Quantitative Nonclassicality of Mediated Interactions

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In a plethora of physical situations, one can distinguish a mediator—a system that couples other, non-interacting, systems. Often, the mediator itself is not directly accessible to experimentation, yet it is interesting and sometimes crucial to understand if it admits nonclassical properties. An example of this sort that has recently been enjoying considerable attention is that of two quantum masses coupled via a gravitational field. It has been argued that the gain of quantum entanglement between the masses indicates nonclassicality of the states of the whole tripartite system. Here, we focus on the nonclassical properties of the involved interactions rather than the states. We derive inequalities the violation of which indicates noncommutativity and nondecomposability (open-system generalization of noncommuting unitaries) of interactions through the mediators. The derivations are based on properties of general quantum formalism and make minimalistic assumptions about the studied systems; in particular, the interactions can remain uncharacterized throughout the assessment. Furthermore, we also present conditions that solely use correlations between the coupled systems, excluding the need to measure the mediator. Next, we show that the amount of violation places a lower bound on suitably defined degree of nondecomposability. This makes the methods quantitative and at the same time experiment ready. We give applications of these techniques in two different fields: for detecting the nonclassicality of gravitational interaction and in bounding the Trotter error in quantum simulations.

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I. INTRODUCTION

Mediated interactions are very common and often the mediators are practically inaccessible to direct experimentation. For example, consider a system of unpaired spins interacting via spin chains in solids [1]. The bulk measurements of magnetic properties are argued to be solely determined by the unpaired spins at the end of the chain, making the chain experimentally inaccessible. As another example, consider light modes interacting via mechanical membranes [2]. In this case, usually it is only the light that is being monitored. Furthermore, fundamentally electric charges are coupled via an electromagnetic field, etc. All these scenarios share a common structure in which systems A and B do not interact directly but are solely coupled via a mediator system, M (see Fig. 1). Already at this general level, one can ask about the properties of the mediator that can be deduced from the dynamics of the coupled systems.

In this line of study, methods have been proposed to witness the nonclassicality of the state of the mediator from the correlation dynamics of the coupled probes. In particular, conditions have been derived under which the gain of quantum entanglement implies that the mediator must have explored nonorthogonal states during the dynamics [3,4]. Similar ideas, applied to more general models than the canonical quantum formalism, have been used to argue that the entanglement gain between quantum masses witnesses nonclassical gravity [5,6] and have motivated a number of concrete proposals aimed at experimental demonstration of

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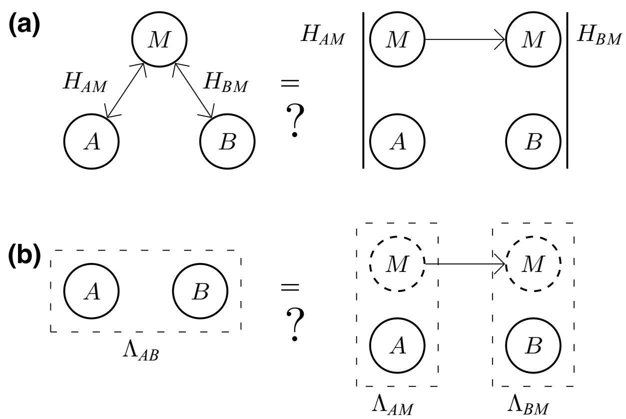

FIG. 1. Mediated interactions. (a) Systems A and B are coupled via mediator M, i.e., the underlying Hamiltonian is $H_{AM} + H_{BM}$ and it explicitly excludes direct coupling between the systems, i.e., H_{AB}. We present methods based on correlations showing that the interaction Hamiltonians do not commute, i.e., the tripartite dynamics cannot be understood as a sequence of interactions via H_{AM} and then H_{BM} or in reverse order. We also quantify this noncommutativity by providing a lower bound on a suitable norm of the commutator $[H_{AM}, H_{BM}]$. These notions are generalized to open systems and we emphasize that the tools make minimalistic assumptions about the whole setup. (b) We extend these techniques to cases in which the mediator is nonaccessible. They are based on correlations in system AB only and show that the tripartite dynamics cannot be understood as a sequence of interactions described by dynamical maps Λ_{AM} and Λ_{BM} or in reverse order. We also quantify this form of nondecomposability.

gravity-induced entanglement (see, e.g., Refs. [7–17]). A considerable advantage of these methods is due to the minimalistic assumptions that they make about the physical systems involved. They are independent of the initial state, the dimensions of the involved systems, or the explicit form of interactions and they also work in the presence of local environments. Accordingly, they are applicable in a variety of fields (see, e.g., Refs. [18,19] for examples in quantum biology and solid-state physics, respectively).

Here, we move on from the nonclassicality of states and develop tools to quantify the amount of nonclassicality of mediated *interactions*, while retaining minimalistic assumptions about the considered physical systems. The notion of nonclassicality that we employ is given by the commutativity of interaction Hamiltonians in the case of closed dynamics, which generalizes to the decomposability of dynamical maps, which also encompasses open systems. Arguments supporting this choice are given in Sec. II. A method to detect the presence of such nonclassicality was first presented in Ref. [20] but it was only qualitative, i.e., it could only witness the presence of noncommutativity. It is intriguing that the methods mentioned earlier, aimed at the nonclassicality of states, are also at this qualitative level at the present moment. Our main contribution here is the development of methods to quantify the amount of nonclassicality. We derive conditions that lower bound the norm of the commutator as well as a suitably defined distance to decomposable maps. These conditions are of two types and the structure of the paper reflects this division. In Sec. III, we assume that the mediator is accessible to experimentation and in Sec. IV, the derived conditions use only data measured on the probes. Nontrivial bounds are derived for any continuous correlation measure. Hence, it is again expected that the methods presented are applicable in a variety of fields. We provide two examples.

The first one is in the field of quantum simulations. Suzuki-Trotter expansion is a common way to simulate arbitrary sums of local Hamiltonians (see, e.g., Refs. [21,22]). It has recently been shown that the number of Trotter steps needed to obtain the required simulation error scales with the spectral norm of the commutator [23]. We link this norm to the correlations in the system, showing a quantitative relation between the complexity of the simulation and the amount of correlations.

As the second example, the methods detect and measure the noncommutativity of the gravitational interaction coupling two quantum masses. The idea of detecting the nonclassicality of gravitational interaction has been discussed very recently in Ref. [24] but there the notion of nonclassicality is different, based on the impossibility of simulating the dynamics via local operations and classical communication. Within the quantum formalism, local operations are modeled by arbitrary local channels and classical communication by sequences of dephasing channels connecting the communicating parties. In the tripartite setting of two masses and gravitational field, this means the following sequence: λ_{AM}, DEPHASING(M), λ_{BM}, DEPHASING(M), etc. In principle, different dephasing maps could even be performed in different bases. In contradistinction, the definition that we adopt in the present work deals with dynamics that are continuous in time and defines classicality at the level of Hamiltonians, as their commutativity. This implies an effective picture in which a *quantum* mediator is transmitted between "communicating" parties, but only one way. So, in the tripartite setting, this means $U_{AM} U_{BM}$ or in reverse order. For other ways of revealing that the evolution cannot be understood in terms of classical (gravitational) field, see also Refs. [25,26], and for general arguments that any system capable of coupling to a quantum system must itself be quantized, see, e.g., Ref. [27]. Our tools show that correlations between the masses exclude gravity as a form of interaction with commuting particle-field couplings.

II. CLASSICALITY AND DECOMPOSABILITY

Let us start with closed systems and explain our choice of the notion of classicality and its relation to the properties of dynamical maps. In this work, classical mediated interactions are defined by commuting Hamiltonians H_{AM} and H_{BM} (see Fig. 1). A high-level motivation for this choice comes from the fact that in classical mechanics, all observables commute; hence a classical mediator would have all its couplings to other systems commuting. The commutativity can also be motivated starting with the notion of classical states as those admitting vanishing quantum discord [28], or vanishing coherence in the case of a single system [29], and asking for the evolution that preserves this form of classicality. The vanishing discord means that the whole tripartite state can be measured on the mediator without disturbing the total state. Mathematically, the state has a block-diagonal form and we assume that at all times there exists a single "preferred" basis of the mediator. We show in Appendix A that such dynamics are generated if and only if the Hamiltonian has a block-diagonal form too, with the same basis on the mediator. Since, here, we consider systems with global Hamiltonian $H = H_{AM} + H_{BM}$, the state classicality is preserved when both H_{AM} and H_{BM} are block diagonal with the same basis on system M, i.e., both Hamiltonians commute $[H_{AM}, H_{BM}] = 0$. Furthermore, for commuting nondegenerate H_{AM} and H_{BM}, the total Hamiltonian admits only product eigenstates and out-of-time-ordered correlators vanish at all times, as shown in Appendix A.

A closely related notion is that of decomposability. A tripartite unitary U is decomposable if there exist unitaries U_{AM} and U_{BM} such that

$$U = U_{BM} U_{AM}. \tag{1}$$

Intuitively, decomposable unitaries are those that can be simulated by first coupling one of the systems to the mediator M and then coupling the other. One can picture that the mediator particle is being transmitted between A and B, which are in separate laboratories, making this setting similar to that in Refs. [30–36]. Although the Suzuki-Trotter formula shows that any unitary can be approximated by a sequence of Trotter steps, decomposable unitaries are special because we can implement the exact unitary with only a single Trotter step. For its relation to the notion of locality in quantum field theory, see Ref. [37].

Clearly, for classical interactions $[H_{AM}, H_{BM}] = 0$, the unitary operator $U(t) = e^{-itH}$ is decomposable for all t. But there exist unitaries that are decomposable and yet are not generated by a classical interaction. A concrete example is given in Appendix A 4 and relies on the fact that the unitary can be written as $U = U_{BM} U_{AM}$, but there exist no unitaries V_{AM} and V_{BM} such that the sequence $V_{AM} V_{BM}$ would be equal to U. This example already suggests that decomposability has to be augmented with commutativity of decompositions to be equivalent to the classicality of interactions, a fact that we prove in Appendix A 5. Therefore, the unitary generated by classical interactions is continuously decomposable, with the added property that the decomposition must commute, i.e., $[U_{AM}(t), U_{BM}(t)] = 0$ for all t. Accordingly, it is irrelevant whether we define the decomposition order as $U_{BM} U_{AM}$ or $U_{AM} U_{BM}$.

Decomposability naturally extends to open systems. In this case, the evolution is described by a map λ giving the state of the system at time t, i.e., $\rho = \lambda(\rho_0)$. We say that a tripartite map λ is decomposable if there exist maps λ_{AM} and λ_{BM} such that

$$\lambda(\rho) = \lambda_{BM} \lambda_{AM}(\rho), \tag{2}$$

for every ρ. In Appendix A 6, we discuss the consistency of this definition and the one based on unitaries. As expected, a unitary operator is decomposable if and only if the corresponding unitary map is decomposable (general maps are not required).

It is this general notion of decomposability that we will exclude and measure the degree of its exclusion in the coming sections. A number of similar concepts have been introduced before and it is instructive to compare the decomposability with them and note where the novelty is. So-called divisibility asks whether map Λ can be written as $\Lambda_1 \Lambda_2$, where neither Λ_1 nor Λ_2 are unitaries [38]. A stronger notion of completely positive (cp) divisibility, studied in the context of Markovian dynamics [39,40], asks whether map Λ_t can be written as the sequence of completely positive maps $\Lambda_t = V_{t,s} \Lambda_s$. Interestingly, the set of cp-divisible maps is not convex [38]. The decomposability that we study here has a specific multipartite structure that has been considered only in Refs. [20,41], which is clearly significant from a physics perspective.

III. ACCESSIBLE MEDIATOR

We first present methods that utilize correlations measured on all three subsystems and devote Sec. IV to eliminating measurements on the mediator. The basic idea is that correlations between subsystem A and subsystems MB together should be bounded in the case of decomposable dynamics because they are effectively established via a process in which the mediator is being transmitted from A to B only *once*. It is therefore expected that the correlations are bounded by the "correlation capacity" of the mediator, i.e., maximal correlations to the mediator alone. Such inequalities for distance-based correlation measures have been derived in Ref. [20] and could also be obtained by manipulating the results of Refs. [3,41]. Our contribution in this section is a generalization to any continuous correlation measure and then quantification of nondecomposability based on the amount of violation of the derived criterion.

A. Detecting nondecomposability

Let us take a correlation quantifier Q that is monotonic under local operations. In Appendix B, we show that the bound in terms of the correlation capacity holds when we additionally assume that the initial state is of the form $\rho_0 = \rho_{AM} \otimes \rho_B$. For such an initial state, the correlations generated by a decomposable map λ admit

$$Q_{A:MB}(\lambda(\rho_0)) \leq \sup_{\sigma_{AM}} Q_{A:M}(\sigma_{AM}), \tag{3}$$

where the bound is derived for any correlation measure Q that is monotonic under local processing. Here, σ_{AM} ranges over all possible joint states of AM. Exchanging the roles of A and B gives rise to another inequality, and the experimenter should choose the one that is violated to detect nondecomposability. This bound is already nontrivial, as we now demonstrate by showing that the maximally entangling map cannot be decomposable. Consider the initial product state $|000\rangle$ and assume that systems A and B are of higher dimension than the mediator, i.e., $d_A = d_B > d_M$. As an exemplary entanglement measure, take the relative entropy of entanglement, E. It is known that its maximum depends on the dimension of the smaller Hilbert space, i.e., $\sup_{\sigma_{AM}} E_{A:M}(\sigma_{AM}) = \log d_M$. According to Eq. (3), no decomposable evolution can produce more entanglement than $\log d_M$. This holds for entanglement $E_{A:MB}$ as well as for $E_{A:B}$ due to the monotonicity of relative entropy under partial trace. Since the dimensions of A and B are larger than the dimension of the mediator, a maximally entangled state between AB cannot be produced by any decomposable map.

Of course, we are interested in extending Eq. (3) to an arbitrary initial state, making the method independent of it. To achieve this aim, we use continuity arguments. Many correlation measures, including relative-entropy-based quantifiers [42], all distance-based measures [43], or convex-roof extensions of asymptotically continuous functions [44], admit a version of continuity in which there exists an invertible monotonically nondecreasing function, g, such that $|Q(x) - Q(y)| \leq g(d(x, y))$, where d is a contractive distance and $\lim_{s\to 0} g(s) = 0$. This is a refinement of the notion of uniform continuity, where we can bound how much the function varies when we perturb the input. A notable example is logarithmic negativity [45], which is not asymptotically continuous and yet fulfills this notion of continuity. For simplicity, we shall call such functions *gd* continuous. We prove in Appendix B that correlation quantifiers that are *gd* continuous are bounded in decomposable dynamics as follows:

$$Q_{A:MB}(\lambda(\rho_0)) \leq \sup_{\sigma_{AM}} Q_{A:M}(\sigma_{AM}) + I_{AM:B}(\rho_0), \tag{4}$$

where $I_{AM:B}(\rho) = \inf_{\sigma_{AM}\otimes\sigma_B} g(d(\rho, \sigma_{AM} \otimes \sigma_B))$ is a measure of the total correlations in the state ρ across the partition $AM : B$. Indeed, from the properties of g and d, it is easy to verify that this quantity is monotonic under local operations and that it is zero if and only if ρ is a product state across the $AM : B$ partition. Again, an independent inequality is obtained by exchanging A and B.

This bound is also nontrivial and its violation has been demonstrated in Ref. [46], which focused on negativity as a concrete correlation (entanglement) measure. The system under consideration involved two cavity modes, A and B, coupled via two-level atom M. This scenario is particularly well suited to demonstrate the violation, because the dimension of the mediator is as small as it can be, whereas the dimensions of the probes are in principle unbounded.

B. Measuring nondecomposability

Having established witnesses of nondecomposability, we now argue that the amount of violation of Eq. (4) quantifies the nondecomposability. As a measure of nondecomposability, we propose a minimal operator distance from an arbitrary map Λ to the set of decomposable maps, which we denote as DEC:

$$d^{\text{DEC}}(\Lambda) = \inf_{\lambda\in\text{DEC}} D(\Lambda, \lambda). \tag{5}$$

We shall refer to this quantity as the "degree of nondecomposability." The operator distance D in its definition could be chosen as the one induced by the distance on states

$$D(\Lambda_1, \Lambda_2) = \sup_{\sigma} d(\Lambda_1(\sigma), \Lambda_2(\sigma)), \tag{6}$$

where Λ_1 and Λ_2 are arbitrary maps and σ is any state from the domain of the map. In Appendix B, we demonstrate that violation of Eq. (4) lower bounds the degree of nondecomposability as follows:

$$d^{\text{DEC}}(\Lambda) \geq g^{-1}(Q_{A:MB}(\Lambda(\rho_0)) - B(\rho_0)), \tag{7}$$

where $B(\rho_0)$ is the right-hand side of Eq. (4). Accordingly, any violation of the decomposability criterion in terms of correlations sets a nontrivial lower bound on the distance between the dynamical map and the set of decomposable maps.

C. Quantum simulations

As the first application of the introduced measure, suppose that we would like to simulate the dynamics generated by the Hamiltonian $H = H_{AM} + H_{BM}$. (In fact, this analysis can be generalized to any 2-local Hamiltonian.) Quantum simulators implement a dynamic close to the desired one by truncating the Suzuki-Trotter formula to r Trotter steps

$$e^{-itH} \approx \left(e^{-i\frac{t}{r}H_{AM}}e^{-i\frac{t}{r}H_{BM}}\right)^r. \tag{8}$$

The error of this approximation can be quantified by the spectral norm (the largest singular value),

$$\left\| e^{-itH} - \left(e^{-i\frac{t}{r}H_{AM}}e^{-i\frac{t}{r}H_{BM}}\right)^r \right\|_\infty, \tag{9}$$

and it has been shown in Ref. [23] that in order to make this error smaller than ϵ, the number of Trotter steps has to scale with the norm of the commutator,

$$r = O\left(\frac{t^2}{\epsilon}||\,[H_{AM}, H_{BM}]\,||_\infty\right). \tag{10}$$

Our aim is to provide a lower bound on the commutator norm in terms of correlations and in this way bound the number of required Trotter steps. Recall, after Ref. [23], that for a single Trotter step, we have

$$||U - U_{AM}U_{BM}||_\infty \leq \frac{t^2}{2}||\,[H_{AM}, H_{BM}]\,||_\infty,$$

where $U = e^{-itH}$ and, e.g., $U_{AM} = e^{-itH_{AM}}$. We need to link our methods to the spectral norm. For finite-dimensional systems, all metrics generate the same topology [47], i.e., for any two distances d_1 and d_2, there exists a constant C such that

$$\frac{1}{C}d_2(\rho,\sigma) \leq d_1(\rho,\sigma) \leq C\,d_2(\rho,\sigma). \tag{11}$$

In particular, there exists a constant that relates any distance to the trace distance $d_{\rm tr}(\rho,\sigma) = \frac{1}{2}||\rho - \sigma||_1$. Therefore, if a correlation quantifier on finite-dimensional systems is *gd* continuous with respect to the trace distance, it is also *gd* continuous with respect to any other distance d. Furthermore, since the trace distance is contractive, Eq. (4) holds for any distance on finite-dimensional systems, at the cost of constants in function g. Accordingly, let us consider the distance induced by the spectral norm $d_\infty(\rho,\sigma) = ||\rho - \sigma||_\infty$. We call the corresponding operator distance $D_\infty(\Lambda_1, \Lambda_2)$ and the degree of nondecomposability $d_\infty^{\rm DEC}(\Lambda)$. For the connection to the Trotter error, we note the following:

$$d_\infty^{\rm DEC}(U) \leq D_\infty(U, U_{AM}U_{BM}) \leq 2||U - U_{AM}U_{BM}||_\infty, \tag{12}$$

where the first inequality follows from the fact that $d_\infty^{\rm DEC}(U)$ is the shortest distance to the set of decomposable maps and $U_{AM}U_{BM}$ is a particular decomposable map. The second inequality is proven in Appendix B 1. Combining the two inequalities, we obtain $d_\infty^{\rm DEC}(U) \leq t^2\,\|[H_{AM}, H_{BM}]\|_\infty$. A concrete example relating the mutual information in a state to the number of Trotter steps is provided in Appendix B 2.

We have therefore shown a direct link between correlations in the system and the number of Trotter steps that one needs to keep the simulation error small. The amount of violation of Eq. (4) lower bounds the degree of nondecomposability and hence the spectral norm of the commutator and, accordingly, sets the number of required Trotter steps. Conversely, if it is possible to simulate U with r Trotter steps to precision ϵ, Eq. (10) shows that the commutator norm is bounded and consequently Eq. (12) implies that the correlations $Q_{A:MB}$ admit an upper bound.

IV. INACCESSIBLE MEDIATOR

An interesting opportunity arises where the nonclassicality of evolution through a mediator could be witnessed without measuring the mediator. Here, we show that this is indeed possible. We start by introducing the necessary concepts and the related mathematical tools and then we present witnesses of nondecomposable evolution based on measurements on AB only. Finally, we establish measures of nondecomposability together with their experimentally friendly lower bounds.

A. Marginal maps

In order to detect nonclassicality of interactions solely through the correlations between the coupled objects, we need the notion of “marginals” of decomposable maps. We propose to introduce it via a related concept of dilation. A dilation of a map $\Lambda : X \to X$ is an ancillary state σ_R and a map $\tilde{\Lambda} : XR \to XR$ acting on the system and ancilla, such that

$$\Lambda(\rho) = {\rm Tr}_R(\tilde{\Lambda}(\rho \otimes \sigma_R)), \tag{13}$$

for all ρ. Accordingly, our aim is to exclude the existence of a decomposable dilation of the dynamics that are observed on systems AB. In principle, the existence of dilations may depend on the dimension of the Hilbert space of the mediator, which motivates us to introduce *decomposable m-dilation* as follows. A map $\Lambda : AB \to AB$ has a decomposable m-dilation if there exists a dilation $\tilde{\Lambda} : ABM \to ABM$ such that $\tilde{\Lambda}$ is decomposable and the dimension of the mediator satisfies $d_M \leq m$. We denote the set of all maps with a decomposable m-dilation as $\overline{\rm DEC}(m)$.

With these definitions, we can state our goal precisely: we wish to infer whether a map on AB admits any decomposable m-dilation and we wish to do this via measurements of correlations only. If no decomposable dilation exists, we conclude that the interaction generating the map is nonclassical.

B. Detecting nondecomposability

It turns out that one can obtain an interesting condition that witnesses nondecomposability as a simple corollary to Eq. (4). In Appendix C, we prove that any gd-continuous correlation measure Q admits the following bound under the evolution generated by $\lambda \in \overline{\text{DEC}}(m)$:

$$Q_{A:B}(\rho_t) \leq \sup_{\sigma_{XM}} Q_{X:M}(\sigma_{XM}) + I_{A:B}(\rho_0), \qquad (14)$$

where $\rho_t = \lambda(\rho_0)$ and we emphasize that $\lambda \in \overline{\text{DEC}}(m)$ acts on AB only. The supremum on the right-hand side runs over all AM or BM states with $d_M \leq m$ and $I_{A:B}(\rho_{AB}) = \inf_{\sigma_A \otimes \sigma_B} g(d(\rho_{AB}, \sigma_A \otimes \sigma_B))$ measures the total correlations across $A : B$. Note that if the correlation measure that we use is not gd continuous, we can still obtain a witness of nondecomposability assuming that we start with a product state. For example, this could be ensured without having access to M by preparing the AB systems in a pure product state.

As an example of using this criterion, note that the maximally entangling maps we have discussed before cannot have any decomposable m-dilation for $m < \min(d_A, d_B)$. A question emerges as to whether there exist evolutions that do not admit decomposable m-dilation even when the dimension of the mediator is unbounded. This is indeed the case. We show in Appendix C 1 that a SWAP operation on two objects (even two qubits) has no decomposable m-dilation for any m. This leads to the conclusion that classical interactions cannot produce a SWAP. The intuitive reason behind this statement is that it takes at least *two* steps to implement swapping with $d_A = d_B = d_M$. We first exchange A and M, then we exchange B and M, and we still must exchange A and M again to complete the implementation. In fact, any AB interaction can be implemented in two steps by first exchanging A and M, applying the interaction on BM, and finally swapping A and M back. The conclusion becomes less unexpected once we realize that SWAP is a highly entangling operation. For example, Alice and Bob can entangle their laboratories by starting with each having local Bell pairs $|\psi^-_{AA'}\rangle \otimes |\psi^-_{BB'}\rangle$ and swapping the AB subsystems.

We wish to give a little more insight into the structure of maps with decomposable dilations. Clearly, the sets are nested: $\overline{\text{DEC}}(m) \subseteq \overline{\text{DEC}}(m+1)$. In fact, the inclusions are strict, as we show in Appendix C 2.

C. Measuring nondecomposability

In the spirit of Sec. III B, we would like to extend Eq. (7) to bound the distance to $\overline{\text{DEC}}(m)$ based solely on correlations measured on systems AB. Of course, the ABM operator distance to DEC and the AB operator distance to $\overline{\text{DEC}}(m)$ are closely related. For contractive distances d on states, we have $D(\Lambda_{ABM}, \lambda_{ABM}) \geq D(\Lambda_{AB}, \lambda_{AB})$, which unfortunately is the opposite of what we need. To overcome this, we use the so-called completely bounded variant of the operator distance [48]:

$$\mathcal{D}(\Lambda_1, \Lambda_2) = \sup_{\sigma_{XY}} d((\Lambda_1 \otimes \mathbb{1}_Y)(\sigma), (\Lambda_2 \otimes \mathbb{1}_Y)(\sigma)), \quad (15)$$

where $\Lambda_1, \Lambda_2 : X \to X$ and Y is a finite-dimensional system. The benefit of the completely bounded operator distance is that it behaves nicely on dilations. This makes it easier to jump from the distance to DEC to the distance to $\overline{\text{DEC}}(m)$. Indeed, the completely bounded distance can be written in terms of the dilations as follows:

$$\mathcal{D}(\Lambda_1, \Lambda_2) = \inf_{\tilde{\Lambda}_i} \mathcal{D}(\tilde{\Lambda}_1, \tilde{\Lambda}_2). \qquad (16)$$

On the one hand, for contractive distances on states, the left-hand side cannot be larger than the right-hand side. On the other, the bound can be achieved by an exemplary dilation $\tilde{\Lambda}_i = \Lambda_i \otimes \mathbb{1}$.

As a measure of nondecomposability that we will link to the violation of Eq. (14), we propose the analogue of the degree of nondecomposability written in terms of the completely bounded distance:

$$\mathcal{D}^{\overline{\text{DEC}}(m)}(\Lambda_{AB}) = \inf_{\lambda_{AB} \in \overline{\text{DEC}}(m)} \mathcal{D}(\Lambda_{AB}, \lambda_{AB}). \qquad (17)$$

With these concepts and tools, it is proven in Appendix C that the amount of violation of Eq. (14) lower bounds the quantity just introduced:

$$\mathcal{D}^{\overline{\text{DEC}}(m)}(\Lambda_{AB}) \geq g^{-1}(Q_{A:B}(\rho_t) - \mathcal{B}(\rho_0)), \qquad (18)$$

where $\mathcal{B}$ is the right-hand side of Eq. (14). Note that all these quantities involve states and maps on AB only.

D. Nonclassical gravity

Our second application of these methods is in foundations. A prime example of an inaccessible mediator is a mediating field. The methods described above allow us to make conclusions about the field from the behavior of objects coupled through it. Gravitational interaction is especially interesting from this perspective, as there is no direct experimental evidence of its quantum properties today. As discussed in Sec. I, observation of quantum entanglement between gravitationally coupled masses is a plausible near-future experiment closing this gap [49]. In this section, we show that our methods allow a concise derivation of the nonclassicality witnesses presented in the literature [3,5,6] and lead to new conclusions about the interactions that can be drawn from the observation of considerable gravitational entanglement.

Assume first a completely classical situation in which both states and interactions are classical. Recall that within our framework, this means a zero-discord state at all times,

$D_{AB|M} = 0$ (with one and the same basis on the mediator at all times), and dynamical maps admitting decomposable dilations. As the correlation measure, consider quantum entanglement, measured by the relative entropy of entanglement. Then, the amount of entanglement $A : B$ that can be produced via these classical maps is

$$E_{A:B}(\rho_t) \leq \sup_{\sigma_{XM}} E_{X:M}(\sigma_{X:M}) + I_{A:B}(\rho_0), \quad (19)$$

where the supremum is over all the states of AM or BM allowed in the theory; here, $d_M \leq m$ and $D_{AB|M} = 0$. It is reasonable to assume that the initial state in the laboratory will be close to a product state and we therefore take $I_{A:B}(\rho_0) = 0$. Furthermore, all states admitting $D_{AB|M} = 0$ are disentangled across $A : M$ and $B : M$ and therefore the supremum is also zero. We therefore arrive at the conclusion that entanglement $A : B$ cannot grow and hence observation of any gain implies nonclassical states or nonclassical interactions or both.

If we assume that the interactions are classical (decomposable) but the state might have nonzero discord, then entanglement still satisfies the bound in Eq. (19). Therefore, the observation of a nonzero value of $E_{A:B}$ means that the supremum on the right-hand side is at least equal to this observed value, i.e., the mediator must be capable of being entangled to A or B, and in fact to AB due to monotonicity, to at least the degree that has been measured. Note that this is stronger than saying that the mediator needs to be discorded.

Finally, by violating the bound in Eq. (19), it is possible to demonstrate in the laboratory that unknown interactions are not decomposable. We stress that it is not sufficient to demonstrate that entanglement grows: we have to demonstrate that the entanglement is above a certain threshold. This threshold depends on the dimension of the mediator and we therefore ask how high entanglement can be generated by gravity. The answer depends on the concrete setup via which gravitational interaction is studied. If we take two nearby harmonically trapped masses initially prepared in squeezed states with squeezing parameters s_A and s_B, it has been shown that the gravitational entanglement in terms of logarithmic negativity can be as large as $E_{A:B}^{\max} = |s_A + s_B|/\ln 2$, which holds for large squeezing [8]. Since, in principle, $s_i \to \infty$, this already shows that gravity cannot be understood as a classical interaction with any finite-dimensional mediator. More practically, the highest optical squeezing achieved today is $s_{A,B} = 1.73$ [50] and assuming that it can be transferred to mechanical systems gives entanglement $E_{A:B}^{\max} \approx 5$ ebits (entangled bits), which would restrict still possible decomposable dilations to use mediators with dimension $m > 2^5$. It is rather unlikely that this amount of entanglement will be observed in the near future, as the time it takes the discussed system to reach $E_{A:B}^{\max}$ in the absence of dissipation is $t_{\max} = \pi\omega L^3/4Gm$, independently of high squeezing, where L is the separation between the masses and ω is the frequency of the trapping potential [8]. For Laser Interferometer Gravitational Wave Observatory (LIGO)–like parameters of masses in the order of $m \sim 1$ kg, $\omega \sim 0.1$ Hz and $L \sim 1$ cm, this time is already in the order of hours and dissipation pushes it further, to tens of hours. Yet, a violation of the unit bound, and hence disproval of classical interactions via a two-level system, which would already be interesting, could be achieved within a second [5,8,11,50].

Another route would be to use gravity to execute dynamics that by other means are known to be nondecomposable. For example, we have shown below Eq. (14) that maximally entangling maps do not admit decomposable dilations for $d_M \leq \min(d_A, d_B)$. The schemes in Refs. [5,6] indeed use gravity to implement maximal entanglement but only between two-level quantum systems encoded in the path degree of freedom. It would therefore be interesting to determine whether gravity could be used to maximally entangle masses in more paths. Along the same line, we have shown that SWAP does not admit any decomposable dilation, even with an infinite-dimensional mediator. Interestingly, Ref. [24] argues that gravity could implement the SWAP gate. In addition, the time it takes to implement the gate is twice as long as the time it takes to implement the maximally entangling unitary, showing that it is not much more demanding than the entanglement-based method. This provides an alternative witness of quantum properties of gravitational interaction that does not rely on the dimension of the mediator.

V. CONCLUSIONS

We have proposed notions of classicality of mediated interactions (commutativity of Hamiltonians and decomposability of dynamical maps) and introduced their mathematical measures. Our main results are inequalities in terms of any continuous correlation quantifiers with the property that their violations place lower bounds on the amount of introduced nonclassicality. These quantitative methods are therefore experiment ready and applicable in a variety of physical situations due to the minimalistic assumptions under which they are derived. As examples, we have shown that accurate simulations of dynamics with high correlations necessarily require a large number of Trotter steps and that gravitational interaction cannot be understood with the help of commuting particle-field couplings.

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APPENDIX A: CLASSICALITY AND DECOMPOSABILITY

1. Classical states

For completeness, let us start with elementary relations. A state is said to be classical (or incoherent) if it is diagonal in a preferred basis $\{|m\rangle\}$. A multipartite state is called quantum classical (qc)—or admits vanishing discord, $D_{AB|M} = 0$—if it can be written as $\rho_{qc} = \sum_m \rho_{AB|m} \otimes \Pi_m$, where $\Pi_m = |m\rangle\langle m|$ is the projector on the preferred basis and the systems are enumerated as in Fig. 1. In words, the whole tripartite state explores only one basis in the Hilbert space of the mediator. Let us introduce a measurement map along the preferred basis, Π, the action of which on an arbitrary input state is to produce an average postmeasurement state: $\Pi(\rho) = \sum_m \Pi_m \rho \Pi_m$. A state ρ is qc if and only if $\rho = \Pi(\rho)$. Alternatively, the definition of classicality can be phrased in terms of commutation with the basis elements.

Proposition 1.—Let $\Pi(X) = \sum_m \Pi_m X \Pi_m$ be a projection map, where $\Pi_m \Pi_{m'} = \delta_{mm'} \Pi_m$. Then,

$$X = \Pi(X) \iff \forall m,\ [X, \Pi_m] = 0. \tag{A1}$$

Proof.—The "if" direction is trivial. For the "only if" direction, consider the following argument:

$$X\Pi_m = \Pi_m X, \tag{A2}$$
$$X\Pi_m = \Pi_m X \Pi_m, \tag{A3}$$
$$X = \sum_m \Pi_m X \Pi_m = \Pi(X), \tag{A4}$$

where we have multiplied the first equation by Π_m from the right and used $\Pi_m^2 = \Pi_m$, and then we have summed the second equation over m and used the completeness relation $\sum_m \Pi_m = \mathbb{1}$. ■

2. Classical interactions

The definition of classicality of interactions in terms of commutativity is justified by the following proposition. It shows that the Hamiltonians preserving classicality of states are invariant under dephasing in the preferred basis. The commutativity is then a corollary.

Proposition 2.—Let H be a time-independent Hamiltonian. Then, $H = \Pi(H)$, if and only if for any classical initial state ρ_0, $\rho_t = e^{-itH}\rho_0 e^{itH}$ is also classical.

Proof.—For the "only if" direction, let us write the assumption explicitly:

$$e^{-itH}\rho_0 e^{itH} = \Pi\left(e^{-itH}\rho_0 e^{itH}\right), \tag{A5}$$
$$e^{-itH}[\rho_0, H]e^{itH} = \Pi\left(e^{-itH}[\rho_0, H]e^{itH}\right), \tag{A6}$$

where the second line is the time derivative of the first one and ρ_0 denotes the initial (classical) state. By evaluating at $t = 0$, we find that the commutator is invariant:

$$[\rho_0, H] = \Pi([\rho_0, H]). \tag{A7}$$

In particular, taking $\rho_0 = \Pi_m$ shows that for all the basis states:

$$[\Pi_m, H] = \Pi([\Pi_m, H]) = 0, \tag{A8}$$

where the last equation is simple to verify. Applying Proposition 1 proves the claim.

For the "if" direction, from the assumption, the Hamiltonian has the block form $H = \sum_m h_m \otimes \Pi_m$, where h_m acts on all the systems other than the mediator. In this case, the orthonormality of the preferred basis implies

$$e^{\pm itH} = \sum_m e^{\pm ith_m} \otimes \Pi_m. \tag{A9}$$

Accordingly, the initially classical mediator stays classical at all times and the remaining systems evolve conditionally depending on the state of the mediator. ■

In the case of the tripartite systems that we consider, where $H = H_{AM} + H_{BM}$, this shows that classicality is preserved when both H_{AM} and H_{BM} are block diagonal with the same basis on system M, i.e., they commute.

3. Simple eigenstates

As another argument to justify our definition of classicality, we show that it constrains the eigenstates of the Hamiltonian to be fully product, at least when the local terms are nondegenerate.

Proposition 3.—Let H_{AM}, H_{BM} be nondegenerate Hamiltonians. Then, $[H_{AM}, H_{BM}] = 0$ implies that $H = H_{AM} + H_{BM}$ can be diagonalized with fully product states.

Proof.—Let us assume that $[H_{AM}, H_{BM}] = 0$. Note that when a Hermitian matrix A has a nondegenerate spectrum, then all eigenvectors of $A \otimes \mathbb{1}$ must be of the form $|\psi_A\rangle \otimes |\psi_B\rangle$, where $|\psi_A\rangle$ is an eigenvector of A and $|\psi_B\rangle$ is an arbitrary vector. Since $[H_{AM}, H_{BM}] = 0$ implies that there is a common eigenbasis between H_{AM} and H_{BM}, this means that there exists a common eigenbasis for $H = H_{AM} + H_{BM}$ that is a product on $A : MB$ and $AM : B$ at the same time, which proves the claim. ∎

4. One-way decomposability

The following proposition gives an example of decomposable unitary that nevertheless cannot be generated by classical interactions.

Proposition 4.—There are no two-qubit unitaries V_{AM}, V_{BM} such that $U_{AM}U_{BM} = V_{BM}V_{AM}$, where

$$U_{AM} = \frac{1}{\sqrt{2}}(\mathbb{1} + iZ_A X_M), \tag{A10}$$

$$U_{BM} = \frac{1}{\sqrt{2}}(\mathbb{1} + iZ_B Z_M), \tag{A11}$$

and Z and X denote Pauli matrices.

Proof.—We prove by contradiction. Suppose that there exist unitaries V_{AM}, V_{BM} such that $U_{AM}U_{BM} = V_{BM}V_{AM}$. Note that we can write U_{AM}, U_{BM} as

$$U_{AM} = |0\rangle\langle 0|_A \otimes \frac{1}{\sqrt{2}}(\mathbb{1}_M + iX_M) + |1\rangle\langle 1|_A \otimes \frac{1}{\sqrt{2}}(\mathbb{1}_M - iX_M), \tag{A12}$$

$$U_{BM} = |0\rangle\langle 0|_B \otimes \frac{1}{\sqrt{2}}(\mathbb{1}_M + iZ_M) + |1\rangle\langle 1|_B \otimes \frac{1}{\sqrt{2}}(\mathbb{1}_M - iZ_M). \tag{A13}$$

Therefore, the product $U_{AM}U_{BM}$ is given by

$$|00\rangle\langle 00|_{AB} \otimes \frac{1}{2}(\mathbb{1} + iX_M + iY_M + iZ_M) + |01\rangle\langle 01|_{AB} \otimes \frac{1}{2}(\mathbb{1} + iX_M - iY_M - iZ_M) + |10\rangle\langle 10|_{AB} \otimes \frac{1}{2}(\mathbb{1} - iX_M - iY_M + iZ_M) + |11\rangle\langle 11|_{AB} \otimes \frac{1}{2}(\mathbb{1} - iX_M + iY_M - iZ_M). \tag{A14}$$

Observe that we can always write $V_{AM} = \sum_{i,j=0}^{1} |i\rangle\langle j|_A \otimes V_M^{A,ij}$ for some matrices $V_M^{A,ij}$ and similarly for V_{BM}. However, because we have assumed $V_{BM}V_{AM} = U_{AM}U_{BM}$ and the AB part in Eq. (A14) is expressed solely in terms of projectors, we can express $V_{BM}V_{AM}$ as

$$V_{BM}V_{AM} = \sum_{i,j} |ij\rangle\langle ij|_{AB} \otimes V_M^{B,jj} V_M^{A,ii}, \tag{A15}$$

where each product $V_M^{B,jj} V_M^{A,ii}$ is a unitary on M. Comparing Eqs. (A14) and (A15), we find that

$$V_M^{B,00} V_M^{A,00} = \frac{1}{2}(\mathbb{1} + iX_M + iY_M + iZ_M), \tag{A16}$$

$$V_M^{B,11} V_M^{A,00} = \frac{1}{2}(\mathbb{1} + iX_M - iY_M - iZ_M), \tag{A17}$$

$$V_M^{B,00} V_M^{A,11} = \frac{1}{2}(\mathbb{1} - iX_M - iY_M + iZ_M), \tag{A18}$$

$$V_M^{B,11} V_M^{A,11} = \frac{1}{2}(\mathbb{1} - iX_M + iY_M - iZ_M). \tag{A19}$$

However, this leads to the contradiction

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \left(V_M^{B,00} V_M^{A,00}\right)\left(V_M^{B,11} V_M^{A,00}\right)^\dagger = \left(V_M^{B,00} V_M^{A,11}\right)\left(V_M^{B,11} V_M^{A,11}\right)^\dagger = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{A20}$$

which completes the proof. ∎

5. Classicality and commuting decompositions

Here, we show the relation between the classicality of an interaction and decomposability of the corresponding unitary. In particular, we show the equivalence between the classicality $[H_{AM}, H_{BM}] = 0$ and the existence of a continuous commuting decomposition $U(t) = U_{AM}(t)U_{BM}(t) = U_{BM}(t)U_{AM}(t)$.

Proposition 5.—A one-parameter continuous group of unitaries $U(t) = e^{-itH}$ has a commuting decomposition $U(t) = U_{BM}(t)U_{AM}(t) = U_{AM}(t)U_{BM}(t)$ such that the map $t \mapsto (U_{AM}(t), U_{BM}(t))$ is continuous if and only if there exist Hamiltonians H_{AM} and H_{BM} such that $H = H_{AM} + H_{BM}$ and $[H_{AM}, H_{BM}] = 0$.

Proof.—Using the Baker-Campbell-Haussdorf (BCH) formula [52], one easily sees that if such H_{AM}, H_{BM} exists, then $U(t) = e^{-itH_{AM}}e^{-itH_{BM}} = e^{-itH_{BM}}e^{-itH_{AM}}$, showing that the unitary has a continuous commuting decomposition.

To show the other direction, suppose that the unitary e^{-itH} has a continuous commuting decomposition. Now, let us take t small enough such that $\|U_{AM}(t) - \mathbb{1}\|_\infty, \|U_{BM}(t) - \mathbb{1}\|_\infty < 1$. This ensures that $H_{AM} = i\log U_{AM}(t)/t, H_{BM} = i\log U_{BM}(t)/t$ can be defined through the power series for a matrix logarithm. Using the series representation $\log(\mathbb{1} - X) =$

$-\sum_{n=1}^{\infty}(1/n)X^n$, we note that these interaction Hamiltonians must commute:

$$\begin{aligned}[H_{AM}, H_{BM}] &= -\frac{1}{t^2}[\log U_{AM}, \log U_{BM}] \\ &= -\frac{1}{t^2}\left[\sum_{n=1}^{\infty}\frac{(\mathbb{1}-U_{AM})^n}{n}, \sum_{m=1}^{\infty}\frac{(\mathbb{1}-U_{BM})^m}{m}\right] \\ &= 0. \end{aligned} \tag{A21}$$

Using the BCH formula, we obtain

$$e^{-itH} = e^{-itH_{AM}}e^{-itH_{BM}} = e^{-it(H_{AM}+H_{BM})}. \tag{A22}$$

Differentiating the above expression with respect to t and using the identity $(d/dt)e^{(tA)}\big|_{t=0} = A$ shows that $H = H_{AM} + H_{BM}$, which proves the claim. ∎

6. Consistency

Let us start by recalling the two definitions of decomposability given in the main text.

Definition 1 (unitary).—Let U be a unitary acting on a tripartite system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_M$. U is decomposable if there exist unitaries U_{AM}, U_{BM} such that

$$U_{ABM} = U_{BM}U_{AM}. \tag{A23}$$

Definition 2 (map).—Let λ be a map acting on a tripartite system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_M$. λ is decomposable if there exist maps λ_{AM} and λ_{BM} such that

$$\lambda(\rho) = \lambda_{BM}\lambda_{AM}(\rho). \tag{A24}$$

The following proposition shows that these two definitions are consistent.

Proposition 6.—A unitary U is decomposable if and only if the map $\lambda(\rho) = U\rho U^\dagger$ is decomposable.

Proof.—If U is decomposable, choosing $\lambda_{AM}(\rho) = U_{AM}\rho U_{AM}^\dagger$ and $\lambda_{BM}(\rho) = U_{BM}\rho U_{BM}^\dagger$ shows that λ is also decomposable.

To show the other implication, suppose that there exist two maps λ_{AM} and λ_{BM} such that $U\rho U^\dagger = \lambda_{BM}\lambda_{AM}(\rho)$. It is enough to show that we can choose the maps λ_{AM} and λ_{BM} to be unitaries. This is indeed possible by the following argument. Since $U\rho U^\dagger = \lambda_{BM}\lambda_{AM}(\rho)$, we see that $\sigma \mapsto U^\dagger\lambda_{BM}(\sigma)U$ is a completely positive trace-preserving (CPTP) inverse of λ_{AM}. Since the only CPTP maps that have a CPTP inverse are unitaries [53], we conclude that λ_{AM} must be a unitary map. The fact that λ_{BM} is also unitary follows from $\lambda_{BM}(\rho) = U\lambda_{AM}^\dagger(\rho)U^\dagger$. ∎

Another question regarding the consistency between the two definitions concerns unitary dilations: is decomposability of a map equivalent to the existence of a decomposable unitary dilation? This would be desirable, since this would imply that any decomposable map is generated by some "classical" interaction on a larger system. Here, we show that the implication holds in at least one direction.

Proposition 7.—Let λ be a decomposable map. Then, there exists a Stinespring dilation of λ,

$$\lambda(\rho_{ABM}) = \mathrm{Tr}_R U_{ABMR}(\rho_{ABM} \otimes \sigma_R)U_{ABMR}^\dagger, \tag{A25}$$

such that U_{ABMR} is decomposable.

Proof.—Since λ is decomposable, there exist maps λ_{AM} and λ_{BM} such that $\lambda = \lambda_{BM}\lambda_{AM}$. Let us denote a Stinespring dilation of λ_{AM} as

$$\lambda_{AM}(\rho_{AM}) = \mathrm{Tr}_{R_A} U_{AMR_A}(\rho_{AM} \otimes \sigma_{R_A})U_{AMR_A}^\dagger, \tag{A26}$$

where R_A is the purifying system for λ_{AM}. Similarly, λ_{BM} must have a dilation with purifying system R_B. We prove the claim by identifying $R = R_AR_B$, $U_{ABMR} = U_{BMR_B}U_{AMR_A}$, and $\sigma_R = \sigma_{R_A} \otimes \sigma_{R_B}$. ∎

7. Out-of-time-ordered correlator

Finally, we comment on the notion of the out-of-time-ordered correlator (OTOC) and its relation to the decomposability. The OTOC is often used to study the spread of correlations in a many-body system [54,55]. Given two observables V and W (usually chosen to be commuting at time $t = 0$), the OTOC is defined as

$$C(t) = -\mathrm{Tr}\left(\rho_\beta\left([V, W(t)]\right)^2\right), \tag{A27}$$

where ρ_β is the thermal state at inverse temperature β and $W(t) = e^{-iHt}We^{iHt}$. Intuitively, it measures the effect of time evolution on the commutator between two initially commuting observables. We show that the OTOC witnesses the nondecomposability, providing an alternative to our methods. In particular, let us choose V as an observable on system A and W on system B. Let us assume that the dynamics are decomposable, i.e., for any t, there exist U_{AM} and U_{BM} such that $e^{-iHt} = U_{BM}U_{AM}$. Noting that $[W, U_{AM}] = 0$, an explicit calculation shows that

$$[V, W(t)] = \left[V, U_{BM}WU_{BM}^\dagger\right] \tag{A28}$$

$$= U_{BM}[V, W]U_{BM}^\dagger, \tag{A29}$$

which is zero, since V and W act on different subsystems. Therefore, the measurement of a nonzero OTOC can witness the nondecomposability of the dynamics. It remains to be shown whether such an approach can be extended to quantify the degree of nondecomposability.

APPENDIX B: ACCESSIBLE MEDIATOR

The following proposition proves the "correlation-capacity" bound when the initial state is product $\rho_0 = \rho_{AM} \otimes \rho_B$.

Proposition 8.—Let λ be a decomposable map. Any correlation measure satisfies

$$Q_{A:MB}(\lambda(\rho_{AM} \otimes \rho_B)) \leq \sup_{\sigma_{AM}} Q_{A:M}(\sigma_{AM}). \tag{B1}$$

Proof.—By assumption, $\lambda = \lambda_{BM}\lambda_{AM}$. The bound follows solely from monotonicity of correlations under local operations:

$$\begin{aligned} Q_{A:MB}(\lambda(\rho_{AM} \otimes \rho_B)) &= Q_{A:MB}(\lambda_{BM}\lambda_{AM}(\rho_{AM} \otimes \rho_B)) \\ &\leq Q_{A:MB}(\lambda_{AM}(\rho_{AM} \otimes \rho_B)). \end{aligned} \tag{B2}$$

Since Q is monotonic under local operations, it must be invariant under invertible local operations. In particular, adding or discarding an uncorrelated system does not change the value of Q. In our case, system B is completely uncorrelated and therefore $Q_{A:MB}(\lambda_{AM}(\rho_{AM} \otimes \rho_B)) = Q_{A:M}(\lambda_{AM}(\rho_{AM}))$. Of course, the last quantity is upper bounded by the supremum over all states. ■

For a general initial state, we have the following bound by continuity.

Proposition 9.—Let λ be a decomposable map and let ρ be any tripartite quantum state. Any gd-continuous correlation measure satisfies

$$Q_{A:MB}(\lambda(\rho)) \leq \sup_{\sigma_{AM}} Q_{A:M}(\sigma_{AM}) + I_{AM:B}(\rho), \tag{B3}$$

where $I_{AM:B}(\rho) = \inf_{\sigma_{AM}\otimes\sigma_B} g(d(\rho, \sigma_{AM} \otimes \sigma_B))$ is a measure of total correlations in the state ρ across the partition $AM : B$.

Proof.—We bound the difference in correlations between an arbitrary state and the product state using gd-continuity:

$$\begin{aligned} &|Q_{A:MB}(\lambda(\rho)) - Q_{A:MB}(\lambda(\sigma_{AM} \otimes \sigma_B))| \\ &\quad\leq g(d(\lambda(\rho), \lambda(\sigma_{AM} \otimes \sigma_B))) \\ &\quad\leq g(d(\rho, \sigma_{AM} \otimes \sigma_B)), \end{aligned} \tag{B4}$$

where in the last line we have used the fact that g is monotonic and d contractive. The derived inequality holds for any $\sigma_{AM} \otimes \sigma_B$—in particular, for the one achieving the infimum of $I_{AM:B}(\rho)$—leading to

$$Q_{A:MB}(\lambda(\rho)) \leq Q_{A:MB}(\lambda(\sigma_{AM} \otimes \sigma_B)) + I_{AM:B}(\rho). \tag{B5}$$

In the last step, we use Proposition 8 to bound the first term on the right. ■

In order to simplify the notation, let us denote the bound on correlations due to decomposable dynamics as $B(\rho) = \sup_{\sigma_{AM}} Q_{A:M}(\sigma_{AM}) + I_{AM:B}(\rho)$ and the state at time t as $\rho_t = \Lambda(\rho_0)$.

Proposition 10.—The degree of nondecomposability $d^{\mathrm{DEC}}(\Lambda)$ is lower bounded as follows:

$$d^{\mathrm{DEC}}(\Lambda) \geq g^{-1}(Q_{A:MB}(\rho_t) - B(\rho_0)). \tag{B6}$$

Proof.—We will prove the theorem by combining the continuity bounds with the statement of Proposition 9. Consider a fixed, but arbitrary, decomposable map λ. Due to gd-continuity, we write

$$\begin{aligned} Q_{A:MB}(\rho_t) - Q_{A:MB}(\lambda(\rho_0)) &\leq |Q_{A:MB}(\rho_t) - Q_{A:MB}(\lambda(\rho_0))| \\ &\leq g(d(\rho_t, \lambda(\rho_0))). \end{aligned} \tag{B7}$$

We rearrange and use the bound in Proposition 9:

$$\begin{aligned} Q_{A:MB}(\rho_t) &\leq Q_{A:MB}(\lambda(\rho_0)) + g(d(\rho_t, \lambda(\rho_0))) \\ &\leq B(\rho_0) + g(d(\rho_t, \lambda(\rho_0))). \end{aligned} \tag{B8}$$

The amount of violation is now brought to the left-hand side and below we use the fact that g is invertible and take the supremum over states ρ_0 to identify the degree of nondecomposability:

$$\begin{aligned} Q_{A:MB}(\rho_t) - B(\rho_0) &\leq g(d(\rho_t, \lambda(\rho_0))) \\ g^{-1}(Q_{A:MB}(\rho_t) - B(\rho_0)) &\leq d(\rho_t, \lambda(\rho_0)), \\ g^{-1}(Q_{A:MB}(\rho_t) - B(\rho_0)) &\leq d^{\mathrm{DEC}}(\Lambda), \end{aligned} \tag{B9}$$

which proves the claim. ■

1. Spectral norm

We link the operator norm of unitary maps with the spectral distance between them.

Lemma 1.—Let U, V be unitaries. Then $D_\infty(U, V) \leq 2\,\|U - V\|_\infty$.

Proof.—By simple algebra, we verify

$$U\rho U^\dagger - V\rho V^\dagger = \frac{1}{2}(U - V)\rho(U + V)^\dagger \tag{B10}$$

$$+ \frac{1}{2}(U + V)\rho(U - V)^\dagger, \tag{B11}$$

where ρ is a density matrix. Taking the spectral norm on both sides, we obtain

$$\left\| U\rho U^\dagger - V\rho V^\dagger \right\|_\infty \tag{B12}$$

$$= \left\| \frac{1}{2}(U - V)\rho(U + V)^\dagger + \frac{1}{2}(U + V)\rho(U - V)^\dagger \right\|_\infty \tag{B13}$$

$$\leq \frac{1}{2}\left\|(U-V)\rho(U+V)^\dagger\right\|_\infty + \frac{1}{2}\left\|(U+V)\rho(U-V)^\dagger\right\|_\infty \tag{B14}$$

$$\leq \|U-V\|_\infty \|\rho\|_\infty \|U+V\|_\infty \tag{B15}$$
$$\leq 2\|U-V\|_\infty, \tag{B16}$$

where we have used the triangle inequality and submultiplicativity of the spectral norm, $\|AB\|_\infty \leq \|A\|_\infty \|B\|_\infty$. Using the bounds $\|\rho\|_\infty \leq \|\rho\|_1 = 1$ and $\|U+V\|_\infty \leq \|U\|_\infty + \|V\|_\infty = 2$ on the last inequality finishes the proof. ■

2. Correlations and the number of Trotter steps

As a concrete illustration, let us relate the mutual information in a state to number of Trotter steps needed. In this case, we can use continuity bounds for von Neumann entropy to conclude that if $\frac{1}{2}\|\rho-\sigma\|_1 = \epsilon$, then [56,57]

$$|I_{A:MB}(\rho) - I_{A:MB}(\sigma)| \leq 2\epsilon \log(d_A d_M d_B - 1) + 3\eta(\epsilon), \tag{B17}$$

where $\eta(x) = -x\log x - (1-x)\log(1-x)$ is the binary entropy. We now bound the first term using $\epsilon \leq \sqrt{\epsilon}$, recalling that ϵ is small, and the second term using $\eta(\epsilon) \leq \sqrt{\epsilon}$ to arrive at

$$|I_{A:MB}(\rho) - I_{A:MB}(\sigma)| \leq 5\log(d_A d_M d_B)\sqrt{\epsilon}. \tag{B18}$$

Furthermore, since $\|X\|_1 \leq \mathrm{rank}X \cdot \|X\|_\infty$, we have

$$|I_{A:MB}(\rho) - I_{A:MB}(\sigma)| \leq C\sqrt{\|\rho-\sigma\|_\infty}, \tag{B19}$$

where $C = 5\sqrt{2}\log(d_A d_M d_B)\sqrt{d_A d_M d_B}$ is a dimension-dependent constant. This means that we can choose $g(s) = C\sqrt{s}$ to show that mutual information is *gd* continuous with respect to the spectral distance and the inverse is $g^{-1}(s) = (s/C)^2$ when $s \geq 0$. Combining this with the discussion in Sec. III C and Proposition 10, we finally obtain

$$\left(\frac{I_{A:MB}\left(e^{-itH}\rho_0 e^{itH}\right) - B(\rho_0)}{C}\right)^2 \leq t^2 \|[H_{AM}, H_{BM}]\|_\infty, \tag{B20}$$

when $I_{A:MB}\left(e^{-itH}\rho_0 e^{itH}\right) \geq B(\rho_0)$. This means that the number of Trotter steps needed to guarantee an ϵ error is

$$r \geq O\left(\frac{\left(I_{A:MB}\left(e^{-itH}\rho_0 e^{itH}\right) - B(\rho_0)\right)^2}{\epsilon}\right). \tag{B21}$$

Note that while we have used some relaxations to derive this bound, we still obtain nontrivial quantitative statements relating the correlations in the system and the commutator norm. In particular, while the quadratic power in the mutual information is suboptimal, a linear bound cannot exist due to the tightness of the entropic continuity bounds.

APPENDIX C: INACCESSIBLE MEDIATOR

First, we derive a necessary condition on maps admitting a decomposable *m*-dilation.

Proposition 11.—A *gd*-continuous correlation measure Q admits the following bound under the evolution generated by $\lambda \in \overline{\mathrm{DEC}}(m)$:

$$Q_{A:B}(\lambda(\rho_{AB})) \leq \sup_{\sigma_{AM}} Q_{A:M}(\sigma_{AM}) + I_{A:B}(\rho_{AB}), \tag{C1}$$

where the supremum is over all AM states with the dimension $d_M \leq m$ and $I_{A:B}(\rho_{AB}) = \inf_{\sigma_A\otimes\sigma_B} g(d(\rho_{AB}, \sigma_A\otimes\sigma_B))$ measures the total correlations across $A:B$.

Proof.—Consider the following argument:

$$\begin{aligned} Q_{A:B}(\lambda(\rho_{AB})) &\leq Q_{A:MB}(\tilde{\lambda}(\rho_{AB}\otimes\sigma_M)) \\ &\leq \sup_{\sigma_{AM}} Q_{A:M}(\sigma_{AM}) + I_{AM:B}(\rho_{AB}\otimes\sigma_M) \\ &= \sup_{\sigma_{AM}} Q_{A:M}(\sigma_{AM}) + I_{A:B}(\rho_{AB}), \end{aligned} \tag{C2}$$

where the first line follows from the monotonicity of Q and the existence of a decomposable *m*-dilation, the second line restates Proposition 9 restricted to an *m*-dimensional mediator, and the last line follows from the fact that tracing out an uncorrelated particle is a reversible process and hence we have equality. ■

Note that we have assumed that any map with a decomposable dilation starts with the joint ABM state of a product form $\rho_{AB}\otimes\sigma_M$. Although this is a restrictive condition, it has been shown that this is essentially the only consistent choice if we require that the dynamics can start from any AB state and the assignment is linear [58].

Next, we show that the violation of the inequality provides a bound on the degree of nondecomposability.

Proposition 12.—The degree of nondecomposability satisfies the following lower bound:

$$\mathcal{D}^{\overline{\mathrm{DEC}}(m)}(\Lambda_{AB}) \geq g^{-1}(Q_{A:B}(\Lambda_{AB}(\rho_{AB})) - \mathcal{B}(\rho_{AB})),$$

where $\mathcal{B}$ is the two-particle version of the bound B,

$$\mathcal{B}(\rho_{AB}) = \sup_{\sigma_{AM}} Q_{A:M}(\sigma_{AM}) + I_{A:B}(\rho_{AB}),$$

and the supremum over σ_{AM} assumes that the dimension of the mediator satisfies $d_M \leq m$.

Proof.—Consider a fixed but arbitrary dilation $\tilde{\Lambda}$ of the map Λ_{AB} and a decomposable map $\tilde{\lambda}$ (acting on all subsystems) that is a dilation of the map $\lambda_{AB} \in \overline{\mathrm{DEC}}(m)$. The same steps as in Proposition 10 and Eqs. (B7) and (B8) lead to the following inequality:

$$g^{-1}(Q_{A:B}(\Lambda(\rho_{AB})) - \mathcal{B}(\rho_{AB})) \leq d(\tilde{\Lambda}(\rho_{AB}\otimes\sigma_M), \tilde{\lambda}(\rho_{AB}\otimes\sigma_M)), \qquad \text{(C3)}$$

where we have used monotonicity and the definition of dilation to write $Q_{A:B}(\Lambda(\rho_{AB})) \leq Q_{A:MB}(\tilde{\Lambda}(\rho_{AB}\otimes\sigma_M))$ and invariance of total correlations under tracing out an uncorrelated system in the bound B, which therefore becomes $\mathcal{B}$. The left-hand side is accordingly fully expressed in terms of bipartite quantities and we now similarly bound the right-hand side.

To show the claim, it is enough to show that the distance on the right-hand side gives a lower bound to the degree of nondecomposability. By taking the supremum over ρ_{AB}, the right hand side is upper bounded by the operator distance:

$$\sup_{\rho_{AB}} d(\tilde{\Lambda}(\rho_{AB}\otimes\sigma_M), \tilde{\lambda}(\rho_{AB}\otimes\sigma_M)) \leq D(\tilde{\Lambda}, \tilde{\lambda}), \qquad \text{(C4)}$$

where the inequality is due to the optimization over states of AB only, not over all three systems. Analogous reasons show that the operator distance is upper bounded by the completely bounded distance

$$D(\tilde{\Lambda}, \tilde{\lambda}) \leq \mathcal{D}(\tilde{\Lambda}, \tilde{\lambda}). \qquad \text{(C5)}$$

This time, because the right-hand side involves additional optimization over the ancillary states. Finally, note that this reasoning holds for any dilation and the best bound is obtained by taking the dilations producing the infimum: $\inf_{\lambda_{AB}\in\overline{\mathrm{DEC}}(m)} \inf_{\tilde{\Lambda},\tilde{\lambda}} \mathcal{D}(\tilde{\Lambda}, \tilde{\lambda}) = \inf_{\lambda_{AB}\in\overline{\mathrm{DEC}}(m)} \mathcal{D}(\Lambda_{AB}, \lambda_{AB})$. ■

With these tools, we now investigate the structure of maps that admit decomposable m-dilations.

1. Nondecomposability of swapping

Proposition 13.—The map SWAP on two qubits has no decomposable m-dilation, for any m.

Proof.—We will prove this by contradiction. Suppose that SWAP has a decomposable m-dilation. Let us compare the action of SWAP on $|00\rangle_{AB}$ and on $|01\rangle_{AB}$. By definition, there exist two maps, λ_{AM} and λ_{BM}, and some initial state σ_M such that

$$|00\rangle\langle 00|_{AB} = \mathrm{SWAP}(|00\rangle\langle 00|_{AB}) = \mathrm{Tr}_M \lambda_{BM}\lambda_{AM}\left(|00\rangle\langle 00|_{AB}\otimes\sigma_M\right), \qquad \text{(C6)}$$

$$|10\rangle\langle 10|_{AB} = \mathrm{SWAP}(|01\rangle\langle 01|_{AB}) = \mathrm{Tr}_M \lambda_{BM}\lambda_{AM}\left(|01\rangle\langle 01|_{AB}\otimes\sigma_M\right). \qquad \text{(C7)}$$

Let us define $\sigma^0_{AM} = \lambda_{AM}\left(|0\rangle\langle 0|_A\otimes\sigma_M\right)$. By Eqs. (C6) and (C7), we have

$$|0\rangle\langle 0|_A = \mathrm{Tr}_B \mathrm{SWAP}(|00\rangle\langle 00|_{AB}) = \mathrm{Tr}_{BM}\lambda_{BM}\left(|0\rangle\langle 0|_B\otimes\sigma^0_{AM}\right), \qquad \text{(C8)}$$

$$|1\rangle\langle 1|_A = \mathrm{Tr}_B \mathrm{SWAP}(|01\rangle\langle 01|_{AB}) = \mathrm{Tr}_{BM}\lambda_{BM}\left(|1\rangle\langle 1|_B\otimes\sigma^0_{AM}\right). \qquad \text{(C9)}$$

But because λ_{BM} is trace preserving and Tr_B factors out when applied to product states, we have

$$\mathrm{Tr}_{BM}\lambda_{BM}\left(|0\rangle\langle 0|_B\otimes\sigma^0_{AM}\right) = \mathrm{Tr}_{BM}\left(|0\rangle\langle 0|_B\otimes\sigma^0_{AM}\right) = \mathrm{Tr}_M\sigma^0_{AM} = \mathrm{Tr}_{BM}\lambda_{BM}\left(|1\rangle\langle 1|_B\otimes\sigma^0_{AM}\right). \qquad \text{(C10)}$$

Combining this with Eqs. (C8) and (C9), we obtain

$$|0\rangle\langle 0|_A = \mathrm{Tr}_{BM}\lambda_{BM}\left(|0\rangle\langle 0|_B\otimes\sigma^0_{AM}\right) \qquad \text{(C11)}$$

$$= \mathrm{Tr}_{BM}\lambda_{BM}\left(|1\rangle\langle 1|_B\otimes\sigma^0_{AM}\right) \qquad \text{(C12)}$$

$$= |1\rangle\langle 1|_A, \qquad \text{(C13)}$$

which is clearly a contradiction. ■

2. Strict inclusions

Proposition 14.—The inclusion $\overline{\mathrm{DEC}}(m) \subsetneq \overline{\mathrm{DEC}}(m+1)$ is strict for all m.

Proof.—Let us fix m and take $d_A = d_B > d_M = m$. Let $\lambda_m(\rho_{AB}) = \mathrm{Tr}_M \mathrm{SWAP}_{BM}\lambda_{AM}(\rho_{AB}\otimes|0\rangle\langle 0|_M)$, where λ_{AM} is a maximally entangling map. By this construction, λ_m has a decomposable m-dilation, i.e., $\lambda_m \in \overline{\mathrm{DEC}}(m)$. Choosing $\rho_{AB} = |00\rangle\langle 00|_{AB}$ and Q to be the relative entropy of entanglement, we obtain $E_{A:B}(\lambda_m(\rho_{AB})) = \log m$, whereas by Proposition 11, for all maps $\lambda \in \overline{\mathrm{DEC}}(m-1)$ we have (recall that ρ_{AB} is product)

$$E_{A:B}(\lambda(\rho_{AB})) \leq \sup_{\sigma_{AM}} E_{A:M}(\sigma_{AM}) + I_{A:B}(\rho_{AB}) \qquad \text{(C14)}$$

$$= \log(m-1). \qquad \text{(C15)}$$

Therefore, $\lambda_m \notin \overline{\mathrm{DEC}}(m-1)$, and the claim is shown. ■

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