we have

$$\frac{1}{n} \text{Tr} \ln A = \ln(\sigma_a) + \frac{a}{\sigma_a}. \quad (\text{E19})$$

With one-step replica symmetry breaking, we have

$$\begin{aligned} \frac{1}{n} \text{Tr} \ln A &= \ln(\sigma_a) + \frac{a_0}{\sigma_a + x(a_1 - a_0)} \\ &- \frac{1}{x} \ln \frac{\sigma_a}{\sigma_a + x(a_1 - a_0)}. \end{aligned} \quad (\text{E20})$$

Let us now consider $\text{Tr} \ln(A + B)$, where B is a constant matrix, i.e., $\tilde{b} = b(u) = b$ and $\sigma_b = 0$. Then, we have

$$\frac{1}{n} \text{Tr} \ln(A + B) = \frac{1}{n} \text{Tr} \ln A + \frac{b}{\tilde{a} - \langle a \rangle}. \quad (\text{E21})$$

7. Cubic term

According to Eq. (3.72) in Ref. [54], for the matrix q_{ab} with $\tilde{q} = 0$,

$$\frac{1}{n} \text{Tr} q^3 = \int_0^1 du \left[u[q(u)]^3 + 3q(u) \int_0^u dv [q(v)]^2 \right]. \quad (\text{E22})$$

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