Admissible Causal Structures and Correlations

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It is well known that if one assumes quantum theory to hold locally, then processes with indefinite causal order and cyclic causal structures become feasible. Here we study *qualitative* limitations on causal structures and correlations imposed by local quantum theory. We find a *necessary graph-theoretic criterion*—the "siblings-on-cycles" property—for a causal structure to be admissible: only such causal structures admit a realization consistent with local quantum theory. We conjecture that this property is moreover *sufficient*. This conjecture is motivated by an explicit construction of quantum causal models, and is supported by numerical calculations. We show that these causal models, in a restricted setting, are indeed consistent. We identify two sets of causal structures that, in the classical-deterministic case, forbid and give rise to noncausal correlations, respectively.

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I. INTRODUCTION

At heart of Einstein's equivalence principle is the impossibility to detect the gravitational field via local experiments [1]. For general relativity, this principle dictates that physics in sufficiently small, i.e., *local*, space-time regions is described by special relativity. This principle naturally extends to the quantum case: *local experiments are described by quantum theory*. In this quantum formulation, however, the gravitational field, the reference frames, and the space-time regions might necessitate quantum descriptions. While different approaches target these descriptions (see, e.g., Refs. [2–5]), another approach—the process-matrix framework [6]—abstracts away the general-relativistic freight and focuses on the idealized prescription of local quantum experiments in countably many regions only (without imposing any global constraints). Similarly to the various formulations of the equivalence principle, this approach can be used to constrain competing theories of quantum gravity: if a candidate theory of quantum gravity exceeds the limits of the latter, then the results of local experiments in that theory *must* disagree with quantum theory. This, in turn, gives a prescription to experimentally falsify that candidate theory.

The process-matrix framework, i.e., the assumption of local quantum theory, reconciles the inherently *probabilistic* nature of quantum theory with the *dynamical* causal structures of general relativity [7]. It extends quantum indefiniteness of physical degrees of freedom such as position and momentum to causal connections. Exemplarily, while the position of a mass in general relativity determines the causal order among events in its future, the *quantum-switch* process [8] does so coherently [9–11]. Moreover, this framework allows for violations of causal inequalities [6]. Causal inequalities, similarly to Bell inequalities [12], are device-independent tests of a global causal order. If the observed correlations violate such an inequality, then they cannot be causally explained: any explanation where only past data influence future observations fails. These correlations are called "noncausal," and they arise in setups that resemble [13] closed timelike curves [14,15]. As notoriously shown by Gödel [15], closed timelike curves appear in solutions to Einstein's equation of general relativity.

This stipulation of local quantum theory is also of interest in theoretical computer science. A pillar of computer science is that machines (programs) and data are treated on an equal footing. This paradigm finds its climax in Church's notion of computation—the λ calculus [16]—where all data *are* functions, and therefore functions are of higher-order: functions on

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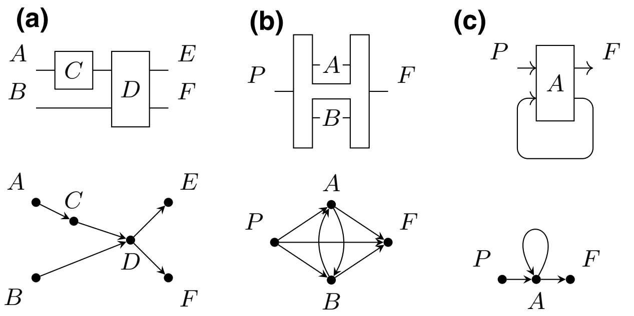

FIG. 1. (a) A quantum circuit and its *acyclic* causal structure. (b) The quantum switch—an instance of the process-matrix framework—has a *cyclic* causal structure: depending on the prepared state at P, a quantum system is sent through the H-shaped region from A to B or from B to A. (c) If A is traversed by a closed timelike curve, then A's output influences the input, and a departure from quantum theory becomes necessary.

functions. The process-matrix framework describes the first level of *higher-order quantum computation* [17,18]: its objects—the process matrices—map quantum gates to quantum gates. For instance, the previously mentioned quantum switch maps two quantum gates A and B to the functionality $(\alpha|0\rangle + \beta|1\rangle) \otimes |\psi\rangle \mapsto \alpha|0\rangle \otimes BA|\psi\rangle + \beta|1\rangle \otimes AB|\psi\rangle$, where the *order* of gate application is controlled by the first qubit. This is achieved, for example, through programmable connections between gates [19]. The quantum switch brings forth a reduction in query complexity when compared with the standard circuit model of computation [8,20–22].

The causal relations among local quantum experiments (gates) are conveniently expressed by *causal structures*, A causal structure is a directed graph where the vertices represent laboratories and where the edges indicate the possibility of a local laboratory directly influencing another local laboratory (see Fig. 1). The causal relations among the gates of any quantum circuit form an *acyclic* causal structure: naturally, a gate at depth d of the circuit has no causal influence on the input to any other gate at the same or smaller depth. This is radically contrasted by processes: the quantum switch, for instance, has a *cyclic* causal structure [23]. Still, not *every* causal structure is compatible with local quantum theory: for the output of a laboratory to influence the same laboratory's input, we require a departure from quantum theory by introducing nonlinear dynamics [24–26].

In this work, we study the causal structures that admit a quantum realization—a question raised in Ref. [27]. In other words, we study the possible causal relations among laboratories under the assumption that within each laboratory no deviation from quantum theory is observable. We find a *necessary graph-theoretic criterion* (the causal structure of every quantum process satisfies this criterion) and conjecture that the criterion is also *sufficient*. The conjecture is motivated by a construction of causal models for the causal structures of interest, and is moreover numerically tested for all directed graphs with up to six nodes. In addition, we provide two graph-theoretic criteria from which, in the classical-deterministic case, only causal or also noncausal correlations arise. Supporting the above-mentioned conjecture, we show that the causal structures that satisfy the criterion for nonviolation are admissible.

The presentation is structured in the following way. First, we provide the mathematical tools necessary for the present treatment. This is followed by our results on admissible and inadmissible causal structures. Thereafter, we relate causal structures with causal inequalities. We conclude with a series of open questions.

II. PRELIMINARIES

We briefly comment on the notation used. If $\mathcal{H}$ is a Hilbert space, then $\mathcal{L}(\mathcal{H})$ is the set of linear operators on $\mathcal{H}$. We use $\mathbb{Z}_n$ for the set $\{0, 1, \dots, n-1\}$. If a symbol is used with and without a subscript from $\mathbb{Z}_n$, then the bare symbol denotes the collection under the natural composition; for example, we use x to denote $(x_k)_{k\in\mathbb{Z}_n}$. If the subscript is a subset of $\mathbb{Z}_n$, then the composition is taken only over those elements. Moreover, we use $\setminus S$ as shorthand for $\mathbb{Z}_n \setminus S$, and we use $\setminus i$ as shorthand for $\setminus\{i\}$. If ε is a completely positive map, then ρ^ε is its Choi operator [28]. For a directed graph $G = (V, E \subseteq V \times V)$, V denotes the set of nodes and E the set of directed edges. A *directed path* $\pi = (v_0, \dots, v_\ell)$ is a sequence of distinct nodes with $\{(v_i, v_{i+1}) \mid 0 \le i < \ell\} \subseteq E$. A *directed cycle* $C = (v_0, \dots, v_\ell)$ is a directed path with $(v_\ell, v_0) \in E$. The induced graph $G[V']$ is the graph $G' = (V', E')$, where $V' \subseteq V$ and all edges $E' \subseteq E$ have endpoints in V', i.e., $(i,j) \in E'$ if and only if $i,j \in V'$ and $(i,j) \in E$. A directed cycle C is called an "induced directed cycle" or a "chordless cycle" if the induced graph $G[C]$ is a directed cycle graph. For a node $k \in V$, we use $\mathrm{Pa}(k)$, $\mathrm{Ch}(k)$, and $\mathrm{Anc}(k)$ to express the set of parents, children, and ancestors of k, respectively, and similarly we use $\mathrm{Pa}(S)$, $\mathrm{Ch}(S)$ and $\mathrm{Anc}(S)$ by taking the union over a set S. The cardinality of the set $\mathrm{Pa}(k)$ is the *in-degree* $\deg_{\mathrm{in}}(k)$. A node with zero in-degree is called a "source." We use $\mathrm{CPa}(S)$ for the union of the common parents of all elements $i \neq j \in S$, i.e., $\mathrm{CPa}(S) := \bigcup_{i\neq j\in S} \mathrm{Pa}(i) \cap \mathrm{Pa}(j)$. Two nodes $i,j \in V$ are called "siblings" if and only if they have common parents, i.e., $\mathrm{CPa}(\{i,j\}) \neq \emptyset$. A directed path π is said to *contain siblings* if and only if $\mathrm{CPa}(\pi) \neq \emptyset$.

A. Correlations

Correlations observed among n parties (regions) $\mathbb{Z}_n$ are expressed by the conditional probability distribution $p(a \mid x)$, where for each party k we have $a_k \in \mathcal{A}_k$ and $x_k \in \mathcal{X}_k$. The set $\mathcal{X}_k$ is the set of experimental *settings* and $\mathcal{A}_k$ is the set of experimental *observations*.

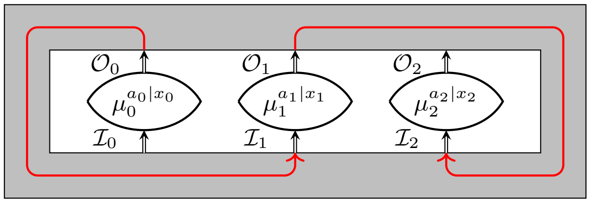

FIG. 2. Three parties and a process. The gray area represents the process: it takes the systems on the future boundaries of the parties and maps it to the past boundaries of the parties. The red connections indicate an example where the parties are causally ordered increasingly. *A priori*, however, the process is not assumed to respect any ordering of the parties.

Definition 1 (Causal correlations [6,9,29]).—The n-party correlations $p(a \mid x)$ are *causal* if and only if they can be decomposed as

$$p(a \mid x) = \sum_{k\in\mathbb{Z}_n} q_k p(a_k \mid x_k) p^{x_k}_{a_k}(a_{\setminus k} \mid x_{\setminus k}), \tag{1}$$

where $\forall k \in \mathbb{Z}_n : q_k \geq 0$, $\sum_{k\in\mathbb{Z}_n} q_k = 1$, and $p^{x_k}_{a_k}(a_{\setminus k} \mid x_{\setminus k})$ are $(n-1)$-party causal correlations. If this decomposition is infeasible, the correlations are called "noncausal."

The motivation behind this definition is that each party can influence its future only, including the causal order of the parties in its future. From this it follows that there exists at least one party k whose observation does not depend on the data of any other party (the value of a_k depends solely on x_k). Moreover, the causal order among the parties might be subject to randomness (e.g., a coin flip).

B. Processes

In the process-matrix framework [6], each party (region) is defined through a past and future spacelike boundary (see Fig. 2). For party k, we denote by $\mathcal{I}_k$ the Hilbert space on the past boundary, by $\mathcal{O}_k$ the Hilbert space on the future boundary, and by CP_k (CPTP_k) the set of all completely positive (trace-preserving) maps from $\mathcal{L}(\mathcal{I}_k)$ to $\mathcal{L}(\mathcal{O}_k)$. A quantum experiment for party k is a quantum channel from $\mathcal{I}_k$ to $\mathcal{O}_k$, equipped with a classical input (the setting) and a classical output (the observation). Hence, it is a family $\{\mu_k^{a_k|x_k} \in \mathrm{CP}_k\}_{(a_k,x_k)\in\mathcal{A}_k\times\mathcal{X}_k}$ of maps, such that for all $x_k \in \mathcal{X}_k$, we have $\sum_{a_k\in\mathcal{A}_k} \mu_k^{a_k|x_k} \in \mathrm{CPTP}_k$. A process interlinks all parties without any assumption on their causal relations, but with the sole assumption that no deviation from quantum theory is locally observable, i.e., the probability distribution over the observations is a multilinear function of the quantum experiments and is well defined (even if the parties share an arbitrary quantum system).

Definition 2 (Process [6]).—An n-party *quantum process* is a positive semidefinite operator $W \in \mathcal{L}(\mathcal{I} \otimes \mathcal{O})$ with

$$\forall\{\mu_k \in \mathrm{CPTP}_k\}_{k\in\mathbb{Z}_n} : \mathrm{Tr}\left[W \bigotimes_{k\in\mathbb{Z}_n} \rho^{\mu_k} \right] = 1. \tag{2}$$

Note that in this definition, the experimental settings and observations are absent, or equivalently, the sets $\mathcal{A}_k$ and $\mathcal{X}_k$ are singletons. Equation (2) states that the total probability of observing this single outcome under this single setting is 1; this holds for any setting and is independent of any resolution of the completely positive trace-preserving map into maps with classical outputs. Oreshkov *et al.* [6] observed that the quantum process W is the Choi operator of a completely positive trace-preserving map from all future boundaries to all past boundaries of the parties (see Fig. 2). For a given choice of n-party quantum process and experiments, the correlations are computed with the generalized Born rule

$$p(a \mid x) := \mathrm{Tr}\left[W \bigotimes_{k\in\mathbb{Z}_n} \rho^{\mu_k^{a_k|x_k}} \right]. \tag{3}$$

If one assumes that the parties perform classical-deterministic experiments, as opposed to quantum experiments, we arrive at the following special case. The spaces on the past and future boundaries $\mathcal{I}_k$ and $\mathcal{O}_k$ are sets (as opposed to Hilbert spaces), and an experiment for party k is a function $\mu_k : \mathcal{I}_k \times \mathcal{X}_k \to \mathcal{O}_k \times \mathcal{A}_k$ (as opposed to a family of maps), where we use $\mu_k^0 : \mathcal{I}_k \times \mathcal{X}_k \to \mathcal{O}_k$ for the first component and $\mu_k^1 : \mathcal{I}_k \times \mathcal{X}_k \to \mathcal{A}_k$ for the second component. As in the quantum case, a classical-deterministic process ω turns out to be a function from the future boundaries to the past boundaries, and Eq. (2) translates into an intuitive condition: for any choice of experiment, there exists a *unique* consistent assignment of values to the input spaces; the map $\omega \circ \mu$ has a unique fixed point.

Theorem 1 (Classical-determintic process [30]).—An n-party *classical-deterministic process* is a function $\omega : \mathcal{O} \to \mathcal{I}$ with

$$\forall\{\mu_k : \mathcal{I}_k \to \mathcal{O}_k\}_{k\in\mathbb{Z}_n}, \quad \exists!(r_k)_{k\in\mathbb{Z}_n} : r = \omega(\mu(r)), \tag{4}$$

where $\exists!$ is the uniqueness quantifier.

Here the correlations among the n parties are computed via

$$p(a\,|\,x) := \omega \star \mu^{a|x} = \sum_{i,o} [\omega(o) = i]\,[(o,a) = \mu(x,i)]\,, \tag{5}$$

where we use $[n = m]$ for the Kronecker delta $\delta_{n,m}$ and $\star$ for the link product [31]. Note that every n-party classical-deterministic process ω also corresponds to an n-party quantum process [27]

$$W_{\omega} := \sum_{o \in \mathcal{O}} |o\rangle\langle o|_{\mathcal{O}} \otimes |\omega(o)\rangle\langle\omega(o)|_{\mathcal{I}}, \tag{6}$$

where $|o\rangle = \bigotimes_{k\in\mathbb{Z}_n} |o_k\rangle$, with $\{|o'_k\rangle\}_{o'_k\in\mathcal{O}_k}$ being a basis for all $k \in \mathbb{Z}_n$, and similarly for $|\omega(o)\rangle$.

We briefly illustrate the above-mentioned theorem in the single-party case ($n = 1$). Assume the function ω is the identity function from $\mathcal{O} = \{0, 1\}$ to $\mathcal{I} = \{0, 1\}$. Here ω describes a closed timelike curve: the output of that single party is identically mapped to the same party's input. Suppose now that this single party implements the negation $\mu(b) = 1 - b$. What is the state of the system on the party's past boundary? In this instance of time-travel antinomy—the grandparent antinomy—no consistent specification is possible: if it is b, then the local experiment specifies the state on the future boundary to be $1 - b$, and in turn, the function ω specifies the state on the past boundary to be $1 - b$, but it is b. In other terms, the function $\omega \circ \mu$ has no fixed point; this function ω is *not* a classical-deterministic process. Another instance of a time-travel antinomy—the information antinomy—arises if the party implements the *identity* experiment $\mu(b) = b$. Then the state on the party's past boundary is not determined: both values, 0 and 1, are equally justifiable; the function $\omega \circ \mu$ has *two* fixed points. It turns out that these two antinomies are equivalent:

Theorem 2 (Equivalence of antinomies [32]).—The n-party function $\omega : \mathcal{O} \to \mathcal{I}$ suffers from the grandparent antinomy; i.e., there exists a choice of experiments $\{\mu_k : \mathcal{I}_k \to \mathcal{O}_k\}_{k\in\mathbb{Z}_n}$ such that $\omega \circ \mu$ has *no* fixed points if and only if ω suffers from the information antinomy, i.e., there exists a μ such that $\omega \circ \mu$ has *two or more* fixed points.

In the single-party case, the only classical-deterministic processes are the constant functions $\omega(b) = c$, the *unique* fixed point being c. For three or more parties, classical-deterministic processes exist that allow for noncausal correlations: any simulation of these correlations requires the abandonment of a causal order, resulting in a form of closed timelike curves [13], or, as recently suggested, time-delocalized systems [33].

In the present treatment, we also use *reduced processes*: if we invoke an experiment μ_k of a single party k but leave all other experiments unspecified, then the reduced function is a process again. To do so, we specify the state on the future boundary of party k, which is $\mu_k(i_k)$, where i_k is the state on the past boundary of party k. The state i_k is well defined because the process is componentwise nonsignaling: $i_k := \omega_k(o)$ is independent of o_k.

Lemma 3 (Componentwise nonsignaling and reduced process [13,32]).—If $\omega : \mathcal{O} \to \mathcal{I}$ is an n-party classical-deterministic process, then it is componentwise nonsignaling, i.e., the component ω_k does not depend on the kth input:

$$\forall k \in \mathbb{Z}_n, \quad \forall\, o \in \mathcal{O}, \quad o'_k \in \mathcal{O}_k : \omega_k(o) = \omega_k\left(o'_k, o_{\setminus k}\right). \tag{7}$$

If $\omega : \mathcal{O} \to \mathcal{I}$ is an $(n \geq 2)$-party classical-deterministic process, then for all $k \in \mathbb{Z}_n$ and $\mu_k : \mathcal{I}_k \to \mathcal{O}_k$, the *reduced function* $\omega^{\mu_k}_{\setminus k} : \mathcal{O}_{\setminus k} \to \mathcal{I}_{\setminus k}$ defined for all $\ell \in \mathbb{Z}_n \setminus \{k\}$ via

$$\omega^{\mu_k}_{\ell} : \mathcal{O}_{\setminus k} \to \mathcal{I}_{\ell} \tag{8}$$

$$o_{\setminus k} \mapsto \omega_{\ell}\left(o_{\setminus k}, \mu_k\left(\omega_k\left(o_{\setminus k}\right)\right)\right) \tag{9}$$

is an $(n - 1)$-party classical-deterministic process.

The process establishes the causal connections among the parties. Such connections—the causal structure—are conveniently expressed by a directed graph where the nodes $\mathbb{Z}_n$ represent the parties and where the absence of an edge from i to j indicates that the process is not signaling from the future boundary of party i to the past boundary of party j. The causal structure of the process schematically represented by the red arrows in Fig. 2 has no edge from 1 to 0, from 2 to 0, and from 2 to 1.

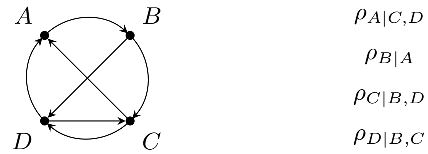

FIG. 3. Example of a four-node causal model. The state on the input space of node A is obtained by evolving the state on the output space of nodes C and D through the channel $\rho_{A|C,D}$, and similarly for the other nodes.

C. Causal models

A causal model is a causal structure (a directed graph) equipped with model parameters (channels along the edges). Traditionally, the nodes of a causal model are random variables [34]. Here, in contrast, we adopt the split-node model [35], where each node is split into an incoming part and an outgoing part, the past boundary and the future boundary (see Fig. 3). We follow the recent work of Barrett *et al.* [23], which unifies processes and causal models.

Definition 3 (Causal model, consistency, and faithfulness [23]).—An n-party *causal model* is a directed graph $G = (\mathbb{Z}_n, E)$ (causal structure) equipped with $\{\rho_{k|\mathrm{Pa}(k)}\}_{k\in\mathbb{Z}_n}$ (model parameters). In the *classical-deterministic* case, the model parameters are functions $\mathcal{O}_{\mathrm{Pa}(k)} \to \mathcal{I}_k$, and they define a *classical map* $\omega := (\rho_{k|\mathrm{Pa}(k)})_{k\in\mathbb{Z}_n}$. In the *quantum* case, the model parameters are the Choi operators of completely positive trace-preserving maps $\mathcal{L}(\mathcal{O}_{\mathrm{Pa}(k)}) \to \mathcal{L}(\mathcal{I}_k)$, such that $\forall i, j \in \mathbb{Z}_n : [\rho_{i|\mathrm{Pa}(i)}, \rho_{j|\mathrm{Pa}(j)}] = 0$, and they define a *quantum map* $W := \prod_{k\in\mathbb{Z}_n} \rho_{k|\mathrm{Pa}(k)}$. The causal model is *consistent* if and only if ω is an n-party classical-deterministic process and W is an n-party quantum process, respectively. The causal model is *faithful* if and only if for all k, the model parameter $\rho_{k|\mathrm{Pa}(k)}$ is signaling from all $\ell \in \mathrm{Pa}(k)$ to k.

We now comment on some aspects of the above definition. First—and as commented on earlier—if ω is a classical-deterministic process, then W_ω [see Eq. (6)] is a quantum process. So the above definition could be adjusted to refer to quantum processes only. However, because in the following we make extensive use of the properties of classical-deterministic processes, we explicitly define classical-deterministic causal models in reference to classical-deterministic processes. Second, the commutativity criterion above ensures Markovianity of the quantum process [23]. Naturally, only Markov processes can be faithfully represented as a causal model; without Markovianity, the representation of a causal structure as a directed graph would be meaningless. Note that not every process admits a description as a faithful causal model. This is the case for the initial two-party process W^{OCB} [6], which does not admit a factorization into commuting Choi operators $\rho_{\mathrm{Alice}|\mathrm{Bob}}, \rho_{\mathrm{Bob}|\mathrm{Alice}}$. Simultaneously, W^{OCB} is not unitarily extensible [36]. It is conjectured that these conditions are equivalent [23].

III. ADMISSIBLE CAUSAL STRUCTURES

The definition of causal models allows us to express precisely the notion of admissible causal structures: A causal structure *admits* a quantum realization whenever it can be amended with model parameters to obtain a *faithful and consistent* causal model. An *inadmissible* causal structure is therefore *incompatible* with local quantum theory. The requirement of faithfulness ensures that all nonsignaling *and signaling* relations are expressed by the causal structure. If we ignore faithfulness, then *every* directed graph is trivially admissible: the constant (state-preparation) model parameters that provide the qubit $|0\rangle$ to each party k satisfy *any* nonsignaling requirement. The siblings-on-cycles property of a graph G turns out to be a relevant graph-theoretic criterion.

Definition 4 (Siblings-on-cycles graph).—A directed graph $G = (V, E)$ is a *siblings-on-cycles* graph if and only if each directed cycle in G contains siblings.

For our first result—the characterization of *inadmissible* causal structures—we use the following lemma, which ensures that signals can be propagated along sibling-free paths. Intuitively, the influences on the parties along the path come from the "previous" party on the path and "outside" parties. Faithfulness then guarantees that the signal can be sent along the path. In contrast, if there were siblings, the common parent might block the signal propagation along the path.

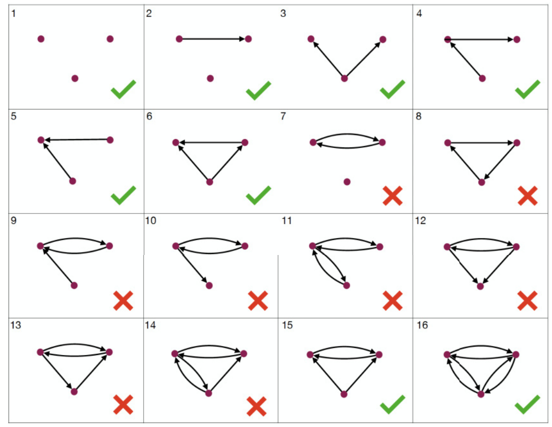

FIG. 4. Characterization of all pairwise nonisomorphic causal structures for three parties. Graphs 7–14 are inadmissible (Theorem 5). In the classical-deterministic case, graph 16 leads to noncausal correlations (Theorem 12), and the other graphs lead to causal correlations only (Theorem 10). Graph 16 is also the causal structure of the Araújo-Feix/Baumeler-Wolf process [37,38]. Graph 15 is the causal structure of the quantum switch without a region in the global future [see Fig. 1(b)].

Lemma 4 (Quantum signaling path).—Consider a faithful quantum causal model with causal structure $G = (V, E)$ and model parameters $\{\rho_{k|\mathrm{Pa}(k)}\}_{k\in V}$. If $\pi = (v, \dots, w)$ is a directed path in G *without siblings,* then there exist local experiments such that party v can signal to party w.

Proof.—As the nodes in π do not have common parents, the quantum map W is $\bigotimes_{k\in\pi} \rho_{k|\mathrm{Pa}(k)} \prod_{k\notin\pi} \rho_{k|\mathrm{Pa}(k)}$. We start by partially fixing the local quantum experiment of each party $k \notin \pi$ to discard the input on $\mathcal{I}_k$, i.e., each such party applies the map $\rho^{\mu'_k} = \mathbb{1}_{\mathcal{I}_k}$. Because $\rho_{k|\mathrm{Pa}(k)}$ is the Choi operator of a completely positive trace-preserving map $\mathcal{L}(\mathcal{O}_{\mathrm{Pa}(k)}) \to \mathcal{L}(\mathcal{I}_k)$, we have $\mathrm{Tr}_{\mathcal{I}_k}[\rho_{k|\mathrm{Pa}(k)}] = \mathbb{1}_{\mathcal{O}_{\mathrm{Pa}(k)}}$, and the resulting reduced quantum map $\mathrm{Tr}_{\mathcal{I}_{V\setminus\pi}} W$ is $\bigotimes_{k\in\pi} \rho_{k|\mathrm{Pa}(k)} \mathbb{1}_{\mathcal{O}_{\mathrm{Pa}(V\setminus\pi)}}$. Note that the causal model we started with is faithful: for each party k and each parent $\ell \in \mathrm{Pa}(k)$, the map $\rho_{k|\mathrm{Pa}(k)}$ is signaling from ℓ to k. In other words, there exists a quantum state $\tau \in \mathcal{L}(\mathcal{O}_{\mathrm{Pa}(k)\setminus\{\ell\}})$ such that $\rho'_{k|\ell} := \mathrm{Tr}_{\mathcal{O}_{\mathrm{Pa}(k)\setminus\{\ell\}}}[\rho_{k|\mathrm{Pa}(k)} \tau^T]$ is the Choi operator of a *signaling* completely positive trace-preserving map from $\mathcal{L}(\mathcal{O}_\ell)$ to $\mathcal{L}(\mathcal{I}_k)$. Because of this and because the nodes in π do not have common parents, we can complete the local experiments of all parties $k \in \mathrm{Pa}(\pi) \setminus \pi$ to prepare a state $\tau \in \mathcal{L}(\mathcal{O}_{\mathrm{Pa}(\pi)\setminus\pi})$ where a signal can be sent from one node to the next along π, i.e., the reduced map is signaling from ℓ to k for all $k \in \pi, \ell \in \mathrm{Pa}(k) \cap \pi$. Finally, a signaling channel from $\mathcal{L}(\mathcal{O}_v)$ to $\mathcal{L}(\mathcal{I}_w)$ is obtained by implementation of an appropriate identity channel for each party $k \in \pi \setminus \{v, w\}$, and by preparation of an arbitrary quantum state for all remaining parties. ■

We now state and prove our first result (see Fig. 4).

Theorem 5 (Inadmissible causal structures).—The causal structure of every faithful and consistent quantum causal model is a siblings-on-cycles graph, or, equivalently, if a graph G is *not* a siblings-on-cycles graph, then the causal structure G is *inadmissible.*

Proof.—We prove the latter formulation of the theorem. Assume that G contains a directed cycle $C = (v, \dots, w)$ *without siblings,* and let $(G, \{\rho_{k|\mathrm{Pa}(k)}\}_{k\in V})$ be an arbitrary faithful quantum causal model with causal structure G. By Lemma 4, there exist local quantum experiments for all parties except v such that the reduced process $W' := \rho_{v|v}$ is the Choi operator of a *signaling* completely positive trace-preserving map from $\mathcal{L}(\mathcal{O}_v)$ to $\mathcal{L}(\mathcal{I}_v)$. This, however, is not a single-party process [6]: there exists a map $\mu_v \in \mathrm{CPTP}_v$ such that $\mathrm{Tr}[W'\rho^{\mu_v}] \neq 1$. ■

For our second result concerning the admissibility of causal structures, we give a construction of model parameters for any causal structure. We conjecture that these model parameters *always* give rise to *consistent* causal models when the relevant causal structure is a siblings-on-cycles graph.

Definition 5 (Model parameters).—Let $G = (V, E)$ be a directed graph. For each party $k \in V$, define the input space $\mathcal{I}_k := \mathbb{Z}_2$, the output space $\mathcal{O}_k := \mathrm{Ch}(k) \cup \{\perp\}$, and the model parameters $\{\rho_{k|\mathrm{Pa}(k)} : \mathcal{O}_{\mathrm{Pa}(k)} \to \mathcal{I}_k\}_{k\in V}$ with

$$\rho_{k|\mathrm{Pa}(k)} : \bigtimes_{\ell\in\mathrm{Pa}(k)} (\mathrm{Ch}(\ell) \cup \{\perp\}) \to \mathbb{Z}_2 \tag{10}$$

$$(t_\ell)_{\ell\in\mathrm{Pa}(k)} \mapsto \prod_{\ell\in\mathrm{Pa}(k)} [k = t_\ell]. \tag{11}$$

The bottom element ($\perp$) is a special element not contained in V.

These model parameters are such that each party ℓ can select *at most* one of its children $k \in \mathrm{Ch}(\ell)$. If all parents of party k select k, then party k receives a system in the state 1, and a system in the state 0 otherwise. Note that this causal model is *faithful:* party $\ell \in \mathrm{Pa}(k)$ can signal a bit to party k whenever k is selected by all parents except ℓ, i.e., with $t_{\ell'} = k$ for all $\ell' \in \mathrm{Pa}(k) \setminus \{\ell\}$, we have $\rho_{k|\mathrm{Pa}(k)}(k, \dots, k, t_\ell) = [k = t_\ell]$. The bottom element $\perp$ in the output space of the parties is required to ensures faithfulness in the case $|\mathrm{Ch}(\ell)| = 1$.

Suppose we amend a siblings-on-cycles graph G with these model parameters. If C is a directed cycle in the graph G, then these model parameters interrupt the signal progression along the cycle C, and therefore no party can send a signal to itself. To see this, let $i, j \in C$ be siblings and $p \in \mathrm{CPa}(\{i, j\})$. If the common parent p selects party $j =: t_p$, then $\rho_{i|\mathrm{Pa}(i)}(t_p, t_{\mathrm{Pa}(i)\setminus\{p\}}) = 0$, i.e., party i receives the constant zero, and therefore the parent of i along the cycle C cannot send a signal to i. Similarly, the signal progression along the cycle C is interrupted at party j if the common parent p selects i. This provides motivation for the following conjecture:

Conjecture 1 (Admissible causal structures).—If $G = (V, E)$ is a siblings-on-cycles graph, then the causal structure G equipped with the model parameters of Definition 5 forms a *consistent* causal model.

Note that the explanation above is *not* a proof of this conjecture. The problem is that the value t_p might depend on the interventions of other parties, and in particular, on the interventions of children of party p.

We have numerically tested this conjecture for all siblings-on-cycles graphs with up to six nodes. For these tests, we used the following recursive function α:

$$\alpha : V \times V^* \to \{0, 1\}, \tag{12}$$

$$\begin{aligned}(k, \pi = (v_1, \dots, v_m)) &\mapsto [k \notin \pi] \\ &\times \prod_{\ell\in\mathrm{Pa}(k)} [k = \mu_\ell\,(\alpha\,(\ell, (k, v_1, \dots, v_m)))],\end{aligned} \tag{13}$$

where V^* is the set of all finite sequences of vertices. We verified that $\hat{i} := (i_k = \alpha(k))_{k\in V}$ is the fixed point of the map $\omega \circ \mu$. Note that $\alpha(k)$ for $k \in V$ is well defined—the recursion terminates after finitely many invocations of α—because the directed graphs are finite, and whenever a vertex is revisited, α returns 0. For these tests, we wrote two C programs: one to generate siblings-on-cycles graphs and the other to verify the fixed point. The source code of these programs can be found in Ref. [39]. We required 132 800-MHz CPU seconds for the generation of the causal structures and around 31 800-MHz CPU hours for the verification of the admissibility of the model parameters.

Because every faithful and consistent classical-deterministic causal model can be lifted to a quantum one [27], this conjecture has as an immediate consequence the completion of Theorem 5 with its converse.

Corollary 6.—If Conjecture 1 holds, then the causal structure G is *admissible* if and only if G is a *siblings-on-cycles* graph.

We have proved a restricted form of Conjecture 1 for a subset of siblings-on-cycles graphs— namely, for all such graphs where every directed cycle is *induced*. We call such graphs “chordless siblings-on-cycles graphs.”

Theorem 7 (Admissible causal structures (chordless)).—If $G = (V, E)$ is a chordless siblings-on-cycles graph, then the causal structure G equipped with the model parameters of Definition 5 forms a *consistent* causal model.

Before we prove this statement, we need to establish the following lemma.

Lemma 8.—Let $G = (V, E)$ be a chordless siblings-on-cycles graph. If G contains a directed cycle C, then the induced graph $G[V']$ with $V' = \mathrm{CPa}(C) \cup \mathrm{Anc}(\mathrm{CPa}(C))$ is a chordless siblings-on-cycles graph, where V' is nonempty and is a strict subset of V.

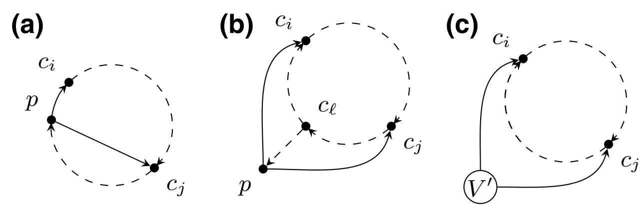

FIG. 5. Cases in which the directed cycle C appears in the graph. Solid lines represent edges and dashed lines represent paths. (a) If the common parent p of c_i, c_j is an element in C, then C is not induced. (b) If there exists a directed path from a node in C to p, then the graph contains a noninduced cycle. (c) The only possibility is that there exists a set V' of nodes without paths from C to V'.

Proof.—Let $C = (c_0, c_1, \dots, c_{m-1})$ be a directed cycle in G. This cycle has siblings $c_i, c_j \in C$ with common parent $p \in \mathrm{Pa}(c_i) \cap \mathrm{Pa}(c_j)$. Because C is induced, we have $p \notin C$ [see Fig. 5(a)].

Moreover, G contains no directed path $\pi = (c_\ell, \dots, p)$ with $c_\ell \in C$. If there were such a path and $i \le \ell < j$, then the directed cycle $(p, c_j, \dots, c_i, \dots, c_\ell, \dots)$ would not be induced [see Fig. 5(b)]. A similar argument holds for $\ell < i$ and $j \le \ell$. Since this holds for any pair of siblings on C and any common parent, we conclude that G does not contain any path from any node in C to any common parent p [see Fig. 5(c)]. The induced graph $G[V']$ is a chordless siblings-on-cycles graph. The reason for this is that the nodes V' have the same incoming edges in $G[V']$ and G. Finally, note that V' is nonempty (it contains at least the node p), and $V' \subsetneq V$ because $V' \cap C = \emptyset$. ■

Proof of Theorem 7.—By Theorem 1, the causal model is consistent if and only if for any choice of experiments $\{\mu_k\}_{k\in\mathbb{Z}_n}$ there exists a *unique* fixed point of $\omega \circ \mu$. Towards a contradiction, assume that ω is *not* a classical-deterministic process. Therefore, and by Theorem 2, let the experiments $\{\mu_k\}_{k\in\mathbb{Z}_n}$ be such that $\omega \circ \mu$ has at least two distinct fixed points, r and r'. We start by observing two implications:

$$r_i \neq r'_i \implies \exists p \in \mathrm{Pa}(i) : r_p \neq r'_p, \tag{14}$$

$$r_i \neq r'_i \wedge r_j \neq r'_j \wedge i \neq j \implies \forall p \in \mathrm{CPa}(\{i,j\}) : r_p \neq r'_p. \tag{15}$$

For the first implication (14), suppose r and r' differ at position i, and assume without loss of generality that $r_i = 1$. By the choice of model parameters, we have

$$r_i = 1 = \prod_{p\in\mathrm{Pa}(i)} [i = \mu_p(r_p)], \tag{16}$$

$$r'_i = 0 = \prod_{p\in\mathrm{Pa}(i)} [i = \mu_p(r'_p)]. \tag{17}$$

This means that for *all* parents $p \in \mathrm{Pa}(i)$ the identity $i = \mu_p(r_p)$ holds. But since $r'_i = 0$, there must exist a parent $p \in \mathrm{Pa}(i)$ such that $i \neq \mu_p(r'_p)$: The fixed points also differ on node p. For the second implication (15), additionally suppose that $r_j \neq r'_j$ for a node j different from i. If i and j do not have common parents, then the implication trivially holds. For the alternative, let $p \in \mathrm{CPa}(\{i,j\})$ be an arbitrary common parent. Since $r_i = 1$ and $p \in \mathrm{Pa}(i)$, we have $i = \mu_p(r_p)$, which implies $j \neq \mu_p(r_p)$, and therefore $r_j = 0$. This, in turn, implies $r'_j = 1$ and $j = \mu_p(r'_p)$; the fixed points also differ for p.

Since the graph G is finite, we conclude from implication (14) that G contains at least one directed cycle on which the fixed points differ. Let $V_{\min} \subseteq V$ be a nonempty set of nodes with minimal cardinality $|V_{\min}|$ such that the induced graph $G[V_{\min}]$ is a chordless siblings-on-cycles graph, and which contains a directed cycle C where the fixed points r and r' differ. Moreover, let $i,j \in C$ be siblings on C and let $p \in \mathrm{CPa}(\{i,j\})$ be a common parent. By Lemma 8, there exists a *smaller* nonempty set of nodes $V' = \mathrm{CPa}(C) \cup \mathrm{Anc}(\mathrm{CPa}(C)) \subsetneq V_{\min}$ such that the induced graph $G[V']$ is a chordless siblings-on-cycles graph. Note that the common parent p and all its ancestors $\mathrm{Anc}(p)$ are in the set V'. Implication (15) now states that $r_p \neq r'_p$, and implication (14) states that V' contains a directed cycle C' where the fixed points r and r' differ: the extremality of $|V_{\min}|$ is violated. ■

IV. CAUSAL CORRELATIONS

In this second part of our work, we derive consequences for the observable correlations from graph-theoretic properties of the causal structure. Because of Theorem 5, we restrict ourselves to siblings-on-cycles graphs.

The following graph-theoretic lemma is helpful for our first result.

Lemma 9.—If $G = (V, E)$ is a chordless siblings-on-cycles graph, then G contains a source node, i.e., $\exists k \in V : \deg_{\mathrm{in}}(k) = 0$.

Proof.—We prove this lemma by contradiction and an extremality argument. Suppose G is a chordless siblings-on-cycles graph, but *without* source nodes. Let $V_{\min} \subseteq V$ be a nonempty set of nodes with *minimal cardinality* $|V_{\min}|$ such that the induced graph $G[V_{\min}]$ has the same properties as G, i.e., $G[V_{\min}]$ is a chordless siblings-on-cycles graph with no source nodes. Since $G[V_{\min}]$ is nonempty and every node has an incoming edge, this induced graph contains a directed cycle $C = (c_0, c_1, \dots, c_{m-1})$. By Lemma 8, however, there exists a *smaller* set of nodes V' such that the induced graph $G[V']$ has the same properties as G. ■

We now relate causal structures with *causal* correlations.

Theorem 10 (Causal correlations).—Let ω be a classical-deterministic process with causal structure $G = (V, E)$. If G is a chordless siblings-on-cycles graph, then

ω always produces causal correlations, i.e., for all experiments μ, the correlations $p(a \mid x) = \omega \star \mu^{a|x}$ are causal.

Proof.—By Lemma 9, the graph G contains a source node $k \in V$, i.e., $\mathrm{Pa}(k) = \emptyset$. Therefore, the kth component of ω is a constant (function) $\omega_k \in \mathcal{I}_k$ ($\omega_k : \emptyset \to \mathcal{I}_k$), and we can express the correlations $p(a \mid x)$ given by Eq. (5) as

$$\begin{aligned} p(a \mid x) = &\sum_{i_k, o_k} [\omega_k = i_k]\,[(a_k, o_k) = \mu_k(x_k, i_k)] \\ &\times \sum_{i_{\setminus k}, o_{\setminus k}} \left[\omega_{\setminus k}(o) = i_{\setminus k}\right] \\ &\times \left[\left(a_{\setminus k}, o_{\setminus k}\right) = \mu_{\setminus k}\left(x_{\setminus k}, i_{\setminus k}\right)\right] \end{aligned} \tag{18}$$

$$\begin{aligned} = &\left[a_k = \mu_k^0(x_k, \omega_k)\right] \times \left(\sum_{i_{\setminus k}, o_{\setminus k}} \left[\omega'_{\setminus k}\left(o_{\setminus k}\right) = i_{\setminus k}\right] \right. \\ &\left. \left[\left(a_{\setminus k}, o_{\setminus k}\right) = \mu_{\setminus k}\left(x_{\setminus k}, i_{\setminus k}\right)\right]\right), \end{aligned} \tag{19}$$

where $\omega'_\ell(o_{\mathrm{Pa}(\ell)})$ is $\omega_\ell(\mu_k^1(x_k, \omega_k), o_{\mathrm{Pa}(\ell)\setminus\{k\}})$ if $k \in \mathrm{Pa}(\ell)$ and is $\omega_\ell(o_{\mathrm{Pa}(\ell)})$ otherwise. This $\omega'_{\setminus k}$ is a *reduced process* (see Lemma 3). The former term of Eq. (19) describes the observed statistics $p(a_k \mid x_k)$ for party k and the latter term has the form $\omega'_{\setminus k} \star \mu_{\setminus k}^{a_{\setminus k}|x_{\setminus k}}$. Therefore, the correlations $p(a \mid x)$ decompose as

$$p(a \mid x) = p(a_k \mid x_k) p^{x_k}(a_{\setminus k} \mid x_{\setminus k}). \tag{20}$$

The causal structure G' of the reduced process $\omega'_{\setminus k}$ remains a chordless siblings-on-cycles graph (no edges are introduced and the reduced process is a classical-deterministic process). By repeating this argument, we end at the decomposition of $p(a \mid x)$ as in Definition 1. ■

Theorem 10 and the proof thereof lead to the following conjecture for the quantum case.

Conjecture 2 (Quantum causal correlations).—Let W be a quantum process with causal structure $G = (V, E)$. If G is a chordless siblings-on-cycles graph, then W always produces causal correlations.

It is well known that the quantum switch mentioned in Sec. I has a chordless siblings-on-cycles causal structure [see Fig. 1(b)] and also that it does not violate causal inequalities [9,11].

V. NONCAUSAL CORRELATIONS

Some causal structures imply that causal inequalities *can* be violated. Before we state and prove these results, we introduce the following causal game.

Causal game ($G_\mathcal{S}^n$).—Consider a scenario with n parties $\mathbb{Z}_n$ and let $\mathcal{S}$ be a nonempty subset of $\mathbb{Z}_n$. For each party $k \in \mathbb{Z}_n$, the set of settings is $\mathcal{X}_k := (\mathbb{Z}_n \cup \{\perp\}) \times (\mathbb{Z}_2 \cup \{\perp\})$ and the set of observations is $\mathcal{A}_k := \mathbb{Z}_2$. A referee uniformly at random picks a party $s \in \mathcal{S}$ and a bit $b \in \mathbb{Z}_2$. Then the referee distributes s to all parties in $\mathcal{S}$, b to all parties in $\mathcal{S} \setminus \{s\}$, and nothing ($\perp$) to the remaining parties, i.e., the settings are $x_s = (s, \perp)$, $x_k = (s, b)$ for $k \in \mathcal{S} \setminus \{s\}$, and $x_\ell = (\perp, \perp)$ for $\ell \in \mathbb{Z}_n \setminus \mathcal{S}$. The parties win the game $G_\mathcal{S}^n$ whenever party s correctly guesses b, i.e., whenever $a_s = b$.

This causal game [40] has a nontrivial upper bound on the winning probability for causal correlations.

Theorem 11 (Causal inequality).—If $p(a \mid x)$ are n-party causal correlations, then the winning probability of the game $G_\mathcal{S}^n$ is bounded by

$$\Pr[a_s = b] \le 1 - \frac{1}{2|\mathcal{S}|}. \tag{21}$$

Proof.—Suppose the correlations $p(a \mid x)$ decompose as

$$p(a \mid x) = p(a_k \mid x_k) p_{a_k}^{x_k}(a_{\setminus k} \mid x_{\setminus k}), \tag{22}$$

where $p_{a_k}^{x_k}(a_{\setminus k} \mid x_{\setminus k})$ are $(n-1)$-party causal correlations and $k \in \mathcal{S}$. In the event that $s = k$, which happens with probability $1/|\mathcal{S}|$, the game is won with half probability (the bit b is uniformly distributed and party s has no access to b). In the event that $s \neq k$, the game is won with probability at most 1. Therefore, the winning probability is upper bounded by

$$\frac{1}{|\mathcal{S}|}\left(\frac{1}{2} + |\mathcal{S}| - 1\right) = 1 - \frac{1}{2|\mathcal{S}|}. \tag{23}$$

The same bound holds for any other decomposition of $p(a \mid x)$, and therefore also for convex combinations thereof. ■

This brings us to a graph-theoretic criterion that implies a violation of the above causal inequality.

Theorem 12 (Noncausal correlations).—Let ω be a classical-deterministic process with causal structure $G = (V, E)$. If G contains a directed cycle C where all common parents are in C, i.e., $\mathrm{CPa}(C) \subseteq C$, then ω produces noncausal correlations; i.e., there exists an experiment μ such that the correlations $p(a \mid x) = \omega \star \mu^{a|x}$ violate the causal inequality for the game $G_C^{|V|}$, and therefore the correlations $p(a \mid x)$ are noncausal.

Proof.—Let the directed cycle C be $(c_0, c_1, \dots, c_{m-1})$, σ such that $s = c_\sigma$, and $s^- := c_{\sigma - 1 \pmod m}$, and let D be the set $\mathrm{Pa}(C) \setminus C$. The condition $\mathrm{CPa}(C) \subseteq C$ implies that every party $d \in D$ has a *single* edge to a party in C. Therefore, and since the causal model is faithful, there exists an experiment μ_D for all parties in D such that signals can be sent along the cycle C. Again because of faithfulness, there exists an experiment $\mu_{D'}$ for the parties in $D' := (\mathrm{Pa}(s) \cap C) \setminus \{s^-\}$ such that party s^- can send a signal to s. Finally, there exists an experiment for party s^- such that party s receives an encoding of b on the input space $\mathcal{I}_s$ (the set $\mathcal{I}_s$ contains at least two elements, but is

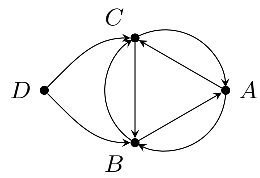

FIG. 6. Example of an admissible causal structure that does not satisfy the constraints of Theorem 10 or Theorem 12.

not necessarily equal to $\mathbb{Z}_2$). By implementing these experiments, an experiment μ_s that decodes b from the input i_s for party s and an arbitrary experiment μ_{rem} for all remaining parties, we obtain $p(a \mid x) = \omega \star \mu^{(a|x)}$ such that a_s *deterministically* takes value b, i.e., $\Pr[a_s = b] = 1$. This violates the causal inequality of Theorem 11. ■

With these results at hand, we can complete Fig. 4: only graph 16 leads to noncausal correlations. This causal structure is actually the causal structure of the Araújo-Feix/Baumeler-Wolf process [37,38]—the first classical-deterministic process known to yield noncausal correlations. An example of a classical-deterministic process ω to deterministically win the game $\mathcal{G}_S^n$ is given in Ref. [27]. The processes described in that article have as causal structure the fully connected graph. Such a graph clearly has a Hamiltonian cycle $\mathcal{C}_H$, and therefore all common parents $\mathrm{CPa}(\mathcal{C}_H)$ are inside that cycle: Theorem 12 is applicable.

VI. CONCLUSION

In the first part of this work, we characterized a set of causal structures for which any faithful causal model is *inconsistent.* We consequently provided, given any directed graph G, a construction of classical-deterministic model parameters. We conjecture—and proved a restricted form thereof—that any causal model with these model parameters and any causal structure not in the inconsistent set is *consistent*. This conjecture complements Theorem 5. It implies that there exists a causal model with causal structure G if and only if G is a siblings-on-cycles graph. A direct consequence of this conjecture is that quantum theory does not allow for more general causal connections when compared with classical theories.

In the second part, we used this characterization to show that two sets of causal structures lead to either *causal* or *noncausal* correlations in the classical-deterministic case. Note that a *decisive* graph-theoretic criterion for (non)causal correlations is impossible. As a simple example, take the causal structure depicted in Fig. 6. If this causal structure is equipped with the model parameters from Definition 5, one obtains a consistent causal model that does *not* violate any causal inequality. Intuitively, this follows because the role of the common parent D in the A-B-C cycle is equal to that of any other common parent. So, effectively, the graph is "chordless." If, however, one uses these model parameters for the parties A, B, and C only, and extends them with identity channels from $\mathcal{O}_D$ to auxiliary input spaces of the parties B and C, then the resulting causal model is consistent and *violates* causal inequalities (in particular, the causal inequality of Theorem 11 with $\mathcal{S} = \{A, B, C\}$). This holds because the induced graph with nodes A, B, and C is simply graph 16 in Fig. 4.

Proving Conjecture 1 would unlock many possibilities as it provides explicit examples of processes producing *noncausal* correlations for any number of parties and any admissible causal structure that contains a directed cycle of which all its common parents are part. It can therefore be a powerful tool to explore further noncausal correlations of different strength in the multiparty case. Then again, Conjecture 2 would provide a better understanding of cyclic causal models that give rise to causal correlations. The difficulty in proving this conjecture stems from the fact that the causal structure of a *reduced* quantum process might not necessarily be representable by a directed graph.

We briefly comment on how our results connect to various facets of other studies. Recently, several constructive bottom-up approaches to quantum processes with indefinite causal order have been explored (see, e.g., Refs. [41–43]). These approaches are contrasted with our complementary top-bottom approach. Especially, exploration of the connection to "routed circuits" and "sectorial decompositions" [44–46] might be insightful, also in light of proving the conjectures presented. Gogioso and Pinzani [47] studied causal order in a theory-independent fashion. Theorem 5, however, imposes a restriction on causal orders for Markovian quantum processes. For instance, the totality of two events in an indefinite causal order is inadmissible [23,36,48]. It is expected that Theorem 5 imposes certain constraints on the "join decomposition" of Markovian quantum causal orders. Apadula *et al.* [49] derived admissibility constraints for the composition of higher-order maps: two higher-order maps can be composed whenever no signaling loop is created. If one wishes to compose the result with a third higher-order map, however, a more in-depth analysis is required. In this light, our results can be understood as such an analysis of a slice within the hierarchy of higher-order computation. Namely, we studied the admissibility of *simultaneous* composition of channels (experiments) with processes. Recently, Eftaxias *et al.* [50] characterized the set of effects in the generalized probabilistic theory of "box world." This theory admits effects that are *not* wirings [51], and it turns out that these nonwiring effects are classical processes. Our graph-theoretic criterion thus specifies the most general signaling structures of these effects. Any operator with a cyclic signaling structure without siblings must therefore be excluded as an effect.

A series of open questions—apart from proving the conjectures and the precise connection to related work—emerge. A central question is to what extent our results are theory independent. As suggested by Conjecture 1, the set of admissible quantum causal structures coincides with the set of admissible classical causal structures. Does this potential coincidence extend to other theories? Another question is how to embed admissible causal structures in space-time geometries of general relativity and quantum gravity [13,52], and how they can be achieved in a time-delocalized formulation [33] (note that some *fine-tuned* cyclic causal structures are known to be embeddable in Minkowski space-time [53]). Finally, this limitation on causal connections might unlock new information-processing protocols, e.g., in the presence of local quantum theory and classical communication without causal order [54].

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[1] This means that if a nongravitational experiment is conducted in a sufficiently small space-time region $\mathcal{R}$ with a gravitational field, then for any space-time region $\mathcal{R}'$ *free* of gravitation there exists a suitable reference frame where the same experimental procedure yields the identical experimental data. This statement and its variations are discussed in Ref. [55].
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